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# A CHARACTERIZATION IN CLOSED IDEALS OF $C(X)$

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Abstract. In this paper, we establish the characterization of closed ideals of $C(X)$ in terms of the topology on $X$. We also discuss an application of multiplicative linear functionals on $C(X)$.

## 1. Introduction and preliminaries

The space of all continuous functions $C(X)$ has always inspired many mathematicians to extend their valuable research in the area of analysis. Vast literature is available dealing with different aspects of continuous functions in $C(X)$ (one can refer [1],[3],[5],[8]). W.E.Dietrichjr [2] has extensively characterized the ideal structure of $C(X)$. P.Nanzetta and D Plank [7] extended some characterization to an arbitrary completely regular Hausdorff space $X$ and derived some corollaries. Recently Liaqat Ali Khan et al [6], has given a short proof of a weaker form of the Stone-Weierstrass Theorem for $C(X)$. In this paper, we establish the characterization of closed ideals of $C(X)$ in terms of the topology on $X$ and providing a trivial application of multiplicative linear functionals on $C(X)$.

Throughout the following the letter $X$ denotes a Compact Hausdorff space. Let $C(X)$ be the space of all continuous, complex valued functions on $X$. For any subset $F$ of $X$, we define

$$M_F := \{f \in C(X) : f(u) = 0, \text{for all } u \in F\} \tag{1.1}$$

Then by simple consequences of ideal, we see that $M_F$ is an ideal in $C(X)$. Also, since the uniform limit of a sequence of functions in $M_F$ is again in $M_F$, clearly, it is a closed ideal on $C(X)$. In fact, we show that every closed ideal is of the above form. We require several intermediate results to establish the above mentioned form. These are given in the next section.

**Lemma 1.1.** *Let $I$ be a proper ideal of $C(X)$, then*

$$\bigcap_{f\in I} Z(f) \neq \phi \tag{1.2}$$

*where*

$$Z(f) := \{x \in X : f(x) = 0\} \tag{1.3}$$

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*Proof.* For if there exists $f \in I$ such that $Z(f) = \phi$. Then $1/f \in C(X)$ and since $I$ is an ideal, we obtain $1 = f(1/f) \in I$ implies that $I \supset C(X)$, which contradicts our assumption that $I$ is a proper ideal. Therefore $Z(f) \neq \phi$, for all $f \in I$. Now let $\{f_1, f_2, ..., f_n\}$ be a finite subset of $I$. Since $I$ is an ideal, $f \in I$, we define

$$f := \sum_{i=1}^{n} f_i \bar{f}_i,$$

and hence $Z(f) \neq \phi$. Consequently

$$\bigcap_{i=1}^{n} Z(f) \neq \phi$$

follows. For each $f \in C(X)$, $Z(f)$ is a closed set, since $Z(f) = f^{-1}(0)$. Hence $\{Z(f) : f \in I\}$ is a collection of closed sets in $X$ such that the intersection of any finite number of them is non-empty. Since $X$ is compact and Hausdorff, the entire collection should have a non-empty intersection and this completes the proof of lemma. □

**Definition 1.2.** Let $I$ be an ideal and $f \in C(X)$, then

$$ker(I) = \bigcap_{f \in I} Z(f)$$

is called the kernel of $I$, where $Z(f)$ is defined in 1.3.

## 2. Main Results

In this section, we characterize the closed ideals of $C(X)$ interms of topology on $X$.

**Theorem 2.1.** *Let $F = ker(I)$ and $I$ be a closed ideal in $C(X)$, then*

$$I = M_F,$$

*where $M_F$ is given in 1.1.*

*Proof.* Clearly $I \subset M_F$ and by Lemma 1.1, $\bigcap_{f \in I} Z(f)$ is a closed subset of $X$. In order to prove $M_F \subset I$, we use the fact that $I$ is closed.
Let $f \in M_F$ and $\epsilon > 0$ be given. Let $g \in I$ such that $\|f - g\| < \epsilon$. Since $\epsilon$ is chosen arbitrarily, we conclude that $f \in c\ell(I) = I$. To this end, we define

$$F_\epsilon := \{u \in X : |f(u)| \geq \epsilon\} \tag{2.1}$$

By using 1.2, $f$ vanishes on $F$. We have

$$F_\epsilon \cap F = \phi.$$

For each $u \in F_\epsilon$, there exists $f_u \in I$ such that $f_u(u) \neq \phi$. Hence $f_u \bar{f}_u$ is strictly positive at $u$. Since $f_u \bar{f}_u$ is continuous, there exists an open neighborhood $V_u$ of $u$ on which it is strictly positive.
Now $\{V_u : u \in F_\epsilon\}$ forms an open cover for $F_\epsilon$. Since $f$ is continuous and

$F_\epsilon$ is closed, we obtain $F_\epsilon$ is compact in $X$. Hence there exists a finite set $\{u_1, u_2, ..., u_n\} \subset F_\epsilon$ such that $\{V_{u_i}\}_{i=1}^k$ covers $F_\epsilon$. Therefore

$$\lambda := \sum_{i=1}^{n} f_{u_i} F_{u_i} \tag{2.2}$$

is non-negative on $X$ and strictly positive on $F_\epsilon$. Since each $f_{u_i}$ is in $I$, $\lambda$ belongs to $I$ as well. Since $F_\epsilon$ is compact, $\lambda$ attains its infimum ($m > 0$) on $F_\epsilon$. Thus the function $k$ defined by

$$k(x) := \sup_{x\in X}\{h(x), m\} \tag{2.3}$$

satisfies the following

i. $k(x) > 0$, for all $x \in X$,
ii. $k(x) \geq h(x)$, for all $x \in X$ and
iii. $k(x) = h(x)$, for all $x \in F_\epsilon$,

Now let $g := k^{-1}fh$. Then $g \in I$, since $h \in I$ and it also satisfies $x \in F_\epsilon$ implies that $g(x) = h(x)$ and $x \notin F_\epsilon$ implies that

$$|f(x) - g(x)| < \epsilon|1 - k^{-1}h(x)| \leq \epsilon.$$

This implies that $\|f - g\| < \epsilon$, since $0 \leq k^{-1}h(x)\| \leq 1$. Hence for every $\epsilon > 0$, there exists $g \in I$ such that $\|f - g\| < \epsilon$. Therefore $f \in c\ell(I) = I$ and this is true for all $f \in M_F$. This shows that $M_F \subset I$ and this is completes the proof of theorem. □

Consequently the above characterization influences the closed sets of $X$ which can be viewed in the following:

**Corollary 2.2.** *If $F$ is a closed subset of $X$, then $F = ker(M_F)$, where $M_F$ is as defined in 1.1.*

*Proof.* It is trivial that

$$F \subset \ker(M_F). \tag{2.4}$$

To prove the converse, we choose an element $x \notin F$, then $\{x\} \cap F = \phi$. Therefore the closed subset of $X$ are $F$ and $\{x\}$, because of $X$ being Hausdorff. By applying Urysohn's lemma [4], we get a function $g \in C(X)$ which is 0 on $F$ and 1 at $\{x\}$. Hence $g \in M_F$ but $x \notin Z(g)$, where $Z(g)$ is as defined in 1.3, which implies $x \notin \ker(M_f)$. Therefore

$$F^c \subset \ker(M_F)^c \Rightarrow \ker(M_F) \subset F. \tag{2.5}$$

From 2.4 and 2.5, the proof follows. □

As an interesting upshot of the characterization of closed ideals of $C(X)$, we characterize all multiplicative linear functionals on $C(X)$.

**Definition 2.3.** A linear functional $\psi$ on $C(X)$ is said to be multiplicative if

$$\psi(fg) = \psi(f)\psi(g),$$

for any $f, g \in C(X)$.

**Definition 2.4.** If $\psi$ is any non-zero function on $C(X)$, then

$$N(\psi) := \{x \in X, \psi(x) = 0\}$$

is a maximal ideal in $C(X)$.

**Lemma 2.5.** *Let $I$ be a proper ideal of $C(X)$, then $c\ell(I)$ is a proper ideal. In particular, every maximal ideal is closed.*

*Proof.* In order to prove $c\ell(I)$ is a proper ideal, it is enough to show that $1 \notin c\ell(I)$. Let $f \in I$, then by Lemma 2.1, $Z(f) \neq \phi$. Then, for $x \in Z(f)$, we obtain

$$\|f - 1\| \geq |f(x) - 1| = 1$$

This is true for each $f \in I$ and hence $1 \notin c\ell(I)$. If $I$ is a maximal ideal, then $c\ell(I) = I$ which proves $I$ is closed in $C(X)$. □

**Theorem 2.6.** *Let $\psi$ be a non-zero multiplicative linear functional on $C(X)$, then there exists a unique element $x_0 \in X$ such that $\psi(f) = f(x_0)$, for any $f \in C(X)$*

*Proof.* Since $N(\psi)$ is a maximal ideal, consequently $N(\psi)$ is closed in $C(X)$. Therefore $N(\psi)$ is a closed maximal ideal in $C(X)$. Let $F = \ker(N(\psi))$ and $x_0 \in F$. Now $M_{\{x_0\}}$ is a proper ideal containing $N(\psi)$ and hence

$$N(\psi) = M_{\{x_0\}} \tag{2.6}$$

holds. For each $f \in C(X), (f - f(x_0)) \in M_{\{x_0\}}$ and using 2.6, we have

$$0 = \psi(f - f(x_0).1) = \psi(f) - f(x_0)\psi(1).$$

This implies that

$$\psi(f) = f(x_0)\psi(1). \tag{2.7}$$

Also if $\psi \neq 0$, then there exists $g \in C(X)$ such that $\psi(g) \neq 0$.
Now

$$\psi(1) = \psi(gg^{-1}) = \psi(g)\psi(g)^{-1} = 1$$

We have

$$\psi(f) = f(x_0).$$

To prove the uniqueness of the above relation, if there is a $y_0 \in X$ such that $\psi(f) = f(y_0)$, for every $f \in C(X)$, then by using Corollary 2.2, we have

$$\{x_0\} = \ker(N(\psi)) = \{y_0\}$$

and this completes the proof. □

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