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# INNERNESS $\sigma$-DERIVATIONS AND INNERNESS $\sigma$-HIGHER DERIVATIONS

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Abstract. Let $\mathcal{M}$ be a $W^*$-algebra and $\mathcal{A} \subseteq \mathcal{M}$ a commutative $W^*$-subalgebra of $\mathcal{M}$ containing the identity element of $\mathcal{M}$. Let $\sigma : \mathcal{M} \to \mathcal{M}$ be a continuous homomorphism and $\delta : \mathcal{M} \to \mathcal{M}$ a $\sigma$-derivation. In this paper, it is proved that there exists a $x_0 \in \mathcal{M}$ such that $\delta(a) = \sigma(a)x_0 - x_0\sigma(a)$, for each $a \in \mathcal{A}$. Also, it is proved that if $\{d_n\}$ is a continuous strongly $\sigma$-higher derivationon on $\mathcal{M}$, then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ such that

$$d_n(a) = \sigma^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) - x_{n-2}d_2(\sigma^{n-2})(a) - ... - x_1d_{n-1}(\sigma(a))$$

for each $a \in \mathcal{A}$.

## 1. INTRODUCTION

Let $\mathcal{A}$ be an algebra. We say that a linear mapping $d$ on $\mathcal{A}$ is a derivation if it satisfies

$$d(ab) = ad(b) + d(a)b \qquad (\forall a, b \in \mathcal{A}).$$

For example, suppose $x_0 \in \mathcal{A}$ is an arbitrarary element; one can prove that a linear mapping $d$ on $\mathcal{A}$ with the property

$$d(a) = ax_0 - x_0a, \quad (\forall a \in \mathcal{A})$$

is a derivation; in general, a linear mapping with such a property is called an inner derivation.

A sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a higher derivation if it satifies

$$d_n(ab) = \sum_{k=0}^{n} d_k(a)d_{n-k}(b) \tag{1.1}$$

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for all $a, b \in \mathcal{A}$ and each nonnegative integer $n$; we say that it is strongly, if $d_0 = I$. A typical example of a higher derivation is $\{\frac{\delta^n}{n!}\}$ where $\delta : \mathcal{A} \to \mathcal{A}$ is a derivation. It seems that this example has been the first motivation to define higher derivations. Higher derivations were introduced, for first time, by Hasse and Schmidt in [3].

Now in what follows, assume that $\sigma, \tau$ be two homomorphism on $\mathcal{A}$. A linear mapping $d : \mathcal{A} \to \mathcal{A}$ is said to be a $(\sigma, \tau)$-derivation if it satisfies the generalized Leibniz rule $d(xy) = d(x)\sigma(y) + \tau(x)d(y)$ for each $x, y \in \mathcal{A}$. By a $\sigma$-derivation we mean a $(\sigma, \sigma)$-derivation. An ordinary derivation is an $I$-derivation where $I$ is the identity mapping on $\mathcal{A}$. Specially, if $\delta : \mathcal{A} \to \mathcal{A}$ is an ordinary derivation and $\sigma : \mathcal{A} \to \mathcal{A}$ is a homomorphism, then $d = \delta\sigma$ is a $\sigma$-derivation.

In view of [4], a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a $\sigma$-higher derivation if it satifies

$$d_n(ab) = \sum_{k=0}^{n} d_k(\sigma^{n-k}(a))d_{n-k}(\sigma^k(b)) \tag{1.2}$$

for all $a, b \in \mathcal{A}$ and each nonnegative integer $n$; we say that it is strongly, if $d_0 = I$. A typical example of a $\sigma$-higher derivation is $\{\frac{\delta^n}{n!}\}$ where $\delta : \mathcal{A} \to \mathcal{A}$ is a $\sigma$-derivation in which $d\sigma = \sigma d$.

In view [6, theorem 2.5.1], if $\mathcal{M}$ is a $W^*$-algebra and $\mathcal{A} \subseteq \mathcal{M}$ is a commutative $W^*$-subalgebra containing the identity element of $\mathcal{M}$ and $d : \mathcal{M} \to \mathcal{M}$ a derivation, then there exists a $x_0 \in \mathcal{M}$ such that $d(a) = ax_0 - x_0a$ for each $a \in \mathcal{A}$. Also in light of [6, theorem 2.5.3], every derivation on a $W^*$-algebra is inner.

In view [5], if $\mathcal{M}, \mathcal{A}$ are as in the same sets stated above and $\{d_n\}$ a stongly higher derivation, then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ such that

$$d_n(a) = ax_n - x_na - x_{n-1}d_1(a) - x_{n-2}d_2(a) - ... - x_1d_{n-1}(a), \ \forall a \in \mathcal{A}.$$

In this paper, by supposing that $\mathcal{M}, \mathcal{A}$ are as in the same sets stated above and that $\sigma : \mathcal{M} \to \mathcal{M}$ a continuous homomorphism and $\delta : \mathcal{M} \to \mathcal{M}$ a $\sigma$-derivation, it is showed that there exists a $x_0 \in \mathcal{M}$ such that $\delta(a) = \sigma(a)x_0 - x_0\sigma(a)$, for each $a \in \mathcal{A}$. Also, it is demonstrated that if $\{d_n\}$ is a continuous and strongly $\sigma$-higher derivation on $\mathcal{M}$, then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ such that

$$d_n(a) = \sigma^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) - x_{n-2}d_2(\sigma^{n-2})(a) - ... - x_1d_{n-1}(\sigma(a))$$

for each a ∈ A.

## 2. INNERNESS $\sigma$-DERIVATIONS

Let $\mathcal{A}$ be an algebra. A linear operator $d$ is said to be a deivation if it satisfies the Libnitz Rule $d(ab) = d(a)b + ad(b)$ for each $a, b \in \mathcal{A}$. We say that it is inner if there exists $x_0 \in \mathcal{A}$ such that $d(a) = ax_0 - x_0a$ for each $a \in \mathcal{A}$. If $\sigma$ is a homomorphism on $\mathcal{A}$, a linear mapping $d$ with the property

$$d(ab) = d(a)\sigma(a) + \sigma(a)d(b), \quad (\forall a, b \in \mathcal{A})$$

is called a $\sigma$-derivation. We say that it is inner, if there exists $x_0 \in \mathcal{A}$ such that $d(a) = \sigma(a)x_0 - x_0\sigma(a)$ for each $a \in \mathcal{A}$.

**Theorem 2.1.** *Let $\mathcal{M}$ is a $W^*$-algebra and $\mathcal{A} \subseteq \mathcal{M}$ a commutative $W^*$-subalgebra of $\mathcal{M}$ containing the identity element of $\mathcal{M}$. Let $\sigma : \mathcal{M} \to \mathcal{M}$ be a continuous homomorphism and $d : \mathcal{M} \to \mathcal{M}$ a $\sigma$-derivation. There exists an element $x_0 \in \mathcal{M}$ with the property $\| x_0 \| \leq \| \sigma \| \| d \|$ such that $d(a) = \sigma(a)x_0 - x_0\sigma(a)$ for each $a \in \mathcal{A}$.*

*Proof.* In view of [4], $d$ is continuous. From [1, proposition 14] (or [7, proposition 1.4.5]), $\mathcal{A}$ is the linear span of its unitary elements (i.e. any element of $\mathcal{A}$ is a finite linear combination of the unitary elements). Suppose $\mathcal{A}^u$ be the set of all unitary elements of $\mathcal{A}$. It is enough to show that the result holds for $\mathcal{A}^u$. For each $u \in \mathcal{A}^u$, we define the linear mapping

$$T_u : \mathcal{M} \to \mathcal{M}$$
$$T_u(x) = (\sigma(u))^{-1}(x\sigma(u) + d(u)), \quad \forall\, x \in \mathcal{M}.$$

Suppose $a, b \in \mathcal{A}^u$. We have

$$\begin{aligned} T_aT_b(x) &= T_a\Big((\sigma(b))^{-1}(x\sigma(b)+d(b))\Big) \\ &= (\sigma(a))^{-1}[\Big((\sigma(b))^{-1}(x\sigma(b)+d(b))\Big)\sigma(a)+d(a)] \\ &= (\sigma(a))^{-1}(\sigma(b))^{-1}x\sigma(b)\sigma(a)+(\sigma(a))^{-1}(\sigma(b))^{-1}d(b)\sigma(a) \\ &+ (\sigma(a))^{-1}(\sigma(b))^{-1}\sigma(b)d(a) \\ &= (\sigma(a))^{-1}(\sigma(b))^{-1}\Big(x\sigma(b)\sigma(a)+d(b)\sigma(a)+\sigma(b)d(a)\Big) \\ &= \sigma(ba)^{-1}\Big(x\sigma(ba)+d(ba)\Big) \\ &= T_{ba}(x)=T_{ab}(x)=T_aT_b(x) \end{aligned}$$

Now consider $\Omega = \{T_u : u \in \mathcal{A}^u\}$; for each $u \in \mathcal{A}^u$, the function $T_u$ is $\sigma$-continuous (see [6, page 2]). Suppose $\mathcal{K}$ is the $\sigma$-closed convex subset of $\mathcal{M}$ generated by $\{T_u(0) : u \in \mathcal{A}^u\}$; since

$$\begin{aligned} \| T_u(0) \| &= \| (\sigma(u))^{-1}d(u) \| \\ &= \| (\sigma(u^{-1}))d(u) \| \\ &\leqslant \| \sigma \| \| d \| . \end{aligned}$$

In light of the Banach-Alaogla theorem, $\mathcal{K}$ is $\sigma$-compact. Also since $T_{vu} = T_vT_u$, for each $u, v \in \mathcal{A}^u$, it is immediate that $T(\mathcal{K}) \subseteq \mathcal{K}$. Thus in view the Markov -Kakutani (see [2, Theorem VII.2.1; page 185]), there exists $x_0 \in \mathcal{K}$ such that $T(x_0) = x_0$ for each $T \in \Omega$; actually, for each $u \in \mathcal{A}^u$, $T_u(x_0) = x_0$. In fact, for each $u \in \mathcal{A}^u$,

$$d(u) = \sigma(u)x_0 - x\sigma(u).$$

Moreover that, obviously

$$\| x_0 \| \leq \| \sigma \| \| d \| .$$

□

*Remark* 2.2. Let $\sigma, \tau : \mathcal{M} \to \mathcal{M}$ be continuous homomorphisms and $\delta : \mathcal{M} \to \mathcal{M}$ a $(\sigma, \tau)$-derivation. Then, similar to the proof of Theorem 2.1, and by considering that

$$\begin{aligned} &T_u : \mathcal{M} \to \mathcal{M} \\ &T_u(x) = (\sigma(u))^{-1}(x\tau(u) + d(u)), \quad \forall\, x \in \mathcal{M}, \end{aligned}$$

one can show that there exists an element x0 ∈ M such that d(a) = $\tau$ (a)x0  x0 $\sigma$ (a) for each a ∈ A.

## 3. INNERNESS $\sigma$-HIGHER DERIVATIONS

Let $\mathcal{A}$ be an algebra. A sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a higher derivation if it satifies

$$d_n(ab) = \sum_{k=0}^{n} d_k(a)d_{n-k}(b) \tag{3.1}$$

for all $a, b \in \mathcal{A}$ and each nonnegative integer $n$; it is siad to be strongly if $d_0 = I$. We say that a stongly higher derivation $\{d_n\}$ is inner if ther exist $x_1, x_2, x_3, ...x_n \in \mathcal{A}$ such that

$$d_n(a) = ax_n - x_n a - x_{n-1}d_1(a) - x_{n-2}d_2(a) - ... - x_1 d_{n-1}(a).$$

In view [5], if $\mathcal{M}, \mathcal{A}$ are as in Theorem 2.1 and $\{d_n\}$ a stongly higher derivation, then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ such that

$$d_n(a) = ax_n - x_n a - x_{n-1}d_1(a) - x_{n-2}d_2(a) - ... - x_1 d_{n-1}(a).$$

**Definition 3.1.** Let $\sigma : \mathcal{M} \to \mathcal{M}$ be a homomorphism; in view of [4], a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a $\sigma$-higher derivation if it satifies

$$d_n(ab) = \sum_{k=0}^{n} d_k(\sigma^{n-k}(a))d_{n-k}(\sigma^k(b)) \tag{3.2}$$

for all $a, b \in \mathcal{A}$ and each nonnegative integer $n$; it is strongly if $d_0 = I$. We say that a stongly $\sigma$-higher derivation $\{d_n\}$ is inner if ther exist $x_1, x_2, x_3, ...x_n \in \mathcal{A}$ such that

$$d_n(a) = \sigma^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) - x_{n-2}d_2(\sigma^{n-2}(a) - ... - x_1 d_{n-1}(\sigma(a)).$$

If moreover $\tau : \mathcal{M} \to \mathcal{M}$ be a homomorphism, a sequence $\{d_n\}$ of linear mappings on $\mathcal{A}$ is called a $(\sigma, \tau)$-higher derivation if it satifies

$$d_n(ab) = \sum_{k=0}^{n} d_k(\tau^{n-k}(a))d_{n-k}(\sigma^k(b)) \tag{3.3}$$

for all $a, b \in \mathcal{A}$ and each nonnegative integer $n$; we say that a stongly $(\sigma, \tau)$-higher derivation $\{d_n\}$ is inner if ther exist $x_1, x_2, x_3, ...x_n \in \mathcal{A}$ such that

$$d_n(a) = \tau^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) - x_{n-2}d_2(\sigma^{n-2}(a) - ... - x_1 d_{n-1}(\sigma(a))$$

for each $a \in \mathcal{A}$.

**Theorem 3.2.** *Let $\mathcal{M}$ be are as in Theorem 2.1. Let $\{d_n\}$ be a continuous and strongly $\sigma$-higher derivation on $\mathcal{A}$. Then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ with the property*

$$\| x_n \| \leqslant \| d_n \| \| \sigma \| + \Big( \| x_1 \| \| d_{n-1} \| + ... + \| x_{n-1} \| \| d_1 \| \Big) \| \sigma \|^2 .$$

*such that*

$$d_n(a) = \sigma^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) - x_{n-2}d_2(\sigma^{n-2}(a) - ... - x_1 d_{n-1}(\sigma(a))$$

*for each $a \in \mathcal{A}$.*

*Proof.* For $n = 1$, it is clear that $d_1(ab) = d_1(a)\sigma(b) + \sigma(a)d_1(b)$; i.e. $d_1$ is a $\sigma$-derivation and hence from Theorem 2.1, there exists an element $x_0 \in \mathcal{M}$ such that $d(a) = \sigma(a)x_0 - x_0\sigma(a)$. By assumming that the theorem holds for $n-1$, we continue the proof by induction and prove the fact for $n$. Similar to the proof of Theorem 2.1, it is enough to show that the result holds for the unitary elements. Suppose $\mathcal{A}^u$ be the set of all unitary elements of $\mathcal{A}$. For each $u \in \mathcal{A}^u$, we define the linear mapping

$$\begin{aligned} &T_u : \mathcal{M} \to \mathcal{M} \\ &T_u(x) = \Big( \sigma^n(u)x - d_n(u) - x_1 d_{n-1}(\sigma(u)) - ... - x_{n-1}d_1(\sigma^{n-1}(u)) \Big) (\sigma(u))^{-n}, \forall x \in \mathcal{M}. \end{aligned}$$

Suppose $u, v \in \mathcal{A}^u$; we have

$$\begin{aligned} T_uT_v(x) &= T_u\Big( \Big( \sigma^n(v)x - d_n(v) - x_1 d_{n-1}(\sigma(v)) - ... - x_{n-1}d_1(\sigma^{n-1}(v)) \Big) (\sigma^{-n}(v)) \Big) \\ &= \Big( \sigma^n(u)[\Big( \sigma^n(v)x - d_n(v) - x_1 d_{n-1}(\sigma(v)) - ... - x_{n-1}d_1(\sigma^{n-1}(v)) \Big) (\sigma^{-n}(v))] \\ &- d_n(u) - x_1 d_{n-1}(\sigma(u)) - ... - x_{n-1}d_1(\sigma^{n-1}(u)) \Big) \sigma^{-n}(u) \\ &= [\Big( \sigma^n(u)\sigma^n(v)x - \sigma^n(u)d_n(v) - \sigma^n(u)x_1 d_{n-1}(\sigma(v)) - ... - \sigma^n(u)x_{n-1}d_1(\sigma^{n-1}(v)) \Big) (\sigma^{-n}(v) \\ &- d_n(u) - x_1 d_{n-1}(\sigma(u)) - ... - x_{n-1}d_1(\sigma^{n-1}u)]\sigma^{-n}u \end{aligned}$$

$$
\begin{aligned}
= & \Big( \ \sigma^n(u)\sigma^n(v)x - d_n(uv) + d_n(u)\sigma^n(v) + d_{n-1}(\sigma(u))d_1(\sigma^{n-1}v) + d_{n-2}(\sigma^2(u))d_2(\sigma^{n-2}(v)) \\
+ & \ ... + d_2(\sigma^{n-2}(u))d_{n-2}(\sigma^2(v)) + d_1(\sigma^{n-1}(u))d_{n-1}(\sigma(v))\Big)\sigma^{-n}(v)\sigma^{-n}(u) \\
+ & \Big(\sigma^n(u)x_1 d_{n-1}(\sigma(v)) - \sigma^n(u)x_2 d_{n-2}(\sigma^2(v)) - ... - \sigma^n(u)x_{n-1}d_1(\sigma^{n-1}(v))\Big)\sigma^{-n}(v)\sigma^{-n}(u) \\
+ & \Big( - d_n(u)\sigma^n(v) - x_1 d_{n-1}(\sigma(u))\sigma^n(v) - ... - x_{n-1}d_1(\sigma^{n-1}(u))\sigma^n(v)\Big)\sigma^{-n}(v)\sigma^{-n}(u) \\
= & \ \sigma^n(u)\sigma^n(v)x\sigma^{-n}(v)\sigma^{-n}(u) - d_n(uv)\sigma^{-n}(v) \\
+ & \ (d_{n-1}(\sigma u)) - \sigma^n(u)x_{n-1})d_1(\sigma^{n-1}(v))\sigma^{-n}(v)\sigma^{-n}(u) \\
+ & \ (d_{n-2}(\sigma^2(u)) - \sigma^n(u)x_{n-2})d_2(\sigma^n(v))\sigma^{-n}(v)\sigma^{-n}(u) \\
+ & \ ... + (d_2(\sigma^{n-2}(u)) - \sigma^n(u)x_2)d_{n-2}(\sigma^2(v))\sigma^{-n}(v)\sigma^{-n}(u) \\
+ & \ (d_1(\sigma^{n-1}(u)) - \sigma^n(u)x_1)d_{n-1}(\sigma(v))\sigma^{-n}(v)\sigma^{-n}(u) \\
- & \ x_1 d_{n-1}(\sigma(u))\sigma^{-n}(u) - x_2 d_{n-2}(\sigma^2(u))\sigma^{-n}(u) \\
- & \ ... - x_{n-2}d_2(\sigma^{n-2}(u))\sigma^{-n}(u) - x_{n-1}d_1(\sigma^{n-1}(u))\sigma^{-n}(u) \\
= & \ \sigma^n(u)\sigma^n(v)x\sigma^{-n}(v)\sigma^{-n}(u) - d_n(uv)\sigma^{-n}(v)\sigma^{-n}(u) \\
+ & \Big( - x_{n-1}\sigma^n(u) - x_{n-2}d_1(\sigma^{n-1}(u)) - x_{n-3}d_2(\sigma^{n-2}(u)) \\
- & \ ... - x_2 d_{n-3}(\sigma^3(u)) - x_1 d_{n-2}(\sigma^2(u))\Big)d_1(\sigma^{n-1}(v))\sigma^{-n}(v)\sigma^{-n}(u) \\
+ & \Big( - x_{n-2}\sigma^n(u) - x_{n-3}d_1(\sigma^{n-1}(u)) - x_{n-4}d_2(\sigma^{n-2}(u)) \\
- & \ ... - x_2 d_{n-4}(\sigma^4(u)) - x_1 d_{n-3}(\sigma^3(u))\Big)d_2(\sigma^{n-2}(v))\sigma^{-n}(v)\sigma^{-n}(u) \\
+ & \ ... + \Big( - x_2\sigma^n(u) - x_1 d_1(\sigma^{n-1}(u))\Big)d_{n-2}(\sigma^2(v))\sigma^{-n}(v)\sigma^{-n}(u) \\
+ & \Big( - x_1\sigma^n(u)\Big)d_{n-1}(\sigma(v))\sigma^{-n}(v)\sigma^{-n}(u) \\
- & \ x_1 d_{n-1}(\sigma(u))\sigma^n(v)\sigma^{-n}(v)\sigma^{-n}(u) - x_2 d_{n-2}(\sigma^2(u))\sigma^n(v)\sigma^{-n}(v)\sigma^{-n}(u) \\
- & \ ... - x_{n-2}d_2(\sigma^{n-2}(u))\sigma^n(v)\sigma^{-n}(v)\sigma^{-n}(u) - x_{n-1}d_1(\sigma^{n-1}(u))\sigma^n(v)\sigma^{-n}(v)\sigma^{-n}(u) \\
= & \ \sigma^n(u)\sigma^n(v)x\sigma^{-n}(v)\sigma^{-n}(u) - d_n(uv)\sigma^{-n}(v)\sigma^{-n}(u) \\
- & \ x_{n-1}\Big(\sigma^n(u)d_1(\sigma^{n-1}(v)) + d_1(\sigma^{n-1}(u))\sigma^n(v)\Big)\sigma^{-n}(v)\sigma^{-n}(u)
\end{aligned}
$$

$$\begin{aligned}
&- x_{n-2}\Big( d_1(\sigma^{n-1}(u))d_1(\sigma^{n-1}(v)) + \sigma^n(u)d_2(\sigma^{n-2}(v)) \\
&+ d_2(\sigma^{n-2}(u))\sigma^n(v)\Big)\sigma^{-n}(v)\sigma^{-n}(u) \\
&- x_{n-3}\Big(\sigma^n(u)d_3(\sigma^{n-3}(v)) + d_1(\sigma^{n-1}(u))d_2(\sigma^{n-2}(v)) \\
&+ d_2(\sigma^{n-2}(u))d_1(\sigma^{n-1}(v)) + d_3(\sigma^{n-3}(u))\sigma^n(v)\Big)\sigma^{-n}(v)\sigma^{-n}(u) \\
&- ... - x_1\Big(\sigma^n(u)d_{n-1}(\sigma(v)) + ... + d_{n-1}(\sigma(u))\sigma^n(v)\Big)\sigma^{-n}(v)\sigma^{-n}(u) \\
&= \Big(\sigma^n(uv)x - d_n(uv) - x_{n-1}d_1(uv) - x_{n-2}d_2(uv) \\
&- x_{n-3}d_3(uv) - ... - x_1 d_{n-1}(uv)\Big)(\sigma^{-n}(uv)) \\
&= T_{uv}(x)
\end{aligned}$$

Now consider $\Omega = \{T_u : u \in \mathcal{A}^u\}$; for each $u \in \mathcal{A}^u$, since $\{d_n\}$ is continuous, the function $T_u$ is $\sigma$-continuous (see [6, page 2]). Suppose $\mathcal{K}$ is the $\sigma$-closed convex subset of $\mathcal{M}$ generated by $\{T_u(0) : u \in \mathcal{A}^u\}$; since

$$\begin{aligned}
\| T_u(0) \| &= \| \Big(d_n(u) + x_1 d_{n-1}(\sigma(u)) + ... + x_{n-1}d_1(\sigma^{n-1}(u))\Big)(\sigma(u))^{-n} \| \\
&\leqslant \| d_n(u) \| + \| x_1 d_{n-1}(\sigma(u)) \| +...+ \| x_{n-1}d_1(\sigma^{n-1}(u))(\sigma(u))^{-n} \| \\
&\leqslant \Big( \| d_n \| + \| x_1 \|\| d_{n-1} \|\| \sigma \| +...+ \| x_{n-1} \|\| d_1 \|\| \sigma \| \Big) \| \sigma \| \\
&= \| d_n \|\| \sigma \| + \Big( \| x_1 \|\| d_{n-1} \| +...+ \| x_{n-1} \|\| d_1 \| \Big) \| \sigma \|^2,
\end{aligned}$$

in light of the Banach-Alaogla theorem, $\mathcal{K}$ is $\sigma$-compact. Also since $T_{vu} = T_v T_u$, it is clear that $T(\mathcal{K}) \subseteq \mathcal{K}$. Thus in view the Markov-Kakutani theorem (see [2, theorem VII.2.1]), there exists $x_0 \in \mathcal{K}$ such that $T(x_0) = x_0$ for each $T \in \Omega$; actually, for each $u \in \mathcal{A}^u$, $T_u(x_0) = x_0$ and hence the proof is finished. Moreover that, obviously

$$\| x_0 \| \leqslant \| d_n \|\| \sigma \| + \Big( \| x_1 \|\| d_{n-1} \| +...+ \| x_{n-1} \|\| d_1 \| \Big) \| \sigma \|^2 .$$

□

**Corollary 3.3.** *Let $\mathcal{M}, \mathcal{A}$ be as in Theorem 2.1. Let $\sigma : \mathcal{M} \to \mathcal{M}$ be a continuous homomorphism with the condition $\sigma^2 = \sigma$. Let $\{d_n\}$ be a strongly $\sigma$-higher derivation on $\mathcal{A}$. Then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ such that*

$$d_n(a) = \sigma^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) - x_{n-2}d_2(\sigma^{n-2}(a) - ... - x_1 d_{n-1}(\sigma(a))$$

*for each $a \in \mathcal{A}$.*

**Proof. From [3], {dn} is continuous and in view of Theorem 3.2, the desired fact is easily**

proved. □

**Theorem 3.4.** *Let $\mathcal{M}, \mathcal{A}$ be as in Theorem 2.1. Let $\sigma, \tau : \mathcal{M} \to \mathcal{M}$ be continuous homomorphisms. Let $\{d_n\}$ be a continuous and strongly $(\sigma, \tau)$-higher derivation on $\mathcal{A}$. Then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ such that*

$$d_n(a) = \tau^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) - x_{n-2}d_2(\sigma^{n-2}(a) - ... - x_1d_{n-1}(\sigma(a))$$

*for each $a \in \mathcal{A}$.*

**Corollary 3.5.** *Let $\mathcal{M}, \mathcal{A}$ be as in Theorem 2.1. Let $\sigma, \tau : \mathcal{M} \to \mathcal{M}$ be continuous homomorphisms with the condition $\sigma^2 = \sigma$ and $\tau^2 = \tau$ . Let $\{d_n\}$ be a strongly $(\sigma, \tau)$-higher derivation on $\mathcal{A}$. Then there exist $x_1, x_2, x_3, ...x_n \in \mathcal{M}$ such that*

$$d_n(a) = \tau^n(a)x_n - x_n\sigma^n(a) - x_{n-1}d_1(\sigma^{n-1}(a)) - x_{n-2}d_2(\sigma^{n-2}(a) - ... - x_1d_{n-1}(\sigma(a))$$

*for each $a \in \mathcal{A}$.*

*Proof.* From [3], $\{d_n\}$ is continuous and in view of Theorem 3.4, the desired fact is easily proved. □

## References

[1] F. F. Bonsall, J. Duncan, *Complete Normed Algebras*, Springer-Verlag Berlin Heidelberg New York 1973.

[2] Kenneth R. Davidson, *C*-Algebras by Example*, American Mathematical Society, 1996.

[3] H. Hasse and F. K. Schmidt, Noch eine Begrdung der theorie der hheren differential quotienten in einem algebraischen funtionenkrper einer unbestimmeten, *J. Reine Angew. Math.* **177** (1937), 215-237.

[4] S. Hejazian, H. Mahdavian Rad, M. Mirzavaziri, $(\sigma, \tau)$-Higher Derivation, Journal of Advanced Research in Pure Mathematics, **4** (2012), no. 4, 67-77.

[5] S. Hejazian, T. L. Shatry, Characterization of higher derivations on Banach algebras, to appear.

[6] S. Sakai, Operator algebras in dynamical systems, CAMBRIDGE UNIVERSITY PRESS, 1991.

[7] S. Sakai, $C^*$-algebras and $W^*$-algebras, Springer-Verlag Berlin Heidelberg New York 1971.