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APPLICATIONS OF THE ROPER-SUFFRIDGE EXTENSION OPERATOR FOR SOME SUBCLASSES OF STARLIKE MAPPINGS

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Abstract. Let $S^*_{\Omega_{n,p_2,\ldots,p_n}}(\beta, A, B)$ be some new subclasses of starlike mappings on Rienhardt domain $\Omega_{n,p_2,\ldots,p_n}$, where $-1 \le A < B < 1$ and $p_j \ge 1$, $j = 2, \ldots, n$ be a positive integer. Some different conditions for a_j are established such that these classes are preserved under the following modified Roper-Suffridge operator $F(z) = \left(f(z_1) + f'(z_1) \sum_{j=2}^{n} a_j z_j^{p_j}, (f'(z_1))^{\frac{1}{p_2}} z_2, \ldots, (f'(z_1))^{\frac{1}{p_n}} z_n \right)$, where f is a normalized locally biholomorphic function on the unit disc U. On the other hand, almost starlike mapping of complex order λ on Rienhardt domain $\Omega_{n,p_2,\ldots,p_n}$ is defined. A necessary and sufficient condition for a_j are established such that under which the above modified Roper-Suffridge operator preserves an almost starlike mapping of complex order λ. These results generalize the modified Roper-Suffridge extension operator from the unit ball to Reinhardt domains. Our result reduce to many well-known results.

1. Introduction and Preliminaries

Let n be a positive integer and $\mathbb{C}^n$ denote the space of n complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w}_j$ and Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$, where $z, w \in \mathbb{C}^n$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by B_r^n and the unit ball B_1^n by B^n. The closed ball $\{z \in \mathbb{C}^n : \|z\| \le r\}$ is denoted by $\bar{B}_r^n$, and the unit sphere is denoted by $\partial B^n = \{z \in \mathbb{C}^n : \|z\| = 1\}$. In the case of one complex variable, B^1 is denoted by U. For $n \ge 2$, let $\hat{z} = (z_2, \ldots, z_n)$ so that $z = (z_1, \hat{z}) \in \mathbb{C}^n$.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of complex linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$ with the standard operator norm,

$$\| A \| = \sup\{\| A(z) \| : \| z \| = 1\},$$

and let I_n be the identity in $L(\mathbb{C}^n, \mathbb{C}^n)$. Let Ω be a domain in $\mathbb{C}^n$ and $H(\Omega)$ be the set of holomorphic mappings from Ω into $\mathbb{C}^n$. A mapping $f \in H(\Omega)$ is called normalized if $f(0) = 0$ and $J_f(0) = I_n$, where $J_f(0)$ is the complex Jacobian matrix of f at the origin. A mapping $f \in H(\Omega)$ is said to be locally biholomorphic if $\det J_f(z) \ne 0$ for every $z \in \Omega$. Let $LS(\Omega)$ be the set of normalized locally biholomorphic mappings on Ω and let $S(\Omega)$ denote the set of normalized biholomorphic mappings on Ω. In the case of one complex variable, the set $S(B^1)$ is denoted by S and $LS(B^1)$ is denoted by LS. A mapping $f \in S(\Omega)$ is called starlike (respectively convex) if its image is a starlike domain with respect to origin (respectively convex domain). The class of starlike (respectively convex) mappings on Ω will be denoted by $S^*(\Omega)$ (respectively $K(\Omega)$). In the case of one complex variable $S^*(B^1)$(respectively $K(B^1)$) is denote by S^* (respectively K). A normalized mapping $f \in H(\Omega)$ is said to be ε starlike if there exists a positive number ε, $0 \le \varepsilon \le 1$, such that $f(B^n)$ is starlike with respect to every point in $\varepsilon f(B^n)$.

A domain Ω is called a circular domain if $e^{i\theta} z \in \Omega$ holds for any $z \in \Omega$ and $\theta \in \mathbb{R}$. A domain $\Omega \subset \mathbb{C}^n$ is said to be a complete Reinhardt if $(z_1, z_2, \ldots, z_n) \in \Omega$ implies that $(e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_n} z_n) \in \Omega$ for all $\theta_j \in \mathbb{R}$, $j = 1, 2, \ldots, n$. The Minkowski functional $\rho(z)$ of a bounded circular convex domain Ω in $\mathbb{C}^n$ is defined as

$$\rho(z) = \inf\left\{t > 0, \frac{z}{t} \in \Omega\right\}, \qquad z \in \mathbb{C}^n.$$

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If Ω is a bounded circular convex domain, then Ω is a Banach space in $\mathbb{C}^n$ with respect to this norm, and $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 1\}$. Also, The Minkowski functional $\rho(z)$ is C^1 on $\overline{\Omega}$ except for a lower dimensional manifold. Moreover, the Minkowski functional $\rho(z)$ has the following properties of (see [16]):

$$\frac{\partial\rho}{\partial z}(\lambda z) = \frac{\partial\rho}{\partial z}(z), \qquad \lambda \in [0, +\infty), \ z \in \Omega\backslash\{0\},$$

$$\frac{\partial\rho}{\partial z}(e^{i\theta} z) = e^{-i\theta}\frac{\partial\rho}{\partial z}(z), \qquad \theta \in \mathbb{R}, \ z \in \mathbb{C}^n\backslash\{0\}. \tag{1.1}$$

Definition 1.1. [31] *Suppose that Ω is a bounded convex circular domain which contains the origin in $\mathbb{C}^n$. Its Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let f be a normalized locally biholomorphic mapping on Ω. If $\alpha \in (0, 1)$, $\beta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, and*

$$\left| e^{-i\beta}\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_f^{-1}(z)f(z) - \left(\frac{\cos\beta}{2\alpha} - i\sin\beta\right)\right| < \frac{\cos\beta}{2\alpha}, \qquad z \in \Omega\backslash\{0\},$$

then f is said to be spirallike mapping of type β and order α on Ω.

Definition 1.2. [2] *Suppose that Ω is a bounded convex circular domain which contains the origin in $\mathbb{C}^n$. Its Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let f be a normalized locally biholomorphic mapping on Ω. If $\alpha \in (0, 1)$, $\beta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, and*

$$\left| i\tan\beta + (1 - i\tan\beta)\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_f^{-1}(z)f(z) - \frac{1+\alpha^2}{1-\alpha^2}\right| \le \frac{2\alpha}{1-\alpha^2}, \qquad z \in \Omega\backslash\{0\}$$

then f is said to be strongly spirallike mapping of type β and order α on Ω.

Wang [27] introduced the following classes of starlike mappings from the unified perspective which contains the above two definitions.

Definition 1.3. [27] *Suppose that Ω is a bounded complete circular domain which contains the origin in $\mathbb{C}^n$. Its Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let f be a normalized locally biholomorphic mapping on Ω. If $-1 \le A < B < 1$, $\beta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, and*

$$\left| i\tan\beta + (1 - i\tan\beta)\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_f^{-1}(z)f(z) - \frac{1-AB}{1-B^2}\right| < \frac{B-A}{1-B^2}, \qquad z \in \Omega\backslash\{0\}, \tag{1.2}$$

then we say that $f \in S^*_\Omega(\beta, A, B)$.

When $\Omega = U$, the inequality (1.2) becomes

$$\left| i\tan\beta + (1 - i\tan\beta)\frac{f(z)}{zf'(z)} - \frac{1-AB}{1-B^2}\right| \le \frac{B-A}{1-B^2}, \qquad z \in U.$$

Remark 1.4. *When $A = -1$, $B = 1 - 2\alpha$, Definition 1.3 reduces to Definition 1.1.*
When $A = -\alpha$, $B = \alpha$, Definition 1.3 reduces to Definition 1.2.
The geometric property of (1.2) *shows that the image of mapped by the mapping*

$$i\tan\beta + (1 - i\tan\beta)\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_f^{-1}(z)f(z)$$

is an open disk with diameter end points $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$. Hence, when $B \to 1^-$, the image of Ω reduces to the half plan $\{z : Re\ z \ge \frac{1+A}{2}\}$.

Definition 1.5. [26] *Suppose Ω is a bounded convex circular domain which contains the origin in $\mathbb{C}^n$, and let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $Re\langle A(z), z\rangle > 0$. A normalized biholomorphic mapping f on Ω is spirallike with respect to A if $e^{-tA}f(\Omega) \subset \Omega$ for all $t > 0$, where*

$$e^{-tA} = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}t^kA^k.$$

Note that any spirallike mapping with respect to a linear operator A such that $Re\langle A(z), z\rangle > 0$ for $z \in \mathbb{C}^n \setminus \{0\}$ is biholomorphic [26].

Remark 1.6. *If $A = e^{-i\beta} I_n$, $\beta \in (\frac{-\pi}{2}, \frac{\pi}{2})$, , in Definition 1.5, we obtain the class of spirallike mappings of type β, studied by Hamada and Kohr* [11]*. Hence $f \in S^*(\Omega)$ if and only if f is spirallike mappings of type zero.*

In the following we introduce the almost starlike mapping of complex order λ on Rienhardt domain Ω.

Definition 1.7. *Suppose that Ω is a bounded complete circular domain in $\mathbb{C}^n$. Its Minkowski functional $\rho(z)$ is C^1 except for a lower-dimensional manifold. Let $\lambda \in \mathbb{C}$, with $Re\lambda \le 0$. A normalized locally biholomorphic mapping $f \in H(\Omega)$ is said to be an almost starlike mapping of complex order λ if*

$$Re\left((1-\gamma)\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}J_f^{-1}(z)f(z)\right) > -Re\ \lambda\|z\|^2, \qquad z \in \Omega\backslash\{0\}.$$

It is easy to see that in the case of one variable, the above inequality reduces to the following

$$Re\left(1-\lambda)\frac{f(z)}{zf'(z)}\right) > -Re\ \lambda, \qquad z \in U.$$

The interest of the study of almost starlikeness of complex order λ arises from the fact that every almost starlike mapping f of complex order λ is also spirallike with respect to the operator $A = (1-\lambda)I_n$, and hence f is biholomorphic on Ω (see [1]).

Remark 1.8. *If we take $\lambda = i\tan\beta$, in Definition 1.7, where $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we obtain the usual notion of spirallike of type β and when $\lambda = 0$, we obtain the usual notion of starlike (see* [33]*).*

In 1995, Roper and Suffridge [25] introduced an extension operator which gives a way of extending a locally biholomorphic function on the unit disc U in $\mathbb{C}$ to a locally biholomorphic mapping of B^n into $\mathbb{C}^n$. For fixed $n \ge 2$, the Roper-Suffridge extension operator (see [10] and [25]) is defined as follows

$$[\Phi_n(f)](z) = (f(z_1), \sqrt{f'(z_1)}\hat{z}), \qquad z \in B^n,$$

where f is a normalized biholomorphic mapping on the unit disc U in $\mathbb{C}$, $z = (z_1, \hat{z})$ belonging to the unit ball B^n in $\mathbb{C}^n$ and the branch of the power function is chosen so that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

The following results illustrate the important and usefulness of the Roper-Suffridge extension operator

$$\Phi_n(K) \subseteq K(B^n), \qquad \Phi_n(S^*) \subseteq S^*(B^n).$$

The first was proved by Roper and Suffridge when they introduced their operator [25], while the second result was given by Graham and Kohr [9]. Until now, it is difficult to construct the concrete convex mappings, starlike mappings on B^n. By making use of the Roper-Suffridge extension operator, we may easily give many concrete examples about these mappings. This is one important reason why people are interested in this extension operator. A good treatment of further applications of the Roper-Suffridge extension operator can be found in the recent book by Graham and Kohr [10].

In 2005, Muir [18] modified the Roper- Suffridge extension operator as follows

$$[\Phi_{n,Q}(f)](z) = \left(f(z_1) + f'(z_1)Q(\hat{z}), \sqrt{f'(z_1)}\hat{z}\right), \qquad z = (z_1, \hat{z}) \in B^n,$$

where $Q(\hat{z})$ is a homogeneous polynomial of degree 2 with respect to $\hat{z}$, and f, z_1 and $\hat{z}$ are defined as above. He proved that this operator preserves starlikeness and convexity if and only if $\|Q\| \le 1/4$ and $\|Q\| \le 1/2$, respectively. Also Rahrovi et all [22] proved that this operator preserves spirallike mapping of type β if and only if $\|Q\| \le 1/4$. The modified operator $\Phi_{n,Q}$ plays a key role to study the extreme points of convex mappings on B^n (see [19], [20]). Later, Kohr [12], Muir [17] and Rahrovi et all [23] used the Loewner chain to study the modified Roper-Suffridge extension operator. Recently, the modified Roper-Suffridge extension operator on the unit ball is also studied by Wang and Liu [29] and Feng and Yu [5].

On the other hand, people also considered the generalized Roper-Suffridge extension operator on the general Reinhardt domains. For example, Gong and Liu [7], [14] introduced the definition of $\varepsilon-$starlike mappings and obtained that the operator

$$[\Phi_{n,\frac{1}{p}}(f)](z) = \left(f(z_1), (f'(z_1))^{\frac{1}{p}}\hat{z}\right),$$

maps the $\varepsilon-$starlike functions on U to the $\varepsilon-$starlike mappings on the Reinhardt domain $\Omega_{n,p} = \left\{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^{n} |z_j|^p < 1\right\}$, where $p \geq 1$, and f, z_1 and $\hat{z}$ are defined as above. When $\varepsilon = 0$ and $\varepsilon = 1$, $\Phi_{n,\frac{1}{p}}$ maps the starlike function and convex function on U to the starlike mapping and the convex mapping on $\Omega_{n,p}$, respectively. Furthermore, Gong and Liu [8] proved that the operator

$$[\Phi_{n,\frac{1}{p_2},\ldots,\frac{1}{p_n}}(f)](z) = \left(f(z_1), (f'(z_1))^{\frac{1}{p_2}} z_2, \ldots, (f'(z_1))^{\frac{1}{p_n}} z_n\right),$$

maps the $\varepsilon-$starlike functions on U to the $\varepsilon-$starlike mappings on the Reinhardt domain $\Omega_{n,p_2,\ldots,p_n} = \left\{z \in \mathbb{C}^n : |z_1|^2 + \sum_{j=2}^{n} |z_j|^{p_j} < 1\right\}$, where $p_j \geq 1$, and f, z_1 and $\hat{z}$ are defined as above. Also, Liu and Liu [15] proved that this operator preserves starlikeness of order α on the domain $\Omega_{n,p_2,\ldots,p_n}$. On the other hand, Feng and Liu [5] proved that this operator preserves almost starlikeness of order α on the domain $\Omega_{n,p_2,\ldots,p_n}$.

In contrast to the modified Roper-Suffridge extension operator on the unit ball, it is natural to ask if we can modify the Roper-Suffridge extension operator on the Reinhardt domain.

In 2011, Wang and Gao [28] introduced the following extension operator:

$$F(z) = \left(f(z_1) + f'(z_1)\sum_{j=2}^{n} a_j z_j^{p_j}, (f'(z_1))^{\frac{1}{p_2}} z_2, \ldots, (f'(z_1))^{\frac{1}{p_n}} z_n\right), \tag{1.3}$$

on the Reinhardt domain $\Omega_{n,p_2,\ldots,p_n}$, where p_j are positive integer and $p_j \geq 1$, the branch are chosen such that $(f'(z_1))^{\frac{1}{p_j}}|_{z_1=0}= 1$, $j = 2, \ldots, n$. For $|a_j| \leq \frac{(1-\alpha)}{4}$, $j = 2, \cdots, n$, they proved that this operator preserves almost starlike function of order α and on the Renihardt domain $\Omega_{n,p_2,\cdots,p_n}$. In this paper, we will give some necessary and sufficient conditions for a_j under which the above Roper-Suffridge operator preserves the classes $S^*_{\Omega_{n,p_2,\ldots,p_n}}(\beta, A, B)$. Also, under special condition for a_j, $j = 2, \ldots, n$, we will show that f is an almost starlike function of complex order λ on U if and only if F is an almost starlike function of complex order λ on $\Omega_{n,p_1,\cdots,p_n}$.

In order to prove the main results, we need the following lemmas.

Lemma 1.9. [10]. *(Schwarz-Pick Lemma) Suppose that $g \in H(U)$ satisfies $g(U) \subset U$, then*

$$|g'(\xi)| \leq \frac{1 - |g(\xi)|^2}{1 - |\xi|^2},$$

for each $\xi \in U$.

Lemma 1.10. [21]. *Let f be a normalized biholomorphic function on U, then*

$$\left|(1 - |z|^2)\frac{f''(z)}{f'(z)} - 2\bar{z}\right| \leq 4, \qquad z \in U.$$

Lemma 1.11. [21]. *Let p be a holomorphic function on U. If $Rep(z) > 0$ and $p(0) > 0$, then*

$$|p'(z)| \leq \frac{2Rep(z)}{1 - |z|^2}.$$

Lemma 1.12. [32]. *If $\rho(z)$ is a Minkowski function of the domain $\Omega_{n,p_2,\ldots,p_n}$, $z \neq 0$, then*

$$\frac{\partial\rho}{\partial z_1}(z) = \frac{\bar{z}_1}{\rho(z)\left[2\left|\frac{z_1}{\rho(z)}\right|^2 + \sum\limits_{j=2}^{n} p_j \left|\frac{z_j}{\rho(z)}\right|^{p_j}\right]},$$

$$\frac{\partial\rho}{\partial z_j}(z) = \frac{p_j \bar{z}_j \left|\frac{z_j}{\rho(z)}\right|^{p_j-2}}{2\rho(z)\left[2\left|\frac{z_1}{\rho(z)}\right|^2 + \sum\limits_{j=2}^{n} p_j \left|\frac{z_j}{\rho(z)}\right|^{p_j}\right]}, \qquad j = 2, \ldots, n.$$

2. Main Results

We begin this section with the main results of this paper.

Theorem 2.1. *Let $\alpha \in (0,1)$ and $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Suppose that the operator $F(z)$ is defined by (1.3). If complex numbers a_j satisfy the condition $|a_j| \leq \frac{(B-A)(1-|B|)\cos\beta}{4(1-B^2)}$, $j = 2, \ldots, n$, then $F \in S^*_{\Omega_{n,p_2,\ldots,p_n}}(\beta, A, B)$ if and only if $f \in S^*_U(\beta, A, B) = S^*(\beta, A, B)$.*

Proof. By the definition 1.3, we only need to prove that the following inequality

$$\left|\frac{1-B^2}{B-A}\left[i\tan\beta + (1 - i\tan\beta)\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right] - \frac{1-AB}{B-A}\right| < 1$$

holds for all $z \in \Omega_{n,p_2,\ldots,p_n}$ and $z \neq 0$ and $|a_j| \leq \frac{(B-A)(1-|B|)\cos\beta}{4(1-B^2)}$. For $z = (z_1, \hat{z}) \in B^n$, we have two cases

First, if $\hat{z} = 0$, then we can get the conclusion easily.

Second, suppose $\hat{z} \neq 0$. Obviously, the mapping F is holomorphic at every point $z = (z_1, \hat{z}) \in \overline{\Omega}_{n,p_2,\ldots,p_n}$. Let us write $z = \lambda u = |\lambda|e^{i\theta}u$ for $u \in \partial\Omega_{n,p_2,\ldots,p_n}$ and $\lambda \in \overline{U}\backslash\{0\}$, then from we have

$$\left|\frac{1-B^2}{B-A}\left[i\tan\beta + (1 - i\tan\beta)\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right] - \frac{1-AB}{B-A}\right| < 1$$

$$\Leftrightarrow \left|\frac{1-B^2}{B-A}\left[i\tan\beta + (1 - i\tan\beta)\frac{2}{\rho(|\lambda|e^{i\theta}u)}\frac{\partial\rho}{\partial z}(|\lambda|e^{i\theta}u)J_F^{-1}(|\lambda|e^{i\theta}u)F(|\lambda|e^{i\theta}u)\right] - \frac{1-AB}{B-A}\right| < 1$$

$$\Leftrightarrow \left|\frac{1-B^2}{B-A}\left[i\tan\beta + (1 - i\tan\beta)\frac{2}{|\lambda|}\frac{e^{-i\theta}\partial\rho}{\partial z}(u)J_F^{-1}(|\lambda|e^{i\theta}u)F(|\lambda|e^{i\theta}u)\right] - \frac{1-AB}{B-A}\right| < 1$$

$$\Leftrightarrow \left|\frac{1-B^2}{B-A}\left[i\tan\beta + (1 - i\tan\beta)\frac{2}{\lambda}\frac{\partial\rho}{\partial z}(u)J_F^{-1}(\lambda u)F(\lambda u)\right] - \frac{1-AB}{B-A}\right| < 1.$$

The expression

$$\frac{1-B^2}{B-A}\left[i\tan\beta + (1 - i\tan\beta)\frac{2}{\lambda}\frac{\partial\rho}{\partial z}(u)J_F^{-1}(\lambda u)F(\lambda u)\right] - \frac{1-AB}{B-A}$$

is holomorphic with respect to λ. Thus, the maximum modules principle for holomorphic functions yield that it attains its maximum on $|\lambda| = 1$. Therefor we need only to prove for all $z = (z_1, \hat{z}) \in \partial\Omega_{n,p_2,\ldots,p_n}$ such that $\hat{z} \neq 0$. Hence, $\rho(z) = 1$, and it is suffice to show that

$$\left|\frac{1-B^2}{B-A}\left[i\tan\beta + (1 - i\tan\beta)\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right] - \frac{1-AB}{B-A}\right| < 1,$$

holds for $z \in \partial\Omega_{n,p_2,\ldots,p_n}$ and $\hat{z} \neq 0$.

Since

$$F(z) = \left(f(z_1) + f'(z_1)\sum_{j=2}^{n} a_j z_j^{p_j}, (f'(z_1))^{\frac{1}{p_2}} z_2, \ldots, (f'(z_1))^{\frac{1}{p_n}} z_n\right),$$

we have

$$J_F(z) = \begin{bmatrix} f'(z_1) + f''(z_1) \sum_{j=2}^{n} a_j z_j^{p_j} & a_2 p_2 f'(z_1) z_2^{p_2-1} & \cdots & a_n p_n f'(z_1) z_n^{p_n-1} \\ a_2 & (f'(z_1))^{1/p_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n & 0 & \cdots & (f'(z_1))^{1/p_n} \end{bmatrix},$$

where

$$a_j = \frac{1}{p_j}(f'(z_1))^{(1/p_j)-1} f''(z_1) z_j, \qquad j = 2, \ldots, n$$

Suppose that $J_F^{-1}(z)F(z) = A = (x_1, x_2, \ldots, x_n)$, some simple computation shows that

$$x_1 = \frac{f(z_1)}{f'(z_1)} - \sum_{j=2}^{n} a_j (p_j - 1) z_j^{p_j},$$

$$x_j = \left(1 - \frac{f(z_1) f''(z_1)}{p_j (f'(z_1))^2} + \frac{f''(z_1)}{p_j f'(z_1)} \sum_{k=2}^{n} a_k (p_k - 1) z_k^{p_k}\right) z_j, \qquad j = 2, \ldots, n.$$

Therefore we get

$$\begin{aligned} \frac{\partial \rho(z)}{\partial z} J_F^{-1}(z) F(z) &= \frac{f(z_1)}{z_1 f'(z_1)} \frac{\partial \rho(z)}{\partial z_1} z_1 - \sum_{j=2}^{n} a_j (p_j - 1) z_j^{p_j} \frac{\partial \rho(z)}{\partial z_1} \\ &+ \sum_{j=2}^{n} \left(1 - \frac{f(z_1) f''(z_1)}{p_j (f'(z_1))^2} + \frac{f''(z_1)}{p_j f'(z_1)} \sum_{k=2}^{n} a_k (p_k - 1) z_k^{p_k}\right) \frac{\partial \rho(z)}{\partial z_j} z_j. \end{aligned} \tag{2.1}$$

Now, from Lemma 1.12, we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial z_1}(z) &= \frac{\bar{z}_1}{\rho(z)\left[2|z_1/\rho(z)|^2 + \sum_{j=2}^{n} p_j |z_j/\rho(z)|^{p_j}\right]} = \frac{\bar{z}_1}{2|z_1|^2 + \sum_{j=2}^{n} p_j |z_j|^{p_j}}, \\ \frac{\partial \rho}{\partial z_j}(z) &= \frac{p_j \bar{z}_j |z_j/\rho(z)|^{p_j-2}}{2\rho(z)\left[2|z_1/\rho(z)|^2 + \sum_{j=2}^{n} p_j |z_j/\rho(z)|^{p_j}\right]} = \frac{p_j \bar{z}_j |z_j|^{p_j-2}}{2\left[2|z_1|^2 + \sum_{j=2}^{n} p_j |z_j|^{p_j}\right]}. \end{aligned} \tag{2.2}$$

In terms of (2.1) and (2.2), we obtain

$$\frac{1-B^2}{B-A}\left[i \tan\beta + (1 - i\tan\beta) \frac{2\partial\rho}{\partial z}(z) J_F^{-1}(z) F(z)\right] - \frac{1-AB}{B-A} = \frac{G(z)}{2|z_1|^2 + \sum_{j=2}^{n} p_j |z_j|^{p_j}}, \tag{2.3}$$

where

$$G(z) = \frac{1-B^2}{B-A}\Bigg[i\tan\beta\Big(2|z_1|^2 + \sum_{j=2}^{n} p_j|z_j|^{p_j}\Big) + (1-i\tan\beta)\Big[2|z_1|^2\frac{f(z_1)}{z_1 f'(z_1)}$$

$$+ \sum_{j=2}^{n} p_j|z_j|^{p_j}\left(1 - \frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right) + \sum_{j=2}^{n} a_j(p_j-1)z_j^{p_j}\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2) - 2\bar{z}_1\right)\Big]\Bigg]$$

$$- \frac{1-AB}{B-A}\left(2|z_1|^2 + \sum_{j=2}^{n} p_j|z_j|^{p_j}\right)$$

$$= 2|z_1|^2\left[\frac{1-B^2}{B-A}i\tan\beta + \frac{1-B^2}{B-A}(1-i\tan\beta)\frac{f(z_1)}{z_1f'(z_1)} - \frac{1-AB}{B-A}\right]$$

$$+ \frac{1-B^2}{B-A}(1-i\tan\beta)\sum_{j=2}^{n} a_j(p_j-1)z_j^{p_j}\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2) - 2\bar{z}_1\right) \tag{2.4}$$

$$+ \sum_{j=2}^{n} p_j|z_j|^{p_j}\left[\frac{1-B^2}{B-A}(1-i\tan\beta)\left(1 - \frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right) + \frac{1-B^2}{B-A}i\tan\beta - \frac{1-AB}{B-A}\right]$$

Let

$$h(z_1) = \frac{1-B^2}{B-A}i\tan\beta + \frac{1-B^2}{B-A}(1-i\tan\beta)\frac{f(z_1)}{z_1f'(z_1)} - \frac{1-AB}{B-A}. \tag{2.5}$$

Notice that $h \in\in S^*_{\Omega_{n,p_2,\ldots,p_n}}(\beta, A, B)$, hence $|h(z_1)| < 1$. By Schwarz-Pick Lemma, we can obtain that

$$|h'(z_1)| \leq \frac{1-|h(z_1)|^2}{1-|z_1|^2}$$

On the other hand, by some calculations, we can get

$$\frac{f(z_1)f''(z_1)}{f'(z_1)^2} = \frac{B(A-B) - (B-A)h(z_1) - (B-A)z_1h'(z_1)}{(1-B^2)(1-i\tan\beta)}. \tag{2.6}$$

Substituting (2.5) and (2.6) into (2.4), we get

$$G(z) = 2|z_1|^2h(z_1) + \frac{1-B^2}{B-A}(1-i\tan\beta)\sum_{j=2}^{n} a_j(p_j-1)z_j^{p_j}\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2) - 2\bar{z}_1\right)$$

$$+ \sum_{j=2}^{n} p_j|z_j|^{p_j}\Bigg[\frac{1-B^2}{B-A}(1-i\tan\beta)$$

$$- \frac{1-B^2}{B-A}(1-i\tan\beta)\frac{1}{p_j}\frac{B(A-B) - (B-A)h(z_1) - (B-A)z_1h(z_1)}{(1-B^2)(1-i\tan\beta)}$$

$$+ \frac{1-B^2}{B-A}i\tan\beta - \frac{1-AB}{B-A}\Bigg]$$

$$= 2|z_1|^2h(z_1) + \frac{1-B^2}{B-A}(1-i\tan\beta)\sum_{j=2}^{n} a_j(p_j-1)z_j^{p_j}\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2) - 2\bar{z}_1\right)$$

$$+ h(z_1)\sum_{j=2}^{n}|z_j|^{p_j} + z_1h'(z_1)\sum_{j=2}^{n}|z_j|^{p_j} + \sum_{j=2}^{n} p_j\left(\frac{1-B^2}{B-A} + \frac{B}{p_j} - \frac{1-B^2}{B-A}\right)|z_j|^{p_j}$$

By Lemma 1.9 and 1.10, we can get that

$$\begin{aligned}|G(z)| &\le (1+|z_1|^2)|h(z_1)| + \sum_{j=2}^{n} |B|(p_j-1)|z_j|^{p_j} + |z_1|\frac{1-|h(z_1)|^2}{1-|z_1|^2}(1-|z_1|^2)\\ &\quad + \frac{4}{\cos\beta}\frac{1-B^2}{B-A}\sum_{j=2}^{n}|a_j|(p_j-1)|z_j|^{p_j}\\ &= (1+|z_1|^2)|h(z_1)| + |z_1|(1-|h(z_1)|^2) + \sum_{j=2}^{n}\left(|B| + \frac{4}{\cos\beta}\frac{1-B^2}{B-A}|a_j|\right)(p_j-1)|z_j|^{p_j}\\ &\le (1+|z_1|^2)(|h(z_1)|-1) + (1+|z_1|^2) + 2|z_1|(1-|h(z_1)|)\\ &\quad + \sum_{j=2}^{n}\left(|B| + \frac{4}{\cos\beta}\frac{1-B^2}{B-A}|a_j|\right)(p_j-1)|z_j|^{p_j}\\ &= (1+|z_1|^2) + (1-|z_1|)^2(|h(z_1)|-1) + \sum_{j=2}^{n}\left(|B| + \frac{4}{\cos\beta}\frac{1-B^2}{B-A}|a_j|\right)(p_j-1)|z_j|^{p_j}\end{aligned}$$

Therefore, when $|a_j| \le \frac{(B-A)(1-|B|)\cos\beta}{4(1-B^2)}$, $j=2,\dots,n$, we have

$$|G(z)| \le 1+|z_1|^2 + \sum_{j=2}^{n}(p_j-1)|z_j|^{p_j} = 2|z_1|^2 + \sum_{j=2}^{n} p_j|z_j|^{p_j} \tag{2.7}$$

In the terms of (2.3) and (2.7), we obtain

$$\left|\frac{1-B^2}{B-A}\left[i\tan\beta + (1-i\tan\beta)\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right] - \frac{1-AB}{B-A}\right| < 1$$

Hence $F(z) \in S^*_{\Omega_{n,p_2,\dots,p_n}}(\beta,A,B)$.

Conversely, if

$$F(z) = \left(f(z_1) + f'(z_1)\sum_{j=2}^{n} a_j z_j^{p_j}, (f'(z_1))^{\frac{1}{p_2}} z_2, \dots, (f'(z_1))^{\frac{1}{p_n}} z_n\right) \in S^*_{\Omega_{n,p_2,\dots,p_n}}(\beta,A,B),$$

then we prove that $f \in S^*(\beta,A,B)$. In fact $\hat{z} = (z_1,0,\dots,0) \in \Omega_{n,p_2,\dots,p_n}$ with $z_1 \neq 0$, from (2.1) and (2.2), we have

$$\left|\frac{1-B^2}{B-A}\left[i\tan\beta + (1-i\tan\beta)\frac{f(z_1)}{z_1 f'(z_1)}\right] - \frac{1-AB}{B-A}\right| < 1$$

for $z_1 \in U$. This completes the proof. □

In particular, if we take $A=-1$ and $B=1-2\alpha$, in Theorem 2.4, then we can obtain the following corollary.

Corollary 2.2. *Let $\alpha \in (0,1)$ and $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Suppose that the operator $F(z)$ is defined by (1.3). If complex numbers a_j satisfy the condition $|a_j| \le \frac{1-|1-2\alpha|\cos\beta}{8\alpha}$, $j=2,\dots,n$, then F is a spirallike mapping of type β and order α on the domain $\Omega_{n,p_2,\dots,p_n}$ if and only if f is a spirallike mapping of type β and order α on U.*

if we take $A=-\alpha$ and $B=\alpha$, in Theorem 2.4, then we can obtain the following corollary.

Corollary 2.3. [24]. *Let $\alpha \in (0,1)$ and $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Suppose that the operator $F(z)$ is defined by (1.3). If complex numbers a_j satisfy the condition $|a_j| \le \frac{\alpha}{1+\alpha}\cos\beta$, $j=2,\dots,n$, then F is a strongly spirallike mapping of type β and order α on the domain $\Omega_{n,p_2,\dots,p_n}$ if and only if f is a strongly spirallike mapping of type β and order α on U.*

Theorem 2.4. *Let $\lambda \in \mathbb{C}$ with Re $\lambda \le 0$. Suppose that the operator $F(z)$ is defined by (1.3). If the complex numbers a_j satisfies the condition $|a_j| \le \frac{1}{4|1-\lambda|}$, $j=2,3,\dots,n$, then F is an almost starlike function of complex order λ on $\Omega_{n,p_2,\dots,p_n}$ if and only if f is an almost starlike function of complex order λ on U.*

Proof. By the definition of almost starlike mapping of complex order λ, we need to prove that the following inequality

$$Re\left\{(1-\lambda)\frac{2}{\rho(z)}\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right\}\geq -Re\ \lambda. \tag{2.8}$$

Similar to the theorem 2.4 we need only to prove that (2.8) holds for $\rho(z)=1$ and $\hat{z}\neq 0$, according to the minimum modulus theorem for analytic functions. So, it is suffice to show that

$$Re\left\{(1-\lambda)\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)\right\}\geq -Re\ \lambda, \qquad z\in\partial\Omega_{n,p_2,\ldots,p_n},\ \hat{z}\neq 0.$$

In terms of (2.1) and (2.2), we obtain

$$(1-\lambda)\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)+Re\ \lambda=\frac{G(z)}{2|z_1|^2+\sum\limits_{j=2}^{n}p_j|z_1|^{p_j}},$$

where

$$\begin{aligned}G(z)&=2(1-\lambda)\bar{z}_1\left(\frac{f(z_1)}{f'(z_1)}-\sum_{j=2}^{n}a_j(p_j-1)z_j^{p_j}\right)+Re\ \lambda\left(2|z_1|^2+\sum_{j=2}^{n}p_j|z_1|^{p_j}\right)\\&\quad+(1-\lambda)\sum_{j=2}^{n}p_j|z_j|^{p_j}\left(1-\frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}+\frac{f''(z_1)}{p_jf'(z_1)}\sum_{k=2}^{n}a_k(p_k-1)z_k^{p_k}\right)\\&=2(1-\lambda)|z_1|^2\frac{f(z_1)}{z_1f'(z_1)}+(1-\lambda)\sum_{j=2}^{n}p_j|z_j|^{p_j}\left(1-\frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right)\\&\quad+(1-\lambda)\sum_{j=2}^{n}a_j(p_j-1)z_k^{p_k}\left(\frac{f''(z_1)}{f'(z_1)}\sum_{k=2}^{n}|z_k|^{p_k}-2\bar{z}_1\right)\\&\quad+Re\ \lambda\left(2|z_1|^2+\sum_{j=2}^{n}p_j|z_j|^{p_j}\right).\end{aligned}$$

By making use of the equality $|z_1|^2+\sum\limits_{j=2}^{n}|z_j|^{p_j}=1$, then we get

$$\begin{aligned}G(z)&=2|z_1|^2\left((1-\lambda)\frac{f(z_1)}{z_1f'(z_1)}+Re\ \lambda\right)\\&\quad+\sum_{j=2}^{n}p_j|z_j|^{p_j}\left[(1-\lambda)\left(1-\frac{f(z_1)f''(z_1)}{p_j(f'(z_1))^2}\right)+Re\ \lambda\right]\\&\quad+(1-\lambda)\sum_{j=2}^{n}a_j(p_j-1)z_j^{p_j}\left[\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\bar{z}_1\right].\end{aligned} \tag{2.9}$$

Let

$$p(z_1)=(1-\lambda)\frac{f(z_1)}{z_1f'(z_1)}+Re\ \lambda, \tag{2.10}$$

then

$$(1-\lambda)\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}=1-\lambda+Re\ \lambda-p(z_1)-z_1p'(z_1). \tag{2.11}$$

In addition, we know that then $p\in H(U)$. Notice that f is an almost starlike function of complex order λ on the unit disk U, and $Re\ p(z_1)>0$ for $z_1\in U$, then by Lemma 1.9 we can obtain

$$|p'(z_1)|\leq\frac{2Re\ p(z_1)}{1-|z_1|^2}.$$

Substituting (2.10) and (2.11) into (2.9), we get

$$\begin{aligned} G(z) &= p(z_1)\left(2|z_1|^2+\sum_{j=2}^{n}|z_j|^{p_j}\right)+(Re\ \lambda+1-\lambda)\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j} \\ &\quad + z_1p'(z_1)\sum_{j=2}^{n}|z_j|^{p_j}+(1-\lambda)\sum_{j=2}^{n}a_j(p_j-1)z_j^{p_j}\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\bar{z}_1\right) \\ &= (1+|z_1|^2)p(z_1)+(1-|z_1|^2)z_1p'(z_1)+\big(1-iIm\ \lambda\big)\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j} \\ &\quad +(1-\lambda)\sum_{j=2}^{n}a_j(p_j-1)z_j^{p_j}\left(\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\bar{z}_1\right) \end{aligned}$$

Hence

$$\begin{aligned} Re\ G(z) &\geq (1+|z_1|^2)Re\ p(z_1)-|1-\lambda|\sum_{j=2}^{n}|a_j|(p_j-1)|z_j|^{p_j}\left|\frac{f''(z_1)}{f'(z_1)}(1-|z_1|^2)-2\bar{z}_1\right| \\ &\quad -(1-|z_1|^2)|z_1p'(z_1)|+\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j} \end{aligned}$$

By Lemma 1.11 and 1.10, we can get that

$$\begin{aligned} Re\ G(z) &\geq (1+|z_1|^2)Re\ p(z_1)-(1-|z_1|^2)\frac{2|z_1|Re\ p(z_1)}{1-|z_1|^2}+\sum_{j=2}^{n}(p_j-1)|z_j|^{p_j} \\ &\quad -4|1-\lambda|\sum_{j=2}^{n}|a_j|(p_j-1)|z_j|^{p_j} \\ &= (1-|z_1|)^2Re\ p(z_1)+\sum_{j=2}^{n}(p_j-1)\big(1-4|a_j||1-\lambda|\big)|z_j|^{p_j}. \end{aligned}$$

Therefore, when $|a_j|\leq\dfrac{1}{4|1-\lambda|}$, $j=2,\dots,n$, we have

$$Re\left\{(1-\lambda)\frac{2\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)+\lambda\right\}\geq 0.$$

Hence F is an almost starlike mapping on $\Omega_{n,p_2,\dots,p_n}$.

Conversely, if $F(z)=\left(f(z_1)+f'(z_1)\sum_{j=2}^{n}a_jz_j^{p_j},(f'(z_1))^{\frac{1}{p_2}}z_2,\dots,(f'(z_1))^{\frac{1}{p_n}}z_n\right)$ is an almost starlike mapping on $\Omega_{n,p_2,\dots,p_n}$, then we prove that f is an almost starlike mapping on U. In fact $z=(z_1,0,\dots,0)\in\Omega_{n,p_2,\dots,p_n}$ with $z_1\neq 0$, from (2.1) and (2.2), we have

$$Re\left\{(1-\lambda)\frac{f(z_1)}{z_1f'(z_1)}+\lambda\right\}=\frac{2}{\rho(z)}Re\left\{(1-\lambda)\frac{\partial\rho}{\partial z}(z)J_F^{-1}(z)F(z)+\lambda\right\}\geq 0,$$

for $z_1\in U$. This completes the proof. □

Taking $\lambda=i\tan\beta$ in Theorem 2.4, we arrive the following corollary.

Corollary 2.5. *Let $|a_j|\leq\frac{\cos\beta}{4}$ and $F(z)$ is defined by (1.3). Then f is a spirallike function of type β on the unit disk U if and only if $F(z)$ is a spirallike function of type β on $\Omega_{n,p_2,\dots,p_n}$. The result has been obtained by Rahrovi [24].*

Set $\lambda=0$ in Theorem 2.4, then we get the following result due to Wang and Gao [28]:

Corollary 2.6. *Let $|a_j|\leq\frac{1}{4}$ and $F(z)$ is defined by (1.3). Then f is a starlike function on the unit disk U if and only if $F(z)$ is a a starlike mapping on $\Omega_{n,p_2,\dots,p_n}$.*

Remark 2.7. *When $a_2 = a_3 = \ldots = a_n = 0$, The result of corollary 2.6 has been obtained by Liu and Liu* [15].

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