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FAMILIES OF MEROMORPHIC FUNCTIONS INVOLVING A NEW GENERALIZED DIFFERENTIAL OPERATOR

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Abstract. A new differential operator $\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)$ is introduced for functions of the form $f(z)=z^{-p}+\sum\limits_{k=n}^{\infty}a_kz^k$ which are meromorphic in the punctured unit disc $\mathrm{U}^*:=\{z\in\mathbb{C}:0<|z|<1\}$. We introduce the class $\Phi^m_p(\alpha_1,\alpha,\beta,\lambda,\ell,\eta)$ of meromorphically functions that generalize and extend many classes previously studied by different authors, and the main object of this paper is to investigate various important properties and characteristics for this class. In addition, we proved that a special property is preserved by some integral operators.

1. Introduction

Let $\Sigma_{p,n}$, with $n\geq 1-p$, denote the class of functions of the form

$$f(z)=z^{-p}+\sum_{k=n}^{\infty}a_kz^k,\quad (p,n\in\mathbb{N}=\{1,2,\dots\}),\tag{1.1}$$

which are analytic in the punctured open unit disc $\mathrm{U}^*:=\mathrm{U}\setminus\{0\}$, where $\mathrm{U}:=\{z\in\mathbb{C}:|z|<1\}$.

For a function $f\in\Sigma_{p,n}$, given by (1.1) and $g\in\Sigma_{p,n}$ defined by

$$g(z)=z^{-p}+\sum_{k=n}^{\infty}b_kz^k,\ z\in\mathrm{U}^*,$$

we define the *Hadamard (or convolution) product* of f and g by

$$(f*g)(z)=z^{-p}+\sum_{k=n}^{\infty}a_kb_kz^k,\ z\in\mathrm{U}^*.$$

For the complex parameters $\alpha_1,\dots,\alpha_q$ and $\beta_1,\dots,\beta_s$, with $\beta_j\notin\mathbb{Z}_0^-=\{0,-1,-2,\dots\}$, $j=1,2,\dots,s$, let consider the *generalized hypergeometric function* ${}_qF_s(\alpha_1,\dots,\alpha_q;\beta_1,\dots,\beta_s;z)$

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defined by (see, for example, [19, p.19])

$$ {}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k\ldots(\alpha_q)_k}{(\beta_1)_k\ldots(\beta_s)_k}\frac{z^k}{k!},\ z\in\mathrm{U}, $$

$$ \left(q\le s+1,\ q,s\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}\right), $$

where $(\theta)_\nu$ is the *Pochhammer symbol*, defined, in terms of the *Gamma function* Γ, by

$$ (\theta)_\nu:=\frac{\Gamma(\theta+\nu)}{\Gamma(\theta)}=\begin{cases}1, & \text{if}\quad \nu=0,\ \theta\in\mathbb{C}\setminus\{0\},\\ \theta(\theta+1)\ldots(\theta+\nu-1), & \text{if}\quad \nu\in\mathbb{N},\ \theta\in\mathbb{C}.\end{cases} $$

Corresponding to the function $h_{p,q,s}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)$ defined by

$$ h_{p,q,s}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z):=z^{-p}\cdot{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z), $$

we consider a linear operator

$$ H_{p,q,s}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s):\Sigma_{p,n}\to\Sigma_{p,n} $$

which is defined by the following Hadamard product:

$$ H_{p,q,s}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=h_{p,q,s}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z),\ z\in\mathrm{U}^*. $$

Therefore, for a function f of form (1.1), we have

$$ H_{p,q,s}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=z^{-p}+\sum_{k=n}^{\infty}\frac{(\alpha_1)_{k+p}\ldots(\alpha_q)_{k+p}}{(\beta_1)_{k+p}\ldots(\beta_s)_{k+p}}\frac{a_k}{(k+p)!}z^k,\ z\in\mathrm{U}^*, $$

and, for convenience, we write

$$ H_{p,q,s}(\alpha_1):=H_{p,q,s}(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s). $$

Using the Hadamard product, we define the new operator $\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)$ as follows:

Definition 1.1. *For $\alpha_1,\ldots,\alpha_q\in\mathbb{C}$ and $\beta_1,\ldots,\beta_s\in\mathbb{C}\setminus\mathbb{Z}_0^-$, with $q\le s+1$ $(q,s\in\mathbb{N}_0)$, let define the operator $\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1):\Sigma_{p,n}\to\Sigma_{p,n}$ by*

$$ \begin{aligned}\mathcal{F}^{0,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)=&H_{p,q,s}(\alpha_1)f(z),\ z\in\mathrm{U}^*,\\ \mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)=&[1-\beta(\lambda-\alpha)]\cdot\mathcal{F}^{m-1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)\\ &+\frac{\beta(\lambda-\alpha)}{\ell z^{p+\ell-1}}\left[z^{p+\ell}\cdot\mathcal{F}^{m-1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)\right]',\ z\in\mathrm{U}^*,\end{aligned} $$

for $m \in \mathbb{N}$, *where* α, β, $\lambda \in \mathbb{C}$ *and* $\ell \in \mathbb{N}$.

Remarks 1.1. (i) We have

$$\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) = \mathcal{F}^{1,p,q,s}_{\ell,\alpha,\beta,\lambda}\left(\mathcal{F}^{m-1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)\right), \quad m \in \mathbb{N},$$

and for all $f \in \Sigma_{p,n}$ the next formula holds:

(1.2)

$$\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z) = z^{-p} + \sum_{k=n}^{\infty}\left[\frac{\ell+\beta(\lambda-\alpha)(k+p)}{\ell}\right]^m \frac{(\alpha_1)_{k+p}\dots(\alpha_q)_{k+p}}{(\beta_1)_{k+p}\dots(\beta_s)_{k+p}}\frac{a_k}{(k+p)!}z^k,\ z \in \mathrm{U}^*,$$

$$(\alpha, \beta, \lambda \in \mathbb{C},\ \ell \in \mathbb{N} \quad \text{and} \quad m \in \mathbb{N}_0).$$

(ii) For $\beta(\lambda-\alpha) = 0$ or $m = 0$ in (1.2), we have $\mathcal{F}^{0,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) = H_{p,q,s}(\alpha_1)$.

(iii) It easily verified from (1.2) that for all $f \in \Sigma_{p,n}$ we have

(1.3)

$$\beta(\lambda-\alpha)\, z\left[\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)\right]' = \ell\,\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z) - [\ell+\beta(\lambda-\alpha)p]\ \mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)$$

and

$$z\left[\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)\right]' = \alpha_1\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1+1)f(z) - (\alpha_1+p)\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z),\ z \in \mathrm{U}^*.$$

We emphasize here some special cases of the operator $\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)$ previously studied by different authors:

(i) $\mathcal{F}^{0,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) = H_{p,q,s}(\alpha_1)$ (see Liu and Srivastava [15], Raina and Srivastava [18] and Aouf [3]);

(ii) For $q = s+1$, $\alpha_i = 1$ $(i = 1, \dots, s+1)$ and $\beta_j = 1$ $(j = 1, \dots, s)$, $\alpha = 0$ and $\beta = 1$, we obtain the operator $I^m_p(\lambda, \ell)$ introduced and studied by El-Ashwah [10]. The operator $I^m_p(\lambda, \ell)$ contains as special cases the multiplier transformation I^m_p (see Aouf and Hossen [4]), $I(m, \ell)$ (see Cho et al. [8, 9]) and I^m (see Uralegaddi and Somanatha [20] and [21]);

(iii) For $m = 0$, $q = 2$, $s = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$, we have $\mathcal{F}^{0,p,2,1}_{\ell,\alpha,\beta,\lambda}(\alpha_1) = \mathcal{L}_p(a, c)$ $(a, c > 0)$ (see Liu and Srivastava [14]);

(iv) For $m = 0$, $q = 2$, $s = 1$, $\alpha_1 = \nu + p$, $\alpha_2 = p$ and $\beta_1 = p$, we have $\mathcal{F}^{0,p,2,1}_{\ell,\alpha,\beta,\lambda}(\alpha_1) = D^{\nu+p-1}$ $(\nu > -p,\ p \in \mathbb{N})$ (see [1] and [5]);

(v) For $m=0$, $q=2$ and $s=1$, $\alpha_1=\mu$, $\alpha_2=1$ and $\beta_1=\mu+1$, we have $\mathcal{F}^{0,p,2,1}_{\ell,\alpha,\beta,\lambda}(\alpha_1)=F_{\mu,p}$ $(\mu>0,\ p\in\mathbb{N})$ (see [13] and [23]);

(vi) For $m=0$, $q=2$, $s=1$, $\alpha_1=\lambda$ $(\lambda>0)$, $\alpha_2=1$ and $\beta_1=n+p$, we have $\mathcal{F}^{0,p,2,1}_{\ell,\alpha,\beta,\lambda}(\alpha_1)=I_{n+p-1,\lambda}$ $(n>-p,\ p\in\mathbb{N})$, where the operator $I_{n+p-1,\lambda}$ was introduced by Aouf and Xu [6].

Also, by specializing the parameters m, ℓ, α, β, λ, p, q, s, α_i $(i=1,\dots,q)$ and $\beta_j=1$ $(j=1,\dots,s)$ we obtain various new operators:

(i) For $q=2$ and $s=1$, $\alpha_1=n+p$, $\alpha_2=1$ and $\beta_1=1$, we obtain a new operator $\mathcal{F}^{m,p,2,1}_{\ell,\alpha,\beta,\lambda}(n+p)f(z)=z^{-p}+\sum\limits_{k=n}^{\infty}\left[\dfrac{\ell+\beta(\lambda-\alpha)(k+p)}{\ell}\right]^m\dfrac{(n+p)_{k+p}}{(1)_{k+p}}a_kz^k$, where $n>-p$, $p,n\in\mathbb{N}$;

(ii) For $q=2$ and $s=1$, $\alpha_1=a$, $\alpha_2=1$ and $\beta_1=c$, we obtain a new operator $\mathcal{F}^{m,p,2,1}_{\ell,\alpha,\beta,\lambda}(a)f(z)=z^{-p}+\sum\limits_{k=n}^{\infty}\left[\dfrac{\ell+\beta(\lambda-\alpha)(k+p)}{\ell}\right]^m\dfrac{(a)_{k+p}}{(c)_{k+p}}a_kz^k$, where $a\in\mathbb{R}$, $c\in\mathbb{R}\setminus\mathbb{Z}_0^-$;

(iii) For $q=2$ and $s=1$, $\alpha_1=p+1$, $\alpha_2=1$ and $\beta_1=n+p$, we obtain a new operator $\mathcal{F}^{m,p,2,1}_{\ell,\alpha,\beta,\lambda}(p+1)f(z)=z^{-p}+\sum\limits_{k=n}^{\infty}\left[\dfrac{\ell+\beta(\lambda-\alpha)(k+p)}{\ell}\right]^m\dfrac{(p+1)_{k+p}}{(n+p)_{k+p}}a_kz^k$, where $n\in\mathbb{Z}$, $n>-p$, $p,n\in\mathbb{N}$;

(iv) For $q=2$ and $s=1$, $\alpha_1=p+\delta$, $\alpha_2=c$ and $\beta_1=a$, we obtain a new operator $\mathcal{F}^{m,p,2,1}_{\ell,\alpha,\beta,\lambda}(p+\delta)f(z)=z^{-p}+\sum\limits_{k=n}^{\infty}\left[\dfrac{\ell+\beta(\lambda-\alpha)(k+p)}{\ell}\right]^m\dfrac{(p+\delta)_{k+p}(c)_{k+p}}{(a)_{k-p}(1)_{k+p}}a_kz^k$, where $a,c\in\mathbb{R}\setminus\mathbb{Z}_0^-$, $\delta>-p$, $p,n\in\mathbb{N}$;

(v) For $q=2$ and $s=1$, $\alpha_1=p+\delta$, $\alpha_2=1$ and $\beta_1=p+\delta+1$, we obtain a new operator $\mathcal{F}^{m,p,2,1}_{\alpha,\beta,\lambda}(p+\delta)f(z)=z^{-p}+\sum\limits_{k=n}^{\infty}\left[\dfrac{\ell+\beta(\lambda-\alpha)(k+p)}{\ell}\right]^m\dfrac{(p+\delta)_{k+p}}{(p+\delta+1)_{k+p}}a_kz^k$, where $\delta>-p$, $p,n\in\mathbb{N}$.

We introduce the class $\Phi_p^m(\alpha_1,\alpha,\beta,\lambda,\ell,\eta)$ of the functions $f\in\Sigma_{p,n}$ which satisfy the condition

$$\mathrm{Re}\left\{\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)\right\}<-\eta,\ z\in\mathrm{U}, \tag{1.4}$$

where

$$\alpha,\lambda\in\mathbb{C}\text{ with }\lambda\neq\alpha,\ \ \beta\in\mathbb{C}^*\ \ell\in\mathbb{N},\ 0\le\eta<p,\ p\in\mathbb{N},\ m\in\mathbb{N}_0. \tag{1.5}$$

Remark that for the function that appeared in the brackets in the left-hand side of (1.4), the point $z_0 = 0$ is a removable singularity, hence this function is regular in the whole unit disk U.

We note that for the special case $\lambda = \beta = \ell = 1$, $\alpha = 0$ and $q = s+1$, $\alpha_i = 1$ $(i = 1, \dots, s+1)$ and $\beta_j = 1$ $(j = 1, \dots, s)$, the class $\Phi_p^m(\alpha_1, \alpha, \beta, \lambda, \ell, \eta)$ reduces to the class $B_n(\beta)$ studied by Aouf and Hossen [4].

In this paper, known results of Bajpai [7], Goel and Sohi [11], Uralegaddi and Somanatha [20] and Aouf and Hossen [4] are extended.

2. Basic properties of the class $\Phi_p^m(\alpha_1, \alpha, \beta, \lambda, \ell, \eta)$

We begin by recalling the following well-known result (*Jack-Miller-Mocanu's Lemma*), which we shall apply in proving our first inclusion theorems.

Lemma 2.1. [12] *Let the nonconstant function w be analytic in* U*, with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in$ U, then*

$$z_0 w'(z_0) = \rho w(z_0),$$

where ρ is a real number and $\rho \geq 1$.

A generalization of of this lemma was given in [16], and represents one of the most important investigation tools of the theory of differential subordinations (see also [17, p. 19]).

Unless otherwise mentioned, we assume throughout this paper that all parameters satisfy the conditions (1.5) of the definition formula (1.4).

Theorem 2.1. *If we assume that $\beta(\lambda - \alpha) > 0$, then*

$$\Phi_p^{m+1}(\alpha_1, \alpha, \beta, \ell, \lambda, \eta) \subset \Phi_p^m(\alpha_1, \alpha, \beta, \lambda, \ell, \eta), \quad \textit{for all} \quad m \in \mathbb{N}_0.$$

Proof. Considering an arbitrary function $f \in \Phi_p^{m+1}(\alpha_1, \alpha, \beta, \lambda, \ell, \eta)$, then

$$\text{Re}\left\{\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}_{\ell,\alpha,\beta,\lambda}^{m+2,p,q,s}(\alpha_1)f(z)}{\mathcal{F}_{\ell,\alpha,\beta,\lambda}^{m+1,p,q,s}(\alpha_1)f(z)} - \left(p + \frac{\ell}{\beta(\lambda-\alpha)}\right)\right\} < -\eta, \ z \in \text{U}, \tag{2.1}$$

and we have to show that (2.1) implies the inequality (1.4).

Defining the function w regular in U by

$$\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)=-\frac{p+(2\eta-p)w(z)}{1+w(z)},\ z\in \mathrm{U}, \tag{2.2}$$

then $w(0)=0$, and the above relation may be written as

$$\frac{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}=\frac{1+\left(1+\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)\right)w(z)}{1+w(z)},\ z\in \mathrm{U}. \tag{2.3}$$

Differentiating (2.3) logarithmically with respect to z and using (1.3), we obtain

$$\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+2,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)+\eta= \tag{2.4}$$

$$(p-\eta)\left\{\frac{\frac{2\beta(\lambda-\alpha)}{\ell}zw'(z)}{(1+w(z))\left[1+\left(1+\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)\right)w(z)\right]}-\frac{1-w(z)}{1+w(z)}\right\},\ z\in \mathrm{U}.$$

Now, we will prove that $|w(z)|<1$ for $z\in \mathrm{U}$. If not, then there exists a point $z_0\in \mathrm{U}$ such that $\max\{|w(z)|:|z|\le|z_0|\}=|w(z_0)|=1$. According to Lemma 2.1, there exists a real number $\rho\ge 1$, such that $z_0w'(z_0)=\rho w(z_0)$, and taking $z=z_0$ in (2.4) we get

$$\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+2,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z_0)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z_0)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)+\eta=$$

$$(p-\eta)\left\{\frac{\frac{2\beta(\lambda-\alpha)}{\ell}\rho w(z_0)}{(1+w(z_0))\left[1+\left(1+\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)\right)w(z_0)\right]}-\frac{1-w(z_0)}{1+w(z_0)}\right\}.$$

Since $w(z_0)=e^{i\theta}$ for some $\theta\in[0,2\pi]$, from the above relation it follows that

$$\mathrm{Re}\left\{\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+2,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z_0)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z_0)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)+\eta\right\}=$$

$$(p-\eta)\,\mathrm{Re}\left\{\frac{\frac{2\beta(\lambda-\alpha)}{\ell}\rho e^{i\theta}}{(1+e^{i\theta})\left[1+\left(1+\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)\right)e^{i\theta}\right]}-\frac{1-e^{i\theta}}{1+e^{i\theta}}\right\}=$$

$$\frac{2\beta(\lambda-\alpha)(p-\eta)\rho}{\ell}\cdot\frac{2+\frac{2\beta(\lambda-\alpha)(p-\eta)}{\ell}}{|1+e^{i\theta}|^2\left|1+\left(1+\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)\right)e^{i\theta}\right|^2}\cdot(1+\cos\theta)\ge 0,$$

for all $\theta\in[0,2\pi]$, whenever $\beta(\lambda-\alpha)>0$. This inequality contradicts our assumption given by (2.1), and therefore we have $|w(z)|<1$ for all $z\in \mathrm{U}$. Finally, from (2.2) we conclude that $f\in\Phi^m_p(\alpha_1,\alpha,\beta,\lambda,\ell,\eta)$. $\square$

For a number $c > 0$, let recall the well-known integral operator $F_{c,p} : \Sigma_{p,n} \to \Sigma_{p,n}$ defined by

$$F_{c,p}(f)(z) = \frac{c}{z^{c+p}} \int_0^z t^{c+p-1} f(t)\,dt,\ z \in \mathrm{U}^*. \tag{2.5}$$

Remark that the operator $F_{c,p}$ was investigated by many authors, for example, [2], [22], [23], etc. Moreover, for all $f \in \Sigma_{c,p}$ the operator can be written in the convolution product form

$$F_{c,p}(f) = \Phi_{c,p} * f, \quad \text{where} \quad \Phi_{c,p}(z) := z^{-p} + \sum_{k=n}^{\infty} \frac{c}{c+p+k} z^k,\ z \in \mathrm{U}^*,$$

and we could easily check that it satisfy the following differentiation relationships:

$$z\left(\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z)\right)' = c\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) f(z) - (c+p)\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z),\ z \in \mathrm{U}^*, \tag{2.6}$$

$$\begin{aligned} z\left(\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z)\right)' &= \frac{\ell}{\beta(\lambda-\alpha)} \mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z) \\ &- \left(p + \frac{\ell}{\beta(\lambda-\alpha)}\right) \mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z),\ z \in \mathrm{U}^*. \end{aligned} \tag{2.7}$$

Theorem 2.2. *If $f \in \Phi^m_p(\alpha_1, \alpha, \beta, \lambda, \ell, \eta)$, then $F_{c,p}(f) \in \Phi^m_p(\alpha_1, \alpha, \beta, \lambda, \ell, \eta)$, that is*

$$F_{c,p}\left(\Phi^m_p(\alpha_1, \alpha, \beta, \lambda, \ell, \eta)\right) \subset \Phi^m_p(\alpha_1, \alpha, \beta, \lambda, \ell, \eta).$$

Proof. For an arbitrary $f \in \Sigma_{p,n}$, since the right-hand sides of (2.6) and (2.7) coincide, we get

$$\begin{aligned} c\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) f(z) &= \frac{\ell}{\beta(\lambda-\alpha)} \mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z) + \\ &\left(c - \frac{\ell}{\beta(\lambda-\alpha)}\right) \mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z),\ z \in \mathrm{U}^*. \end{aligned}$$

From this last relation, it follows that the assumption $f \in \Phi^m_p(\alpha_1, \alpha, \beta, \lambda, \ell, \eta)$ given by (1.4) is equivalent to

$$\operatorname{Re}\left\{ \frac{\frac{\ell}{\beta(\lambda-\alpha)} \frac{\mathcal{F}^{m+2,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z)} + \left(c - \frac{\ell}{\beta(\lambda-\alpha)}\right)}{1 + \left(\frac{\beta(\lambda-\alpha)}{\ell} c - 1\right) \frac{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z)}} - \left(p + \frac{\ell}{\beta(\lambda-\alpha)}\right) \right\} < -\eta, \tag{2.8}$$

and we have to prove that (2.8) implies the inequality

$$\mathrm{Re}\left\{\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)\right\}<-\eta,\ z\in\mathrm{U}.$$

Defining the function w regular in U by

(2.9)
$$\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)=-\frac{p+(2\eta-p)w(z)}{1+w(z)},\ z\in\mathrm{U},$$

then $w(0)=0$, and the definition relation (2.9) may be written as

$$\frac{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}=\frac{1+\left(1+\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)\right)w(z)}{1+w(z)},\ z\in\mathrm{U}. \tag{2.10}$$

Differentiating (2.10) logarithmically with respect to z and using (2.7), we obtain

$$\frac{\mathcal{F}^{m+2,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}-\frac{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}=$$
$$\frac{\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)zw'(z)}{(1+w(z))\left[1+\left(1+\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)\right)w(z)\right]},\ z\in\mathrm{U}.$$

Using the above relation, the function that appeared in the left-hand side of (2.8) may be written in the form

$$\frac{\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+2,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}+\left(c-\frac{\ell}{\beta(\lambda-\alpha)}\right)}{1+\left(\frac{\beta(\lambda-\alpha)}{\ell}c-1\right)\frac{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)=$$
$$\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)$$
$$+\frac{\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)zw'(z)}{(1+w(z))\left[1+\left(1+\frac{2\beta(\lambda-\alpha)}{\ell}(p-\eta)\right)w(z)\right]}\cdot\frac{1}{1+\left(\frac{\beta(\lambda-\alpha)}{\ell}c-1\right)\frac{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}},$$

which, by using (2.9) and (2.10), reduces to

$$\frac{\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+2,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}+\left(c-\frac{\ell}{\beta(\lambda-\alpha)}\right)}{1+\left(\frac{\beta(\lambda-\alpha)}{\ell}c-1\right)\frac{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}(f)(z)}}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)+\eta=$$
$$-(p-\eta)\frac{1-w(z)}{1+w(z)}+\frac{2(p-\eta)zw'(z)}{(1+w(z))\left[c+(c+2(p-\eta))\,w(z)\right]},\ z\in\mathrm{U}. \tag{2.11}$$

Like in the proof of the previous theorem, we will show that $|w(z)| < 1$ for $z \in \mathrm{U}$. Contrary, there exists a point $z_0 \in \mathrm{U}$ such that $\max\{|w(z)| : |z| \leq |z_0|\} = |w(z_0)| = 1$, and form Lemma 2.1 there exists a real number $\rho \geq 1$, such that $z_0 w'(z_0) = \rho w(z_0)$. Thus, using the fact that $w(z_0) = e^{i\theta}$ for some $\theta \in [0, 2\pi]$, and taking $z = z_0$ in (2.11) we get

$$\mathrm{Re}\left\{ \frac{\frac{\ell}{\beta(\lambda-\alpha)} \frac{\mathcal{F}^{m+2,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z_0)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z_0)} + \left(c - \frac{\ell}{\beta(\lambda-\alpha)}\right)}{1 + \left(\frac{\beta(\lambda-\alpha)}{\ell} c - 1\right) \frac{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z_0)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p}(f)(z_0)}} - \left(p + \frac{\ell}{\beta(\lambda - \alpha)}\right) + \eta \right\} =$$

$$\mathrm{Re}\left\{ -(p-\eta)\frac{1 - e^{i\theta}}{1 + e^{i\theta}} + \frac{2(p-\eta)\rho e^{i\theta}}{(1 + e^{i\theta})\left[c + (c + 2(p-\eta))\, e^{i\theta}\right]} \right\} =$$

$$\frac{2(p-\eta)\rho}{c} \cdot \frac{2 + \frac{2(p-\eta)}{c}}{|1 + e^{i\theta}|^2 \left|1 + \left(1 + \frac{2(p-\eta)}{c}\right) e^{i\theta}\right|^2} \cdot (1 + \cos\theta) \geq 0,$$

for all $\theta \in [0, 2\pi]$. Since this inequality contradicts the assumption (2.8), it follows that $|w(z)| < 1$ for all $z \in \mathrm{U}$, and from (2.9) we get our conclusion. $\square$

Remarks 2.1. (*i*) For $q = s + 1$, $\alpha_i = 1$ $(i = 1, \ldots, s + 1)$, $\beta_j = 1$ $(j = 1, \ldots, s)$, $\alpha = 0$, $\lambda = \beta = \ell = p = c = a_k = 1$ and $m = \eta = 0$, we note that Theorem 2.2 extends a results of Bajpai [7, Theorem 1];

(*ii*) For $q = s + 1$, $\alpha_i = 1$ $(i = 1, \ldots, s + 1)$, $\beta_j = 1$ $(j = 1, \ldots, s)$, $\alpha = 0$, $\lambda = \beta = \ell = p = a_k = 1$ and $m = \eta = 0$, we note that Theorem 2.2 extends a results of Goel and Sohi [11, Corollary 1].

Theorem 2.3. *If we suppose that* $c = \dfrac{\ell}{\beta(\lambda - \alpha)} > 0$, *then* $f \in \Phi^m_p(\alpha_1, \alpha, \beta, \lambda, \ell, \eta)$ *if and only if* $F_{c,p} f \in \Phi^{m+1}_p(\alpha_1, \alpha, \beta, \lambda, \ell, \eta)$.

Proof. Differentiating the definition relation (2.5), we get

$$z\left(F_{c,p} f(z)\right)' + (c + p) F_{c,p} f(z) = c f(z), \ z \in \mathrm{U}^*,$$

therefore

$$(2.12) \quad z\left(\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p} f(z)\right)' + (c+p) \mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) F_{c,p} f(z) = c \mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1) f(z), \ z \in \mathrm{U}^*.$$

Using (1.3), the relation (2.12) becomes

$$\frac{\ell}{\beta(\lambda-\alpha)}\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}f(z)+\left(c-\frac{\ell}{\beta(\lambda-\alpha)}\right)\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}f(z)=$$

$$c\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z),\ z\in \mathrm{U}^*,$$

and according to the assumption $c=\dfrac{\ell}{\beta(\lambda-\alpha)}$, it follows that

$$\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}f(z)=\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z),\ z\in \mathrm{U}^*,\ m\in\mathbb{N}_0. \tag{2.13}$$

From (2.13) we deduce that

$$\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}{\mathcal{F}^{m,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)f(z)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right)=$$

$$\frac{\ell}{\beta(\lambda-\alpha)}\frac{\mathcal{F}^{m+2,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}f(z)}{\mathcal{F}^{m+1,p,q,s}_{\ell,\alpha,\beta,\lambda}(\alpha_1)F_{c,p}f(z)}-\left(p+\frac{\ell}{\beta(\lambda-\alpha)}\right),\ z\in \mathrm{U},$$

and our conclusion follows immediately. □

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