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Certain Coefficient inequalities for transforms of Bounded turning functions

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Abstract

The objective of this paper is to obtain best possible upper bounds for the second and third Hankel functionals associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of normalized analytic function $f(z)$ when it belongs to it belongs to bounded turning functions, defined on the open unit disc in the complex plane, using Toeplitz determinants.

Keywords and phrases: Bounded turning function, upper bound, second and third Hankel functional, positive real function, Toeplitz determinants.

1 Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e., for a univalent function its n^{th} Taylor coefficient is bounded by n (see [3]). The bounds for the coefficients of these functions give information about their geometric properties. The k^{th} root transform for the function f given in (1.1) is defined as

$$F(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.2}$$

Now, we introduce the Hankel determinant for the k^{th} root transform for the function f given in (1.1) for the values $q, n, k \in \mathbb{N} = \{1, 2, 3, ...\}$ defined as

$$[H_q(n)]^{\frac{1}{k}} = \begin{vmatrix} b_{kn} & b_{kn+1} & \cdots & b_{k(n+q-2)+1} \\ b_{kn+1} & b_{k(n+1)+1} & \cdots & b_{k(n+q-1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k(n+q-2)+1} & b_{k(n+q-1)+1} & \cdots & b_{k[n+2(q-1)-1]+1} \end{vmatrix}. \tag{1.3}$$

In particular for $k = 1$, the above determinant reduces to the Hankel determinant defined by Pommerenke [10] for the function f given in (1.1) and this determinant has been investigated by many authors in the literature. For the values $q = 2, n = 1, b_k = 1$ and $q = 2, n = 2$, the k^{th} root Hankel determinant in (1.3) simplifies respectively to

$$[H_2(1)]^{\frac{1}{k}} = \begin{vmatrix} b_k & b_{k+1} \\ b_{k+1} & b_{2k+1} \end{vmatrix} = b_{2k+1} - b_{k+1}^2$$

$$\text{and} \qquad [H_2(2)]^{\frac{1}{k}} = \begin{vmatrix} b_{2k} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix} = b_{2k} b_{3k+1} - b_{2k+1}^2.$$

For a family $\mathcal{T}$ of functions in S, the more general problem of finding sharp estimates for the functional $|a_3 - \mu a_2^2|$ ($\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$) is popularly known as the Fekete-Szegö problem for $\mathcal{T}$. Ali et al. [1] obtained sharp bounds for the Fekete-Szegö functional denoted by $|b_{2k+1}-\mu b_{k+1}^2|$ associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ for the function given in (1.1), when it belongs to certain subclasses of S. We refer to $[H_2(2)]^{\frac{1}{k}}$ as the second Hankel determinant for the k^{th} root transform associated with the function f. For our discussion in this paper, we consider the k^{th} root Hankel determinant of the function f for the values $q = 3, n = 1$ in (1.3), given by

$$[H_3(1)]^{\frac{1}{k}} = \begin{vmatrix} b_k & b_{k+1} & b_{2k+1} \\ b_{k+1} & b_{2k+1} & b_{3k+1} \\ b_{2k+1} & b_{3k+1} & b_{4k+1} \end{vmatrix} (b_k = 1).$$

On expanding the determinant and applying the triangle inequality, we obtain

$$\begin{aligned} |[H_3(1)]^{\frac{1}{k}}| \leq |b_{2k+1}||b_{k+1}b_{3k+1} - b_{2k+1}^2| + |b_{3k+1}||b_{k+1}b_{2k+1} - b_{3k+1}| \\ + |b_{4k+1}||b_{2k+1} - b_{k+1}^2|. \quad (1.4) \end{aligned}$$

In section 3, we seek best possible upper bound to the third Hankel determinant given in (1.4) for the k^{th} root transform of the function f, when it belongs to the subclass denoted by $\Re$ of S, consisting of bounded turning functions (also called as bounded turning functions), defined as follows.

Definition 1.1. Let f be given by (1.1). Then $f \in \Re$, if it satisfies the condition

$$Ref'(z) > 0 \qquad \forall z \in E.$$

The subclass $\Re$ was introduced by Alexander in 1915 and a systematic study of properties these functions was conducted by MacGregor [8] in 1962, who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part (also called functions whose derivative has a positive real part).

2 Preliminary Results

Let $\mathscr{P}$ denote the class of functions consisting of g, such that

$$g(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad (2.1)$$

which are regular (analytic) in the open unit disc E and satisfy Re$g(z) > 0$ for any $z \in E$. Here $g(z)$ is called a Caratheòdory function [4].

Lemma 2.1. *(([9, 11])) If $g \in \mathscr{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. *([5]) The power series for g given in (2.1) converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for $g(z) = \sum_{k=1}^{m} \rho_k g_0(\exp(it_k)z)$, with $\sum_{k=1}^{m} \rho_k = 1$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $g_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

With out loss of generality, we consider that $c_1 > 0$. On using Lemma 2.2, for $n = 2$ and $n = 3$ respectively, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}$$

On expanding the determinant, we get

$$D_2 = [\ 8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4 \mid c_1 \mid^2\ ] \geq 0,$$

Applying the fundamental principles of complex numbers, the above expression is equivalent to

$$2c_2 = c_1^2 + y(4 - c_1^2), \text{ for some complex value } y \text{ with } |y| \leq 1. \tag{2.2}$$

$$\text{and } D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

Simplifying the relations (2.2) and (2.3), we obtain

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta\}, \tag{2.4}$$

for some complex values y and ζ with $|y| \leq 1$ and $|\zeta| \leq 1$ respectively.

To obtain our results, we refer to the classical method devised by Libera and Zlotkiewicz [9], which has been used widely.

3 Main Results

Theorem 3.1. *If $f \in \Re$ and F is the k^{th} root transformation of f given by (1.2) then*

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \frac{4}{9k^2}$$

and the inequality is sharp.

Proof. For $f \in \Re$, by virtue of Definition 1.1, we have

$$f'(z) = g(z) \qquad \forall z \in E. \tag{3.1}$$

Using the series representations for f' and g in (3.1), we get

$$a_{n+1} = \frac{c_n}{n+1} \qquad \forall n \in \mathbb{N}. \tag{3.2}$$

For a function f given by (1.1), a computation shows that

$$\begin{aligned}\left[f(z^k)\right]^{\frac{1}{k}} = \left[z^k + \sum_{n=2}^{\infty} a_n z^{nk}\right]^{\frac{1}{k}} &= \Big[z + \frac{1}{k}a_2 z^{k+1} + \left\{\frac{1}{k}a_3 + \frac{1-k}{2k^2}a_2^2\right\}z^{2k+1} \\ &+ \left\{\frac{1}{k}a_4 + \frac{1-k}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\right\}z^{3k+1} \\ &+ \Big\{\frac{1}{k}a_5 + \frac{1-k}{k^2}(a_3^2 + 2a_2a_4) + \frac{(1-k)(1-2k)}{2k^3}a_2^2a_3 \\ &+ \frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\Big\}z^{4k+1} + \cdots\Big].\end{aligned} \tag{3.3}$$

From the equations (1.2) and (3.3), we obtain

$$\begin{aligned} b_{k+1} &= \frac{1}{k}a_2 \ ; \quad b_{2k+1} = \frac{1}{k}a_3 + \frac{1-k}{2k^2}a_2^2 \ ; \\ b_{3k+1} &= \frac{1}{k}a_4 + \frac{1-k}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3; \\ b_{4k+1} &= \Big[\frac{1}{k}a_5 + \frac{1-k}{2k^2}(a_3^2 + 2a_2a_4) + \frac{(1-k)(1-2k)}{2k^3}a_2^2a_3 \\ &+ \frac{(1-k)(1-2k)(1-3k)}{24k^4}a_2^4\Big].\end{aligned} \tag{3.4}$$

Simplifying the expressions (3.2) and (3.4), we get

$$b_{k+1}=\frac{c_1}{2k}\ ;\quad b_{2k+1}=\frac{c_2}{3k}+\frac{(1-k)}{8k^2}c_1^2\ ;$$
$$b_{3k+1}=\frac{c_3}{4k}+\frac{(1-k)}{6k^2}c_1c_2+\frac{(1-k)(1-2k)}{48k^3}c_1^3.$$
$$b_{4k+1}=\frac{c_4}{5k}+\frac{(1-k)c_2^2}{18k^2}+\frac{(1-k)c_1c_3}{8k^2}+\frac{(1-k)(1-2k)}{24k^3}c_1^2c_2 + \frac{(1-k)(1-2k)(1-3k)}{384k^4}c_1^4. \quad (3.5)$$

Upon using the values of b_{k+1}, b_{2k+1} and b_{3k+1} from (3.5), we obtain

$$|b_{k+1}b_{3k+1}-b_{2k+1}^2|=\frac{1}{576k^4}\left|(72c_1c_3-64c_2^2)k^2+3(k^2-1)c_1^4\ \right|. \quad (3.6)$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2, on the right-hand side of the expression (3.6), we have

$$\left|(72c_1c_3-64c_2^2)k^2+3(k^2-1)c_1^4\right| = \Big|\Big[72c_1\times\frac{1}{4}\big\{c_1^3+2c_1(4-c_1^2)y-c_1(4-c_1^2)y^2 +2(4-c_1^2)(1-|y|^2)\zeta\big\}-64\times\frac{1}{4}\big\{c_1^2+y(4-c_1^2)\big\}^2\Big]k^2+3(k^2-1)c_1^4\Big|.$$

Applying the triangle inequality and the fact $|\zeta|<1$, which simplifies to

$$\left|(72c_1c_3-64c_2^2)k^2+3(k^2-1)c_1^4\ \right| \le \big|(5k^2-3)c_1^4+36k^2c_1(4-c_1^2) +4k^2c_1^2(4-c_1^2)|y|+2(c_1+2)(c_1+16)k^2(4-c_1^2)|y|^2\big|. \quad (3.7)$$

By choosing $c_1=c\in[0,2]$, noting that $(c_1+a)(c_1+b)\ge(c_1-a)(c_1-b)$, where $a,b\ge 0$, applying the triangle inequality and replacing $|y|$ by μ on the right-hand side of (3.7), we obtain

$$\left|(72c_1c_3-64c_2^2)k^2+3(k^2-1)c_1^4\ \right| \le \big[(5k^2-3)c^4+36k^2c(4-c^2)+4k^2c^2(4-c^2)\mu + 2(c-2)(c-16)k^2(4-c^2)\mu^2\big]=F(c,\mu),\ \text{for } 0\le\mu=|y|\le 1. \quad (3.8)$$

Where $$F(c,\mu)=\big[(5k^2-3)c^4+36k^2c(4-c^2)+4k^2c^2(4-c^2)\mu + 2(c-2)(c-16)k^2(4-c^2)\mu^2\big]. \quad (3.9)$$

Next, we need to find the maximum value of the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Let us suppose that there exists a maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. From (3.9), on differentiating $F(c,\mu)$ partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 4k^2\big\{c^2 + (c-2)(c-16)\mu\big\}(4-c^2). \tag{3.10}$$

For $0 < \mu < 1$, for fixed c with $0 < c < 2$ and for $k \in \mathbb{N}$, from (3.10), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2] \times [0,1]$. The maximum value of $F(c,\mu)$ occurs on the boundary i.e., when $\mu = 1$. Therefore, for fixed $c \in [0,2]$, we have

$$\max_{0 \le \mu \le 1} F(c,\mu) = F(c,1) = G(c). \tag{3.11}$$

On replacing μ by 1 in (3.9), we get

$$G(c) = -(k^2+3)c^4 - 40k^2c^2 + 256k^2, \tag{3.12}$$

$$G'(c) = -4(k^2+3)c^3 - 80k^2c. \tag{3.13}$$

From the expression (3.13), we observe that $G'(c) \le 0$ for all values of c in the interval $[0,2]$ and for each $k \in \mathbb{N}$. Therefore, $G(c)$ is a monotonically decreasing function of c in the interval $[0,2]$ and hence it attains the maximum value at $c = 0$ only. From (3.12), the maximum value $G(c)$ at $c = 0$ is given by

$$\max_{0 \le c \le 2} G(0) = 256k^2. \tag{3.14}$$

From the expressions (3.8) and (3.14), we get

$$\left|(72c_1c_3 - 64c_2^2)k^2 + 3(k^2-1)c_1^4\right| \le 256k^2. \tag{3.15}$$

Simplifying the relations (3.6) and (3.15), we obtain

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \le \frac{4}{9k^2}. \tag{3.16}$$

By choosing $c_1 = c = 0$ and selecting $y = 1$ in (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$. Substituting the values $c_1 = c_3 = 0$, $c_2 = 2$ in (3.3) then the obtained values in (3.16), we see that equality is attained, which shows that our result is sharp. For the values $c_1 = c_3 = 0$, and $c_2 = 2$, from (2.1), we derive

$$g(z) = 1 + 2z^2 + 2z^4 - .... = \frac{1+z^2}{1-z^2}. \tag{3.17}$$

Therefore, in this case the extremal function is

$$f'(z) = \frac{1+z^2}{1-z^2}.$$

This completes the proof of our Theorem 3.1. □

Remarks 3.2. By choosing $k = 1$ in (3.16), the result coincides with that of Janteng et al. [6].

The following Theorem is a straight forward verification on applying Theorem 3.1.

Theorem 3.3. *If $f \in \Re$ and F is the k^{th} root transformation of f given by (1.2) then*

$$|b_{2k+1} - b_{k+1}^2| \leq \frac{2}{3k},$$

the inequality is sharp for the values $c_1 = c = 0, \;\; c_2 = 2$ and $\; y = 1$

Theorem 3.4. *If $f \in \Re$ and F is the k^{th} root transformation of f given by (1.2) then*

$$|b_{k+1}b_{2k+1} - b_{3k+1}| \leq \frac{1}{2k}.$$

Proof. Applying the same procedure as we did in Theorem 3.1, we arrive at

$$|b_{k+1}b_{2k+1} - b_{3k+1}| = \frac{1}{48k^3}\Big| - 6k^2c_3 + 4k^2c_1c_2 - (1-k)(2-k)c_1^3\Big|. \quad (3.18)$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 on the right-hand side of the expression (3.18), after simplifying, we get

$$\begin{aligned} 48k^3|b_{k+1}b_{2k+1} - b_{3k+1}| \leq \big|&(3k-2)c_1^3 - 6k^2(4-c_1^2)\zeta - 2k^2c_1(4-c_1^2)|y| \\ &-3k^2(c_1+2)(4-c_1^2)|y|^2\zeta\big|. \quad (3.19)\end{aligned}$$

Using the fact $|\zeta| < 1$ and applying the triangle inequality, we have

$$\begin{aligned} 48k^3|b_{k+1}b_{2k+1} - b_{3k+1}| \leq \big|&(3k-2)c_1^3 + 6k^2(4-c_1^2) + 2k^2c_1(4-c_1^2)|y| \\ &+3k^2(c_1+2)(4-c_1^2)|y|^2\big|. \quad (3.20)\end{aligned}$$

Since $c_1 = c \in [0, 2]$, noting that $c_1 + a \geq c_1 - a$, where $a \geq 0$ and replacing $|y|$ by μ on the right-hand side of the above inequality, we obtain

$$\begin{aligned} 48k^3|b_{k+1}b_{2k+1} - b_{3k+1}| &\leq \big[(3k-2)c^3 + 6k^2(4-c^2) + 2k^2c(4-c^2)\mu \\ &\quad +3k^2(c-2)(4-c^2)\mu^2\big] \\ &= F(c,\mu)\;,\;\; 0 \leq \mu = |y| \leq 1 \text{ and } 0 \leq c \leq 2, \quad (3.21)\end{aligned}$$

$$\text{where } \; F(c,\mu) = \big[(3k-2)c^3 + 6k^2(4-c^2) + 2k^2c(4-c^2)\mu + 3k^2(c-2)(4-c^2)\mu^2\big]. \quad (3.22)$$

Now, we maximize the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. Differentiating $F(c,\mu)$ given in (3.22) partially with respect to μ and c respectively, we obtain

$$\frac{\partial F}{\partial \mu} = 2k^2c(4-c^2)+6k^2(4c-c^3-8+2c^2)\mu. \tag{3.23}$$

and $$\frac{\partial F}{\partial c} = 3(3k-2)c^2-12k^2c+8k^2\mu-6k^2c^2\mu+3k^2(4-3c^2+4c)\mu^2. \tag{3.24}$$

For the extreme values of $F(c,\mu)$, consider

$$\frac{\partial F}{\partial \mu}=0 \qquad \text{and} \qquad \frac{\partial F}{\partial c}=0. \tag{3.25}$$

In view of (3.25), on solving the equations in (3.23) and (3.24), we obtain the only critical point for the function $F(c,\mu)$ which lies in the closed region $[0,2]\times[0,1]$ is $(0,0)$. At the point $(0,0)$, we observe that

$$\frac{\partial^2 F}{\partial \mu^2} = -48k^2<0; \quad \frac{\partial^2 F}{\partial c^2} = -12k^2<0; \quad \frac{\partial^2 F}{\partial c\partial \mu} = 8k^2$$
$$\text{and } \left[\left(\frac{\partial^2 F}{\partial \mu^2}\right)\left(\frac{\partial^2 F}{\partial c^2}\right)-\left(\frac{\partial^2 F}{\partial c\partial \mu}\right)^2\right] = 512k^4>0, \text{ with } k\in\mathbb{N}.$$

Therefore, the function $F(c,\mu)$ has maximum value at the point $(0,0)$, from (3.22), it is given by

$$G_{max} = F(0,0) = 24k^2. \tag{3.26}$$

Simplifying the expressions (3.18), (3.21) and (3.26), we get

$$|b_{k+1}b_{2k+1}-b_{3k+1}| \le \frac{1}{2k}. \tag{3.27}$$

By setting $c_1 = c = y = 0$ and selecting $\zeta = 1$ in the expressions (2.2) and (2.4), we find that $c_2 = 0$ and $c_3 = 2$ respectively. Substituting the values $c_1 = c_2 = 0$, $c_3 = 2$ in (3.3) and then the obtained values in (3.27), we observe that equality is attained, which shows that our result is sharp. For the values $c_1 = c_2 = 0$ and $c_3 = 2$, from (2.1), we derive the extremal function, given by

$$g(z) = 1+2z^3+... = \frac{1+z^3}{1-z^3}.$$

so that from (2.1), we have

$$f'(z) = 1+2z^3+..... = \frac{1+z^3}{1-z^3}.$$

This completes the proof of our Theorem 3.4. □

Remarks 3.5. By choosing $k = 1$ in (3.27), we get $|b_2b_3 - b_4| \leq \frac{1}{2}$. This result coincides with that of Bansal et al. [2].

Theorem 3.6. *If $f \in \Re$ and F is the k^{th} root transformation of f given by (1.2), then we have the following inequalities:*

$$(i)\left|b_{k+1}\right| \leq \frac{1}{k}. \quad (ii)\left|b_{2k+1}\right| \leq \frac{3+k}{6k^2}. \quad (iii)\left|b_{3k+1}\right| \leq \frac{k^2+k+1}{6k^3}.$$
$$(iv)\left|b_{4k+1}\right| \leq \frac{54k^3+405k^2-710k+395}{360k^4}.$$

Proof. Using the fact that $|c_n| \leq 2$, for $n \in \mathbb{N}$, with the help of c_2 and c_3 values given in (2.2) and (2.4) respectively together with the values in (3.5), upon simplification, we at once obtain all the above inequalities. This completes the proof of our Theorem 3.6. □

Substituting the results of Theorems 3.1, 3.3, 3.4 and 3.6 in the inequality given in (1.4), which simplifies to obtain the following Corollary.

Corollary 3.7. *If $f \in \Re$ and F is the k^{th} root transformation of f given by (1.2) then*

$$\left|[H_3(1)]^{\frac{1}{k}}\right| \leq \frac{99k^3+490k^2-545k+395}{540k^5}. \tag{3.28}$$

Remarks 3.8. Choosing $k = 1$ in (3.28), which simplifies to $|H_3(1)| \leq \frac{439}{540}$. This result coincides with that of Bansal et al. [2].

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