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The modified extragradient method for nonexpansive multivalued mappings and variational inequality problems

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Abstract

In this paper, we prove the strong convergence of an approximating common element of the set of fixed points of a nonexpansive multivalued mapping and the set of solutions of a variational inequality problem for a monotone, Lipschitz continuous mapping in a Hilbert space by using the modified extragradient method. As applications, we give the example and numerical results for supporting our main theorem.

Keywords: Extragradient method; Variational inequality; Nonexpansive multi-valued mapping; Iteration; Hilbert space.

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1 Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H . Let $CB(C), K(C)$ and $P(C)$ denote the families of nonempty closed bounded subsets, nonempty compact subsets and nonempty proximinal bounded subset of C, respectively. The *Hausdorff metric* on $CB(C)$ is defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B),\ \sup_{y\in B} d(y,A)\right\}$$

for all $A, B \in CB(C)$ where $d(x,B) = \inf_{b\in B}\|x-b\|$. A singlevalued mapping $S : C \to C$ is said to be *nonexpansive* if

$$\|Sx - Sy\| \leq \|x - y\|$$

for all $x, y \in C$. A multivalued mapping $S : C \to CB(C)$ is said to be *nonexpansive* if

$$H(Sx, Sy) \leq \|x - y\|$$

*

for all $x, y \in C$. An element $z \in C$ is called a *fixed point* of $S : C \to C$ (resp., $S : C \to CB(C)$) if $z = Sz$ (resp., $z \in Sz$). The fixed point set of S is denoted by $F(S)$. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ implies that $\{x_n\}$ converges strongly to x.

Let $S : C \to CB(H)$ be a multivalued mapping, $I - S$ (I is an identity mapping) is said to be *demiclosed* at $y \in C$ if $\{x_n\}_{n=1}^{\infty} \subset C$ such that $x_n \rightharpoonup x$ and $\{x_n - z_n\} \to y$ where $z_n \in Sx_n$ imply $x - y \in Sx$.

Lemma 1.1. [1] *Let C be a nonempty and weakly compact subset of a Hilbert space H and $S : C \to K(H)$ a nonexpansive mapping. Then $I - S$ is demiclosed.*

Recently, many authors have shown the existence of fixed points of multivalued mappings in Hilbert spaces and Banach spaces (see [3, 9, 10, 14, 13]). The study multivalued mapping is much more complicated and difficult more than singlevalued mapping.

Subsequently, Hussain and Khan [5] proved fixed point theorems of a *-nonexpansive multivalued mapping and strong convergence of its iterates to a fixed point defined on a closed and convex subset of a Hilbert space by using the best approximation operator $P_S x$, which is defined by $P_S x = \{y \in Sx : \|y - x\| = d(x, Sx)\}$. For more results, refer to [6, 12, 21]. This is an important tool for studying fixed point theorem for multivalued mapping.

Let $A : C \to H$ be a mapping of C into H. A mapping A is called
(i) *monotone* if

$$\langle Au - Av, u - v\rangle \geq 0 \qquad , \forall u, v \in C;$$

(ii) *k-Lipschitz continuous* if there exists a positive real number k such that

$$\|Au - Av\| \leq k\|u - v\| \qquad , \forall u, v \in C;$$

(iii) *α-inverse-strongly-monotone* if there exists a positive real number α such that

$$\langle Au - Av, u - v\rangle \geq \alpha\|Au - Av\|^2 \qquad , \forall u, v \in C.$$

We know that if $S : C \to C$ is nonexpansive, then $A = I - S$ is $\frac{1}{2}$ -inverse strongly monotone; see[16, 17, 18] for more details.

It is easy to see that an *α-inverse-strongly-monotone* mapping A is monotone and Lipschitz continuous. We consider the following variational inequality problem (VI(A, C)): find a $u \in C$ such that

$$\langle Au, u - v\rangle \geq 0 \qquad , \forall v \in C.$$

The solution set of the variational inequality problem is denoted by Ω. Recently, Takahashi and Toyoda [19] introduced the following iterative scheme for finding an element of $F(S) \cap \Omega$ under the assumption that a set $C \subset H$ is nonempty, closed and convex, a mapping $S : C \to C$ is nonexpansive and a mapping $A : C \to H$ is *α-inverse-strongly-monotone* :

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A x_n) \qquad \forall n \geq 0,$$

where $x_0 = x \epsilon C$, $\{\alpha_n\}$ is a sequence in (0,1) and $\{\lambda_n\}$ is a sequence in (0,2α).They proved that if $F(S) \cap \Omega$ is nonempty, then the sequence $\{x_n\}$ converges weakly to some $z \epsilon F(S) \cap \Omega$.

In 1976, Korpelevich [7] introduced the new method which is called extragradient method for solving the variational inequality problem in a finite-dimensional Euclidean space R^n under the assumption that a set $C \subset R^n$ is nonempty, closed and convex, a mapping $A : C \to R^n$ is monotone and k-Lipschitz continuous and Ω is nonempty. The method as follows :

$$\begin{cases} x_0 = x \in R^n, \\ \bar{x}_n = P_C(x_n - \lambda A x_n), \\ x_1 \ = P_C(x_n - \lambda A \bar{x}_n) \qquad \forall n \geq 0 \end{cases}$$

where $\lambda \in (0, 1/k)$. He showed that the sequences $\{x_n\}$ and $\{\bar{x}_n\}$ converge to the same point $z \in \Omega$.

Subsequently motivated by the idea of Korpelevich's extragradient method [7], Nadezhkina and Takahashi [8] introduced an iterative process for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of a variational inequality problem. They proved the following weak convergence theorem for two sequences generated by this process : Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpasive mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by

$$\begin{cases} x_0 = x \in H, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} \ = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda A y_n) \quad \forall n \geq 0 \end{cases}$$

where $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/k)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $z \in F(S) \cap \Omega$ where

$$z = \lim_{n\to\infty} P_{F(S)\cap\Omega} x_n.$$

Very recently, Lu-Chuan Zeng and Jen-Chih Yao[22]are inspired by Nadezhkina and Takahashi's iterative process[8] , they introduced the following iterative process :

$$\begin{cases} x_0 = x \in H, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} \ = \alpha_n x_0 + (1 - \alpha_n) S P_C(x_n - \lambda A y_n) \quad \forall n \geq 0 \end{cases}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions :
(a) $\{\lambda_n\} k \subset (0, 1 - \delta)$ for some $\delta \in (0, 1)$;
(b) $\{\alpha_n\} \subset (0, 1)$, $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$, $\lim\limits_{n\to\infty} \alpha_n = 0$.
They showed that the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $P_{F(S)\cap\Omega} x_0$ where

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

Motivated by Lu-Chuan Zeng and Jen-Chih Yao [22],we introduce the new iteration for a nonexpansive multivalued as follow:

Let C be a closed convex subset of a real Hilbert space H . Let A be an α-inverse strongly monotone mapping of C into H and let $S : C \to K(C)$ be a nonexpansive multivalued mapping. Let $\{\alpha_n\} \subset (0,1)$ and $\{\lambda_n\} \subset (0,1)$. For any $x_0 \in C$, we find $y_0, t_0, x_1 \in C$ such that

$$\begin{cases} y_0 = P_C(x_0 - \lambda_0 A x_0), \\ t_0 = P_C(x_0 - \lambda_0 A y_0), \\ x_1 \in \alpha_0 x_0 + (1-\alpha_0)\, S t_0. \end{cases}$$

Then we compute $y_1, t_1 \in C$ by $y_1 = P_C(x_1 - \lambda_1 A x_1)$ and $t_1 = P_C(x_1 - \lambda_1 A y_1)$.
Let $c_1 \in S x_1$ from Nadler theorem [9], there exists $g_1 \in S y_1$ such that

$$\|c_1 - g_1\| \leq H(S x_1, S y_1).$$

Again from Nadler theorem [9] , there exists $b_1 \in S t_1$ such that

$$\|g_1 - b_1\| \leq H(S y_1, S t_1),$$

then we can find $x_2 \in C$ such that

$$x_2 = \alpha_1 x_0 + (1-\alpha_1)\, b_1.$$

Next, we can compute $y_2, t_2 \in C$ by $y_2 = P_C(x_2 - \lambda_2 A x_2)$ and $t_2 = P_C(x_2 - \lambda_2 A y_2)$.
Inductively, we can construct the sequence $\{x_n\} \subset C$ by the following manner ;

$$\begin{cases} y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\, b_n, \end{cases}$$

for each $n \in \mathbb{N}$, where $b_n \in S t_n$ such that $t_n = P_C(x_n - \lambda_n A y_n)$, $g_n \in S y_n$ and $c_n \in S x_n$ such that $\|b_n - g_n\| \leq H(S t_n, S y_n)$, $\|g_n - c_n\| \leq H(S y_n, S x_n)$.

2 Preliminaries and lemmas

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. We know that a Hilbert space H satisfies Opial's condition, that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\lim_{n\to\infty} \|x_n - x\| < \lim_{n\to\infty} \|x_n - y\|$$

for all $y \in H$ with $x \neq y$. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that $\|x - P_C x\| \leq \|x - y\| \;\; \forall y \in C$. P_C is called the metric projection of H onto C. It is know that P_C is a nonexpansive mapping of H onto C. Ii is also know that P_C is characterized by the following properties :$P_C x \in C$ and for all $x \in H, y \in C$

$$\langle x - P_C x, P_C x - y \rangle \geq 0, \tag{2.1}$$

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle. \tag{2.2}$$

Let $A : C \to H$ be a mapping. It is easy to see from (2.1) that the following implications hold :

$$u \in \Omega \Leftrightarrow u = P_C(u - \lambda A u) \quad \forall \lambda > 0. \tag{2.3}$$

A setvalued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$, we have $\langle x - y, f - g \rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is know that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H, \langle x-y, f-g\rangle \geq 0$ for all $(y,g) \in G(T)$, then $f \in Tx$. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v \ , \ if \ v \in C, \\ \emptyset \qquad \qquad , if \ v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$.

In order to prove the main result in Section 3, we shall use the following lemmas in the sequel.

Lemma 2.1. [20] *Let $\{s_n\}$ be a sequence of nonnegative numbers satisfying the conditions : $s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n\beta_n, \forall n \geq 0$ where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences of real numbers such that*
(i) $\{\alpha_n\} \subset [0,1]$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$, or equivalently,
$\prod_{n=0}^{\infty}(1-\alpha_n) := \lim_{n\to\infty}\prod_{k=0}^{n}(1-\alpha_k) = 0$;
(ii) $\limsup_{n\to\infty}\beta_n \leq 0$, or
(iii) $\sum_n \alpha_n\beta_n$ is convergent.
Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2. *In a real Hilbert space H, there holds the inequality :*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle \qquad \forall x,y \in H.$$

Lemma 2.3. *Let H be a real Hilbert space. Then the following hold:*

(1) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$ *for all $x,y \in H$;*

(2) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$ *for all $x,y \in H$;*

(3) $\|tx+(1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2$ *for all $t \in [0,1]$ and $x,y \in H$;*

Lemma 2.4. [15] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space and let $\{\beta_n\}$ be a sequence of $[0,1]$ such that $0 < \lim_{n\to\infty}\inf\beta_n \leq \lim_{n\to\infty}\sup\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty}\sup(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

Lemma 2.5. [2] *Let $\{s_n\}$ be a sequence of nonnegative real numbers, $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty}\beta_n < \infty$ and $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty}\gamma_n \leq 0$. Suppose that*

$$s_{n+1} = (1-\alpha_n)s_n + \alpha_n\gamma_n + \beta_n$$

for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} s_n = 0$.

Condition(A). Let H be a Hilbert space and C be a subset of H. A multivalued mapping $S : C \to K(C)$ is said to satisfy *Condition* (A) if $\|x-p\| = d(x, Sp)$ for all $x \in H$ and $p \in F(S)$.

Remark 2.6. We see that S satisfies Condition (A) if and only if $Sp = \{p\}$ for all $p \in F(S)$. It is known that the best approximation operator P_S also satisfies Condition (A).

3 Main results

In this section, we prove strong convergence theorems for a variational inequality problem and a fixed point problem of a nonexpansive multivalued mapping.

Theorem 3.1. *Let C be a nonempty weakly compact and convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $S : C \to K(C)$ a nonexpansive multivalued mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 \in C \qquad \text{chosen arbitrary,} \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1 - \alpha_n) b_n, \end{cases}$$

for each $n \in N$, where $c_n \in Sx_n$, there exist $g_n \in Sy_n$ and $b_n \in SP_c(x_n - \lambda_n Ay_n)$ such that $\|b_n - g_n\| \leq H(SP_C(x_n - \lambda_n Ay_n), Sy_n)$ and $\|g_n - c_n\| \leq H(Sy_n, Sx_n)$.
Assume that $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions :

(a) $\{\alpha_n k\} \subset (0, 1 - \delta)\, for some\ \delta \in (0, 1)$,

(b) $\{\alpha_n\} \subset (0, 1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

If S satisfies Condition (A), then the sequences $\{x_n\}$, $\{y_n\}$ converge strongly to the same point $P_{F(S)\cap\Omega} x_0$ provided

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0.$$

Proof. We divide the proof into five steps.

Step 1. Show that $\{x_n\}$ is bounded. Let $u \in F(S) \cap \Omega$. From the definition of $\{x_n\}$, we have

$$\begin{aligned} \|x_n - \lambda_n Ay_n - u\|^2 &\geq \|x_n - \lambda_n Ay_n - P_C(x_n - \lambda_n Ay_n)\|^2 + \|u - P_C(x_n - \lambda_n Ay_n)\|^2 \\ &\geq \|x_n - \lambda_n Ay_n - t_n\|^2 - \|u - t_n\|^2. \end{aligned}$$

We observe that

$$\begin{aligned} \|u - t_n\|^2 &\leq \|x_n - \lambda_n Ay_n - u\|^2 - \|x_n - \lambda_n Ay_n - t_n\|^2 \\ &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 - 2\langle \lambda_n Ay_n, x_n - u\rangle + 2\langle \lambda_n Ay_n, x_n - t_n\rangle \\ &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - t_n\|^2 \\ &\quad +2\lambda_n(\langle Ay_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\ &\leq \|x_n - u\|^2 - \|x_n - t_n\|^2 + 2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\ &\quad +2\lambda_n\langle Ay_n, y_n - t_n\rangle \\ &= \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle. \end{aligned} \tag{3.1}$$

Further from the property of metric projection, we obtain

$$\begin{aligned}\langle x_n-\lambda_nAy_n-y_n,t_n-y_n\rangle &= \langle x_n-\lambda_nAx_n-y_n,t_n-y_n\rangle+\langle \lambda_nAx_n-\lambda_nAy_n,t_n-y_n\rangle\\ &\le \langle \lambda_nAx_n-\lambda_nAy_n,t_n-y_n\rangle\\ &\le \lambda_nk\|x_n-y_n\|\|t_n-y_n\|. \qquad (3.2)\end{aligned}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}\|t_u-u\|^2 &\le \|x_n-u\|^2-\|x_n-y_n\|^2-\|y_n-t_n\|^2+2\lambda_nk\|x_n-y_n\|\|t_n-y_n\|\\ &\le \|x_n-u\|^2-\|x_n-y_n\|^2+2\lambda_n^2k^2\|x_n-y_n\|^2\\ &\le \|x_n-u\|^2. \qquad (3.3)\end{aligned}$$

For $b_n\in St_n$, we have

$$\begin{aligned}\|x_{n+1}-u\| &= \|\alpha_nx_0+(1-\alpha_n)b_n-u\|,\ \forall b_n\in St_n\\ &\le \alpha_n\|x_0-u\|+(1-\alpha_n)\|b_n-u\|\\ &= \alpha_n\|x_0-u\|+(1-\alpha_n)d(b_n,Su)\\ &\le \alpha_n\|x_0-u\|+(1-\alpha_n)H(St_n,Su)\\ &\le \alpha_n\|x_0-u\|+(1-\alpha_n)\|t_n-u\|\\ &\le \alpha_n\|x_0-u\|+(1-\alpha_n)\|x_n-u\|\\ &\le \alpha_n\|x_0-u\|+(1-\alpha_n)\|x_0-u\|\\ &\le \|x_0-u\|\end{aligned}$$

This implies that $\{x_n\}$ is bounded. It follows from (3.3) that

$$\|t_n-u\|\le\|x_0-u\|,\ \forall n\ge 0. \qquad (3.4)$$

This shows that $\{t_n\}$ is also bounded.

Step 2. Show that $\lim_{n\to\infty}\|x_n-y_n\|=0$. Since S satisfies Condition (A), for each $b_n\in St_n$ we have

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \|\alpha_nx_0+(1-\alpha_n)\,b_n-u\|^2\\ &\le \alpha_n\|x_0-u\|^2+(1-\alpha_n)\|b_n-u\|^2\\ &= \alpha_n\|x_0-u\|^2+(1-\alpha_n)\,d\,(b_n,u)^2\\ &\le \alpha_n\|x_0-u\|^2+(1-\alpha_n)\,H\,(St_n,Su)^2\\ &\le \alpha_n\|x_0-u\|^2+(1-\alpha_n)\|t_n-u\|^2. \qquad (3.5)\end{aligned}$$

It follows from (3.3) that

$$\|t_n-u\|^2 \le \|x_n-u\|^2+\left(\lambda_n{}^2k^2-1\right)\|x_n-y_n\|^2 \qquad (3.6)$$

From (3.5) and (3.6), we have

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \alpha_n\|x_0-u\|^2+(1-\alpha_n)\Big(\|x_n-u\|^2+\left(\lambda_n{}^2k^2-1\right)\|x_n-y_n\|^2\Big)\\ &\le \alpha_n\|x_0-u\|^2+\|x_n-u\|^2+\left(\lambda_n{}^2k^2-1\right)\|x_n-y_n\|^2,\end{aligned}$$

which implies that

$$\begin{aligned}\delta\|x_n-y_n\|^2 &\le \left(1-\lambda_n{}^2k^2\right)\|x_n-y_n\|^2\\ &\le \alpha_n\|x_0-u\|^2+\|x_n-u\|^2-\|x_{n+1}-u\|^2\\ &\le \alpha_n\|x_0-u\|^2+\|x_n-x_{n+1}\|\left(\|x_n-u\|-\|x_{n+1}-u\|\right).\end{aligned}$$

Since $\lim_{n\to\infty}\|x_n-x_{n+1}\|=0$ and $\lim_{n\to\infty}\alpha_n=0$, we have

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.7}$$

Step 3. Show that $\lim_{n\to\infty}\|c_n-x_n\|=0$ for some $c_n\in Sx_n$. Setting $t_n=P_C(x_n-\lambda_nAy_n)$, we have

$$\begin{aligned}\|y_n-t_n\| &= \|P_C(x_n-\lambda_nAx_n)-P_C(x_n-\lambda_nAy_n)\|\\ &\le \|x_n-\lambda_nAx_n-x_n+\lambda_nAy_n\|\\ &= \lambda_n\|Ax_n-Ay_n\|\\ &\le \lambda_nk\|x_n-y_n\|.\end{aligned} \tag{3.8}$$

It follows from (3.7) and (3.8) that

$$\lim_{n\to\infty}\|y_n-t_n\|=0. \tag{3.9}$$

By the definition of $\{x_n\}$, there exists $b_n\in St_n$ such that $\|g_n-b_n\|\le H(Sy_n,St_n)$. For $u\in F(S)$, from (3.5) we have

$$\begin{aligned}\|g_n-x_{n+1}\| &\le \|g_n-b_n\|+\|b_n-x_{n+1}\|\\ &\le H(Sy_n,St_n)+\|b_n-(\alpha_nx_0+(1-\alpha_n)b_n)\|\\ &\le \|y_n-t_n\|+\alpha_n\|b_n-x_0\|\\ &\le \|y_n-t_n\|+\alpha_n(\|b_n-u\|+\|u-x_0\|)\\ &= \|y_n-t_n\|+\alpha_n(d(b_n,Su)+\|u-x_0\|)\\ &\le \|y_n-t_n\|+\alpha_n(H(St_n,Su)+\|u-x_0\|)\\ &\le \|y_n-t_n\|+\alpha_n(\|t_n-u\|+\|u-x_0\|)\\ &\le \|y_n-t_n\|+\alpha_n\|x_0-u\|+\alpha_n\|x_0-u\|\\ &\le \|y_n-t_n\|+2\alpha_n\|x_0-u\|.\end{aligned} \tag{3.10}$$

It follows from (3.9), (3.10) and $\lim_{n\to\infty}\alpha_n$ that

$$\lim_{n\to\infty}\|g_n-x_{n+1}\|=0. \tag{3.11}$$

From the definition of $\{x_n\}$, for each $c_n\in Sx_n$ there exists $g_n\in Sy_n$ such that $\|c_n-g_n\|\le H(Sx_n,Sy_n)$. Observe that

$$\begin{aligned}\|c_n-x_n\| &\le \|c_n-g_n\|+\|g_n-x_{n+1}\|+\|x_{n+1}-x_n\|\\ &\le H(Sx_n,Sy_n)+\|g_n-x_{n+1}\|+\|x_{n+1}-x_n\|\\ &\le \|x_n-y_n\|+\|g_n-x_{n+1}\|+\|x_{n+1}-x_n\|.\end{aligned}$$

From (3.7), (3.13) and $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$, we obtain

$$\lim_{n\to\infty} \|c_n - x_n\| = 0. \tag{3.12}$$

Step 4. Show that $\limsup_{n\to\infty} \langle x_0 - u^*, x_n - u^* \rangle \leq 0$ where $u^* = P_{F(s)\cap\,\Omega} x_0$. Indeed we pick a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ so that

$$\limsup_{n\to\infty} \langle x_0 - u^*, x_n - u^* \rangle \quad = \quad \lim_{n\to\infty} \langle x_0 - u^*, x_{n_i} - u^* \rangle. \tag{3.13}$$

Without loss of generality, we may further assume that $\{x_{ni}\}$ converges weakly to $\tilde{u}$ for some $\tilde{u} \in H$. Hence (3.13) reduces to

$$\limsup_{n\to\infty} \langle x_0 - u^*, x_n - u^* \rangle \quad = \quad \langle x_0 - u^*, \tilde{u} - u^* \rangle. \tag{3.14}$$

In order to prove $\langle x_0 - u^*, \tilde{u} - u^* \rangle \leq 0$, it suffices to show that $\tilde{u} \in F(S) \cap\ \Omega$.
From $\langle x_0 - u^*, \tilde{u} - u^* \rangle = \left\langle x_0 - P_{F(S)\cap\ \Omega} x_0, \tilde{u} - P_{F(S)\cap\ \Omega} x_0 \right\rangle \leq 0$, we have $\tilde{u} \in F(S) \cap\ \Omega$.
By Lemma (1.1) ; it follows from step 3 , we obtain $\tilde{u} \in P(S)$. Now we show $\tilde{u} \in \Omega$. Since from (3.7) and (3.8) we have $t_{n_i} \rightharpoonup \tilde{u}$ and $y_{n_i} \rightharpoonup \tilde{u}$. Let

$$Tv = \begin{cases} Av + N_C v\ , & if\ v \in C, \\ \emptyset\ , & if\ v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \Omega$; see [11]. Let $(v, w) \in G(T)$. Then we have $w \,\epsilon\, Tv = Av + N_C v$ and hence $w - Av \in N_C v$. Therefore we have $\langle v - u, w - A_v \rangle \geq 0$ for all $u \in C$. On the other hand , from $t_n = P_C(x_n - \lambda_n A y_n)$ and $v \in C$ we have

$$\langle x_n - \lambda_n A y_n - t_n, t_n - v \rangle \geq 0$$
$$\langle v - t_n, t_n - x_n + \lambda_n A y_n \rangle \geq 0$$
$$\left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + \frac{\lambda_n A y_n}{\lambda_n} \right\rangle \geq 0$$

and hence

$$\left\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + A y_n \right\rangle \geq 0.$$

Therefore according to the face that $w - Av \in N_C v$ and $t_n \in C$, we have

$$\begin{aligned} \langle v - t_{n_i}, w \rangle &\leq \langle v - t_{n_i}, Av \rangle \\ &\leq \langle v - t_{n_i}, Av \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} + A y_i \right\rangle \\ &\leq \langle v - t_{n_i}, Av - A t_{n_i} \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle + \langle v - t_{n_i}, -A t_{n_i} - A y_{n_i} \rangle \\ &\leq \langle v - t_{n_i}, -A y_{n_i} - A t_{n_i} \rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - x_{n_i}}{\lambda_{n_i}} \right\rangle. \end{aligned}$$

Thus we get $\langle v - \tilde{u}, w \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $\tilde{u} \in T^{-1}0$ and hence $\tilde{u} \in \Omega$. This shows that $\tilde{u} \,\epsilon\, F(S) \cap \Omega$. Therefore by the property of the metric projection, we obtain $\langle x_0 - u^*, \tilde{u} - u^* \rangle \leq 0$.

Step 5. Show that $x_n \to u^*$ and $y_n \to u^*$ as $n \to \infty$ where $u^* \in P_{F(S)\cap\Omega}x_0$. By Lemma 2.2 and (3.3), we get

$$\begin{aligned} \|x_{n+1} - u^*\|^2 &= \|\alpha_n x_0 + (1-\alpha_n)b_n - u^*\|^2 \\ &\le (1-\alpha_n)^2\|b_n - u^*\|^2 + 2\alpha_n\langle x_0 - u^*, x_{n+1} - u^*\rangle \\ &= (1-\alpha_n)^2 d(b_n, Su^*)^2 + 2\alpha_n\langle x_0 - u^*, x_{n+1} - u^*\rangle \\ &\le (1-\alpha_n)H(St_n, Su^*)^2 + 2\alpha_n\langle x_0 - u^*, x_{n+1} - u^*\rangle \\ &\le (1-\alpha_n)\|t_n - u^*\|^2 + 2\alpha_n\langle x_0 - u^*, x_{n+1} - u^*\rangle \\ &\le (1-\alpha_n)\|x_n - u^*\|^2 + \alpha_n\beta_n \end{aligned}$$

where $\beta_n = 2\langle x_0 - u^*, x_{n+1} - u^*\rangle$. Thus an application of Lemma 2.1 combined with Step 4 yields that $x_n \to u^*$. Since $\|x_n - y_n\| \to 0$,we have $y_n \to u^*$. This completes the proof. □

If $Sp = \{p\}$ for all $p \in F(S)$, S satisfies Condition (A) then we obtain the following results.

Theorem 3.2. *Let C be a nonempty weakly compact and convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $S : C \to K(C)$ nonexpansive multivalued mapping such that $F(S) \cap \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 \in C \qquad \textit{chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\, b_n, \end{cases}$$

for each $n \in N$, where $c_n \in Sx_n$, there exist $g_n \in Sy_n$ and $b_n \in SP_C(x_n - \lambda_n A y_n)$ such that $\|b_n - g_n\| \le H\,(SP_C(x_n - \lambda_n A y_n), Sy_n)$ and $\|g_n - c_n\| \le H\,(Sy_n, Sx_n)$.
Assume that $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions :

(a) $\{\alpha_n k\} \subset (0, 1-\delta)\, for some\ \delta \in (0,1)$,

(b) $\{\alpha_n\} \subset (0,1)\,, \sum\limits_{n=1}^{\infty} \alpha_n = \infty,\ \lim\limits_{n\to\infty} \alpha_n = 0.$

If $Sp = \{p\}$ for all $p \in F(S)$, then the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $P_{F(S)\cap\Omega}x_0$ provided $\lim\limits_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Since P_S satisfies condition (A), we also obtain the following result.

Theorem 3.3. *Let C be a nonempty weakly compact and convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $P_S : C \to K(C)$ nonexpansive multivalued mapping such that $F(S) \cap\ \Omega \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 \in C \qquad \textit{chosen arbitrary}, \\ y_n = P_C(x_n - \lambda_n A x_n) \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\, b_n, \end{cases}$$

for each $n \in N$, where $c_n \in P_S x_n$, there exist $g_n \in P_S y_n$ and $b_n \in P_S P_C(x_n - \lambda_n A y_n)$ such that $\|b_n - g_n\| \le H\,(P_S P_C(x_n - \lambda_n A y_n), P_S y_n)$ and $\|g_n - c_n\| \le H\,(P_S y_n, P_S x_n)$.
Assume that $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions :

(a) $\{\alpha_n k\} \subset (0, 1-\delta)\, for some\ \delta \in (0,1)$,
(b) $\{\alpha_n\} \subset (0,1)$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

If Ps is nonexpansive multivalued mapping, then the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $P_{F(S)\cap\Omega}x_0$ provided $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

Proof. By the same proof as in theorem 3.1, we have

$$\lim_{n\to\infty} \|c_n - x_n\| = 0$$

where $c_n \in Psx_n$.
This implies that

$$d(x_n, Sx_n) \leq d(x_n, Psx_n) \leq \|c_n - x_n\| \to 0$$

as $n \to \infty$. From $I - S$ is demiclosed at 0, so we obtain the result. □

4 Examples and Numerical Results

In this section, we give examples with numerical results for supporting our theorem.

Example 4.1 Let $H = \mathbb{R}$ and $C = [-0.5, 0.5]$. Define mappings $A : C \to H$ and $S : C \to K(C)$ by $Ax = 2x$ for all $x \in C$ and $Sx = [0, \frac{x^2}{2}]$ for all $x \in C$, respectively. Choose $\lambda_n = \frac{1}{2(n+1)}$, $\alpha_n = \frac{n}{2+n^2}$. It is easy to check that A satisfy all condition in Theorem 3.1 , S nonexpansive multivalued mapping such that $F(S) = \{0\}$. Since $S(0) = \{0\}$, we have $\|x - 0\| = d(x, \{0\})$. Thus S satisfies condition (A).

Table 4.1. Numerical results of Example 4.1 being randomized $c_n \in Sx_n$ in two times

n	Randomized in the 1st			Randomized in the 2nd		
	c_n	y_n	x_n	c_n	y_n	x_n
0	0.117802458	-0.5	0.5	0.077357373	-0.5	0.5
1	0.002411539	-0.235604916	0.117802458	0.001829127	-0.154714754	0.077357373
2	0.000714444	-0.5	0.168065128	0.007705911	-0.5	0.167821762
3	0.008117840	-0.5	0.166826993	0.006591779	-0.5	0.16881101
4	0.001859387	-0.5	0.149521120	0.005185511	-0.5	0.139377096
5	0.001297219	-0.5	0.112447639	0.001793885	-0.5	0.11881805
6	0.000384351	-0.5	0.093610049	0.002015589	-0.5	0.093528338
7	0.002045022	-0.5	0.079139023	0.002553498	-0.5	0.080024555
8	0.002207204	-0.5	0.06971751	0.000122847	-0.5	0.069324433
9	0.000384815	-0.5	0.06081756	0.001202973	-0.5	0.060701763
...	...	...	...	...	...	...
49	5.1122E-0.5	-0.5	0.010407632	5.29047E-0.5	-0.5	0.010408298

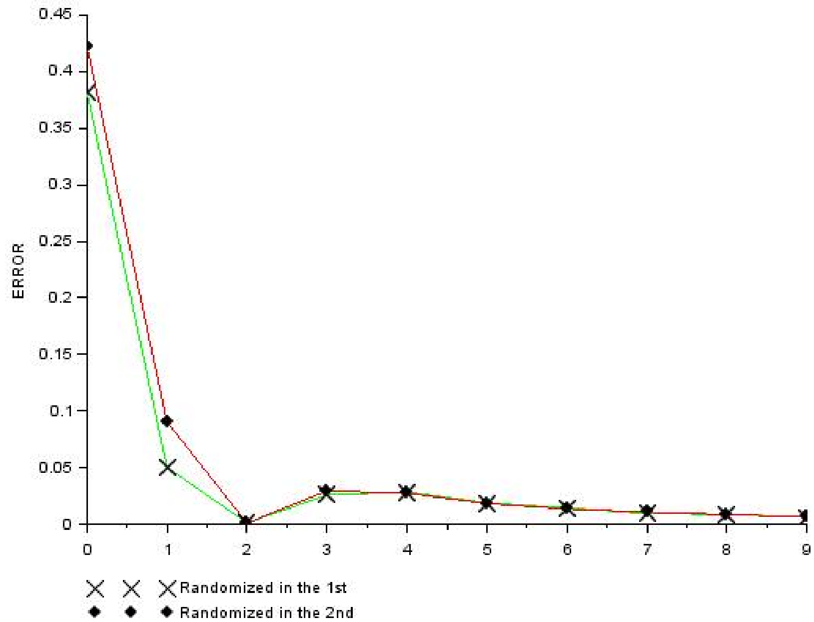

Figure 4.1: Error plots for all sequences $\{x_n\}$ in Table 4.1.

Choosing $x_0 = 0.5$, we can compute the numerical results as in Table 4.1 and Figure 4.1. From Table 4.1 and Figure 4.1, we see that 0 is the solution in Example 4.1.

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