Tamsui Oxford Journal of Informational and Mathematical Sciences 32(2) (2018)
Aletheia University 48-63

SANDWICH RESULTS FOR P–VALENT MEROMORPHIC FUNCTIONS ASSOCIATED WITH HURWITZ-LERECH ZETA FUNCTION

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Received January, 18, 2018, Accepted December, 3, 2018

2010 Mathematics Subject Classification. 30C80, 30C45.
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Abstract. Using the principle of subordination, in the present paper we obtain the sharp subordination and superordination-preserving properties of some convex combinations associated with a linear operator in the open unit disk. The *sandwich-type theorem* on the space of meromophic functions for these operators is also given, together with a few interesting special cases obtained for an appropriate choices of the parameters and the corresponding functions.

1. Introduction

Let denote by $H(\mathrm{U})$ the space of all analytical functions in the unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$, and for $a \in \mathbb{C}$, $n \in \mathbb{N}^*$, we denote

$$H[a, n] = \{f \in H(\mathrm{U}) : f(z) = a + a_n z^n + \dots\}.$$

Let denote the class of functions

$$A_n = \{f \in H(\mathrm{U}) : f(z) = z + a_{n+1} z^{n+1} + \dots\},$$

and let $A \equiv A_1$.

If $f, F \in H(\mathrm{U})$ and F is univalent in U we say that the function f is *subordinate* to F, or F is *superordinate* to f, written $f(z) \prec F(z)$, if $f(0) = F(0)$ and $f(\mathrm{U}) \subseteq F(\mathrm{U})$.

Letting $\varphi : \mathbb{C}^3 \times \overline{\mathrm{U}} \to \mathbb{C}$, $h \in H(\mathrm{U})$ and $q \in H[a, n]$, in [16] the authors determined conditions on φ such that

$$h(z) \prec \varphi(p(z), zp', z^2 p''(z); z) \quad \text{implies} \quad q(z) \prec p(z),$$

for all p functions that satisfy the above superordination. Moreover, they found sufficient conditions so that the q function is the *largest* function with this property, called the *best subordinant* of this superordination.

Using the principle of subordination, Miller et al. [17] investigated some subordination theorems involving certain integral operators for analytic functions in U (see also [2, 18]). Moreover, Miller and Mocanu [16] considered the differential superordinations as the dual concept of differential subordinations (see also [3]).

Let Σ_p be the class of functions of the form

$$f(z) = z^{-p} + \sum_{n=1-p}^{\infty} a_n z^n \quad (n, p \in \mathbb{N} = \{1, 2, \dots\}), \tag{1.1}$$

Key words and phrases. Meromorphic function, convex function, convolution product, differential subordination, differential superordination, integral operator.

which are analytic and p–valent in the punctured unit disc $\dot{\mathrm{U}} = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathrm{U} \setminus \{0\}$. We note that $\Sigma \equiv \Sigma_1$ the class of univalent meromorphic fuctions. For the functions $f \in \Sigma_p$ given by (1.1) and $g \in \Sigma_p$ given by

$$g(z) = z^{-p} + \sum_{n=1-p}^{\infty} b_n z^n \quad (n, p \in \mathbb{N}),$$

the Hadamard (or convolution) product of f and g is given by

$$(f * g)(z) = z^{-p} + \sum_{n=1-p}^{\infty} a_n b_n z^n = (g * f)(z).$$

The *general Hurwitz-Lerech Zeta function* $\Phi(z, s, b)$ is defined by (see [20])

$$\Phi(z, s, d) = \sum_{n=0}^{\infty} \frac{z^n}{(n+d)^s},$$

with $d \in \mathbb{C} \setminus \mathbb{Z}_0^- = \mathbb{C} \setminus \{0, -1, -2, \dots\}$, $s \in \mathbb{C}$ when $|z| < 1$ and $\operatorname{Re} s > 1$ when $|z| = 1$ (all the powers are principal ones).

Several interesting properties and characteristics of the above defined Hurwitz-Lerech Zeta function may be found in the investigations by several authors (see [4], [7], [10], [2]).

Now, defining the function $H_{p,d}^s$ $(d \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ s \in \mathbb{C})$ by

$$H_{p,d}^s(z) = \frac{d^s}{z^p} \Phi(z, s, d),\ z \in \dot{\mathrm{U}},$$

we could introduce the linear operator

$$\mathcal{L}_{p,d}^s : \Sigma_p \to \Sigma_p,$$

defined by

$$\mathcal{L}_{p,d}^s f(z) = H_{p,d}^s(z) * f(z) \quad (d \in \mathbb{C} \setminus \mathbb{Z}_0^-,\ s \in \mathbb{C}).$$

We note that

$$\mathcal{L}_{p,d}^s f(z) = \frac{1}{z^p} + \sum_{n=1-p}^{\infty} \left(\frac{d}{n+p+d} \right)^s a_n z^n,\ z \in \dot{\mathrm{U}}, \tag{1.2}$$

where all the powers are principal ones, and using this form of the operator $\mathcal{L}_{p,d}^s$ it is easy to verify that

$$z(\mathcal{L}_{p,d}^{s+1} f(z))' = d\mathcal{L}_{p,d}^s f(z) - (d+p)\mathcal{L}_{p,d}^{s+1} f(z),\ z \in \dot{\mathrm{U}}. \tag{1.3}$$

Also, we note that

$$(i) \qquad \mathcal{L}_{p,d}^0 f(z) = f(z);$$

$$(ii) \qquad \mathcal{L}_{p,1}^{-1} f(z) = \frac{1}{z^p} + \sum_{n=1-p}^{\infty} (n+p+1)\, a_n z^n = \frac{(z^{p+1} f(z))'}{z^p}.$$

Moreover, we could easily check that for all $f \in \Sigma_p$ we have

$$\mathcal{L}_{p,d}^k f(z) = \frac{d^k}{z^{d+p}} \int_0^z \frac{1}{t_1} \int_0^{t_1} \frac{1}{t_2} \int_0^{t_2} \cdots \frac{1}{t_{k-1}} \int_0^{t_{k-1}} t_k^{d+p-1} f(t_k)\, dt_k\, dt_{k-1} \dots\, dt_2\, dt_1,\ (k \in \mathbb{N})$$

and

$$\mathcal{L}_{p,d}^{s+1} f(z) = \frac{d}{z^{d+p}} \int_0^z t^{d+p-1} \mathcal{L}_d^s f(t)\, dt, \ (s \in \mathbb{C}).$$

We remark the following special cases of the operator $\mathcal{L}_{p,d}^s$:

(i) $\mathcal{L}_{p,\mu}^1 f(z) = F_\mu f(z) = \dfrac{\mu}{z^{\mu+p}} \displaystyle\int_0^z t^{\mu+p-1} f(t)\, dt, \ (\mu > 0)$ (see [14, p. 11 and p. 389]);

(ii) $\mathcal{L}_{p,1}^\alpha f(z) = P^\alpha f(z) = \dfrac{1}{z^p \Gamma(\alpha)} \displaystyle\int_0^z \left(\log \frac{z}{t}\right)^{\alpha-1} t^p f(t)\, dt, \ (\alpha > 0)$

(see Aqlan et al. [1]);

(iii) $\mathcal{L}_{p,\alpha}^\lambda f(z) = J_{p,\alpha}^\lambda f(z) = \dfrac{\alpha^\lambda}{z^{\alpha+p}\Gamma(\lambda)} \displaystyle\int_0^z \left(\log \frac{z}{t}\right)^{\lambda-1} t^{\alpha+p-1} f(t)\, dt, \ (\alpha, \ \lambda > 0)$

(see El-Ashwah and Aouf [6]);

(iv) $\mathcal{L}_{1,d}^s f(z) = \mathcal{L}_d^s f(z) = \dfrac{1}{z} + \displaystyle\sum_{n=0}^\infty \left(\frac{d}{n+1+d}\right)^s a_n z^n, \ (d \in \mathbb{C} \setminus \mathbb{Z}_0^-, \ s \in \mathbb{C})$

(see El-Ashwah [5]).

In the present paper we obtain some type of subordination and superordination preserving properties for the linear operators $\mathcal{L}_{p,d}^s$ defined by (1.2), and the corresponding *sandwich-type theorem.*

2. Preliminaries

To prove our main results, we will need the following definitions and lemmas presented in this section.

A function $L(z;t) : \mathrm{U} \times [0, +\infty) \to \mathbb{C}$ is called *a subordination (or a Loewner) chain* if $L(\cdot;t)$ is analytic and univalent in U for all $t \geq 0$, and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$.

The next well-known lemma gives a sufficient condition so that the $L(z;t)$ function will be a subordination chain.

Lemma 2.1. [12, p. 159] *Let $L(z;t) = a_1(t)z + a_2(t)z^2 + \ldots,$ with $a_1(t) \neq 0$ for all $t \geq 0$ and $\lim\limits_{t\to+\infty} |a_1(t)| = +\infty$. Suppose that $L(\cdot;t)$ is analytic in* U *for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0, +\infty)$ for all $z \in \mathrm{U}$. If $L(z;t)$ satisfies*

$$\mathrm{Re}\left[z \frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0, \ z \in \mathrm{U}, \ t \geq 0,$$

and

$$|L(z;t)| \leq K_0 \, |a_1(t)| \, , \ |z| < r_0 < 1, \ t \geq 0$$

for some positive constants K_0 and r_0, then $L(z;t)$ is a subordination chain.

We denote by $K(\alpha)$, $\alpha < 1$, the class of *convex functions of order* α in the unit disk U, i.e.

$$K(\alpha) = \left\{ f \in A : \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \alpha, \ z \in \mathrm{U} \right\}.$$

In particular, the class $K \equiv K(0)$ represents the class of *convex (and univalent) functions* in the unit disk.

Lemma 2.2. [13], [15, Theorem 2.3i, p. 35] *Suppose that the function* $H : \mathbb{C}^2 \to \mathbb{C}$ *satisfies the condition*

$$\operatorname{Re} H(is, t) \leq 0,$$

for all $s, t \in \mathbb{R}$ *with* $t \leq -n(1+s^2)/2$, *where* n *is a positive integer. If the function* $p(z) = 1 + p_n z^n + \dots$ *is analytic in* U *and*

$$\operatorname{Re} H(p(z), zp'(z)) > 0, \ z \in \mathrm{U},$$

then $\operatorname{Re} p(z) > 0$, $z \in \mathrm{U}$.

The next result deals with the solutions of the Briot–Bouquet differential equation (2.1), and more general forms of the following lemma may be found in [14, Theorem 1].

Lemma 2.3. [14] *Let* β, $\gamma \in \mathbb{C}$ *with* $\beta \neq 0$ *and let* $h \in H(\mathrm{U})$, *with* $h(0) = c$. *If* $\operatorname{Re}[\beta h(z) + \gamma] > 0$, $z \in \mathrm{U}$, *then the solution of the differential equation*

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \tag{2.1}$$

with $q(0) = c$, *is analytic in* U *and satisfies* $\operatorname{Re}[\beta q(z) + \gamma] > 0$, $z \in \mathrm{U}$.

As in [16], let denote by $\mathcal{Q}$ the set of functions f that are analytic and injective on $\overline{\mathrm{U}} \setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial \mathrm{U} : \lim_{z \to \zeta} f(z) = \infty \right\},$$

and such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \mathrm{U} \setminus E(f)$.

Lemma 2.4. [16, Theorem 7] *Let* $q \in H[a, 1]$, *let* $\chi : \mathbb{C}^2 \to \mathbb{C}$ *and set* $\chi(q(z), zq'(z)) \equiv h(z)$. *If* $L(z; t) = \chi(q(z), tzq'(z))$ *is a subordination chain and* $p \in H[a, 1] \cap \mathcal{Q}$, *then*

$$h(z) \prec \chi(p(z), zp'(z)) \quad \textit{implies} \quad q(z) \prec p(z).$$

Furthermore, if $\chi(q(z), zq'(z)) = h(z)$ *has a univalent solution* $q \in \mathcal{Q}$, *then* q *is the best subordinant.*

Like in [13] and [15], let $\Omega \subset \mathbb{C}$, $q \in \mathcal{Q}$ and n be a positive integer. Then, the *class of admissible functions* $\Psi_n[\Omega, q]$ is the class of those functions $\psi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$ that satisfy the *admissibility* condition

$$\psi(r, s, t; z) \notin \Omega,$$

whenever $r = q(\zeta)$, $s = m\zeta q'(\zeta)$, $\operatorname{Re} \dfrac{t}{s} + 1 \geq m \operatorname{Re} \left[\dfrac{\zeta q''(\zeta)}{q'(\zeta)} + 1 \right]$, $z \in \mathrm{U}$, $\zeta \in \partial \mathrm{U} \setminus E(q)$ and $m \geq n$. This class will be denoted by $\Psi_n[\Omega, q]$.

We write $\Psi[\Omega, q] \equiv \Psi_1[\Omega, q]$. For the special case when $\Omega \neq \mathbb{C}$ is a simply connected domain and h is a conformal mapping of U onto Ω, we use the notation $\Psi_n[h, q] \equiv \Psi_n[\Omega, q]$.

Remark 2.1. If $\psi : \mathbb{C}^2 \times \mathrm{U} \to \mathbb{C}$, then the above defined admissibility condition reduces to

$$\psi(q(\zeta), m\zeta q'(\zeta); z) \notin \Omega,$$

when $z \in \mathrm{U}$, $\zeta \in \partial \mathrm{U} \setminus E(q)$ and $m \geq n$.

Lemma 2.5. [13], [15] *Let h be univalent in* U *and* $\psi : \mathbb{C}^3 \times \mathrm{U} \to \mathbb{C}$. *Suppose that the differential equation*

$$\psi(q(z), zq'(z), z^2q''(z); z) = h(z)$$

has a solution q, with $q(0) = a$, and one of the following conditions is satisfied:

(i) $q \in \mathcal{Q}$ *and* $\psi \in \Psi[h, q]$

(ii) q *is univalent in* U *and* $\psi \in \Psi[h, q_\rho]$, *for some* $\rho \in (0, 1)$, *where* $q_\rho(z) = q(\rho z)$, *or*

(iii) q *is univalent in* U *and there exists* $\rho_0 \in (0, 1)$ *such that* $\psi \in \Psi[h_\rho, q_\rho]$ *for all* $\rho \in (\rho_0, 1)$, *where* $h_\rho(z) = h(\rho z)$ *and* $q_\rho(z) = q(\rho z)$.

If $p(z) = a + a_1 z + \ldots \in H(\mathrm{U})$ *and* $\psi(p(z), zp'(z), z^2p''(z); z) \in H(\mathrm{U})$, *then*

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z) \quad \textit{implies} \quad p(z) \prec q(z)$$

and q is the best dominant.

3. Main results

Unless otherwise mentioned, we assume throughout this paper that $d = d_1 + id_2 \in \mathbb{C} \setminus \mathbb{Z}_0^-$, with $d_1,\ d_2 \in \mathbb{R}$, $s \in \mathbb{C}$ and $p \in \mathbb{N}$.

We begin by proving the following subordination theorem:

Theorem 3.1. *Let $\alpha < 1$ and $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$, with* $\operatorname{Re} d > 1 - \alpha$. *For a given function $g \in \Sigma_p$, suppose that*

$$\operatorname{Re}\left[1 + \frac{z\phi''(z)}{\phi'(z)}\right] > -\delta,\ z \in \mathrm{U}, \tag{3.1}$$

where

$$\phi(z) = z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} g(z) + \alpha\mathcal{L}_{p,d}^{s+1} g(z)\right], \tag{3.2}$$

and

(3.3)

$$\delta = \frac{(1-\alpha)^2 + |\alpha - 1 + d|^2 - \sqrt{\left[(1-\alpha)^2 + |\alpha - 1 + d|^2\right]^2 - 4(1-\alpha)^2(\alpha - 1 + \operatorname{Re} d)^2}}{4(1-\alpha)(\alpha - 1 + \operatorname{Re} d)}.$$

If $f \in \Sigma_p$ such that

$$z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} f(z) + \alpha\mathcal{L}_{p,d}^{s+1} f(z)\right] \prec z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} g(z) + \alpha\mathcal{L}_{p,d}^{s+1} g(z)\right],$$

then

$$z^{p+1}\mathcal{L}_{p,d}^{s+1} f(z) \prec z^{p+1}\mathcal{L}_{p,d}^{s+1} g(z),$$

and the function $z^{p+1}\mathcal{L}_{p,d}^{s+1} g(z)$ is the best dominant.

Proof. If we denote

$$\varphi(z) = z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} f(z) + \alpha\mathcal{L}_{p,d}^{s+1} f(z)\right]$$

and

$$F(z) = z^{p+1}\mathcal{L}_{p,d}^{s+1} f(z), \quad G(z) = z^{p+1}\mathcal{L}_{p,d}^{s+1} g(z), \tag{3.4}$$

then we need to prove that $\varphi(z) \prec \phi(z)$ implies $F(z) \prec G(z)$.

Differentiating the second part of the relation (3.4), by using the identity (1.3) we have

$$z^{p+1}\mathcal{L}_{p,d}^{s}g(z) = \frac{1}{d}\left[zG'(z) + (d-1)G(z)\right],$$

and replacing the left-hand side of the above relation in (3.2) we get

$$d\phi(z) = (\alpha - 1 + d)G(z) + (1-\alpha)zG'(z). \tag{3.5}$$

If we let $q(z) = 1 + \dfrac{zG''(z)}{G'(z)}$, by differentiating (3.5) we have

$$dz\phi'(z) = (1-\alpha)zG'(z)\left[q(z) + \frac{\alpha - 1 + d}{1-\alpha}\right],$$

and by computing the logarithmical derivative of the above equality we deduce

$$q(z) + \frac{zq'(z)}{q(z) + \frac{\alpha-1+d}{1-\alpha}} = 1 + \frac{z\phi''(z)}{\phi'(z)} \equiv h(z). \tag{3.6}$$

From (3.1), using the assumptions $\alpha < 1$ and $d_1 = \operatorname{Re} d > 1 - \alpha$, we have

$$\operatorname{Re}\left[h(z) + \frac{\alpha - 1 + d}{1-\alpha}\right] > -\delta + \frac{\alpha - 1 + \operatorname{Re} d}{1-\alpha} \geq 0,\ z \in \mathrm{U},$$

and by using Lemma 2.3 we conclude that the differential equation (3.6) has a solution $q \in H(\mathrm{U})$, with $q(0) = h(0) = 1$.

Now we will use Lemma 2.2 to prove that, under our assumption, the inequality

$$\operatorname{Re} q(z) > 0,\ z \in \mathrm{U}, \tag{3.7}$$

holds. Let us put

$$H(u,v) = u + \cfrac{v}{u + \cfrac{\alpha - 1 + d}{1-\alpha}} + \delta, \tag{3.8}$$

where δ is given by (3.3). From the assumption (3.1), according to (3.6), we obtain

$$\operatorname{Re} H(q(z), zq(z)) > 0,\ z \in \mathrm{U}, \tag{3.9}$$

and we proceed to show that $\operatorname{Re} H(is,t) \leq 0$ for all $s, t \in \mathbb{R}$, with $t \leq -(1+s^2)/2$.

From (3.8), using the assumptions $\alpha < 1$ and $b_1 = \operatorname{Re} b > -\alpha$, we have

$$\operatorname{Re} H(is,t) = \operatorname{Re}\left(is + \frac{t}{is + \frac{\alpha-1+d}{1-\alpha}} + \delta\right) = \frac{\frac{\alpha-1+d_1}{1-\alpha}t}{\left|is + \frac{\alpha-1+d}{1-\alpha}\right|^2} + \delta \leq \frac{E(s)}{-2\left|is + \frac{\alpha-1+d}{1-\alpha}\right|^2},$$

where

$$E(s) = \left(\frac{\alpha - 1 + d_1}{1-\alpha} - 2\delta\right)s^2 - \frac{4d_2\delta}{1-\alpha}s - 2\delta\frac{|\alpha - 1 + d|^2}{(1-\alpha)^2} + \frac{\alpha - 1 + d_1}{1-\alpha},$$

and $d_2 = \operatorname{Im} d$. It is well-known that the second order polinomial function $E(s)$ is non-negative for all $s \in \mathbb{R}$, if and only if

$$\Delta \leq 0 \quad \text{and} \quad \frac{\alpha - 1 + d_1}{1-\alpha} - 2\delta > 0, \tag{3.10}$$

where Δ is the discriminant of $E(s)$, i.e.

$$\Delta = -\frac{4(\alpha-1+d_1)}{(1-\alpha)^2}\left\{4(\alpha-1+d_1)\,\delta^2 - \frac{2\left[(1-\alpha)^2+|\alpha-1+d|^2\right]}{1-\alpha}\,\delta+\alpha+d_1\right\}.$$

We may easily check that the value of δ given by (3.3) is the greater one for which $\Delta \le 0$. Since this value of δ satisfies the second part of the conditions (3.10), it follows that $\operatorname{Re} H(is,t) \le 0$ for all $s,t \in \mathbb{R}$, with $t \le -(1+s^2)/2$.

Form (3.9), according to Lemma 2.2, we deduce that the inequality (3.7) holds, hence $G \in K$, that is G is a convex (and univalent) function in the unit disk, hence the following well-known growth and distortion sharp inequalities (see [8]) are true:

$$\frac{r}{1+r} \le |G(z)| \le \frac{r}{1-r}, \text{ if } |z| \le r,$$
$$\frac{1}{(1+r)^2} \le |G'(z)| \le \frac{1}{(1-r)^2}, \text{ if } |z| \le r.$$

If we let

$$L(z;t) = \frac{\alpha-1+d}{d}\,G(z) + \frac{(1-\alpha)(1+t)}{d}\,zG'(z), \tag{3.11}$$

from (3.5) we have $L(z;0) = \phi(z)$. Denoting $L(z;t) = a_1(t)z + \dots$, then

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = \frac{\alpha-1+d+(1-\alpha)(1+t)}{d}\,G'(0) = \frac{\alpha-1+d+(1-\alpha)(1+t)}{d},$$

hence $\lim\limits_{t\to+\infty} |a_1(t)| = +\infty$, and because $\alpha < 1$ and $\operatorname{Re} d > 1-\alpha$ we obtain $a_1(t) \ne 0$, $\forall t \ge 0$.

From (3.11) we may easily deduce the equality

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = \operatorname{Re}\left[\frac{\alpha-1+d}{1-\alpha} + (1+t)\left(1+\frac{zG''(z)}{G'(z)}\right)\right] = \frac{\alpha-1+\operatorname{Re} d}{1-\alpha} + (1+t)\operatorname{Re} q(z).$$

Using the inequality (3.7) together with the assumptions $\alpha < 1$ and $\operatorname{Re} d > 1-\alpha$, the above relation yields that

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0,\ \forall z \in \mathrm{U},\ \forall t \ge 0.$$

From the definition (3.11), for all $t \ge 0$ we have

$$\frac{|L(z;t)|}{|a_1(t)|} \le \frac{|\alpha-1+d|\,|G(z)| + |1-\alpha||1+t|\,|zG'(z)|}{|d+(1-\alpha)t|}. \tag{3.12}$$

Using the right-hand sides of these inequalities in (3.12), we deduce that

$$\frac{|L(z;t)|}{|a_1(t)|} \le \frac{|\alpha-1+d|}{|1-\alpha|}\frac{r}{1-r}\,\varphi_1(t) + \frac{r}{(1-r)^2}\,\varphi_2(t),\ |z| \le r,\ \forall t \ge 0, \tag{3.13}$$

where

$$\varphi_1(t) = \frac{1}{\left|t+\frac{d}{1-\alpha}\right|} \quad \text{and} \quad \varphi_2(t) = \frac{|t+1|}{\left|t+\frac{d}{1-\alpha}\right|}.$$

Since $\operatorname{Re}\frac{d}{1-\alpha} > 0$ whenever $\operatorname{Re} d > 1-\alpha$ and $\alpha < 1$, it follows

$$\left|t + \frac{d}{1-\alpha}\right| \geq \left|\frac{d}{1-\alpha}\right|, \ \forall t \geq 0,$$

hence

$$\varphi_1(t) \leq \left|\frac{1-\alpha}{d}\right|, \ t \geq 0. \tag{3.14}$$

Moreover, since $\operatorname{Re}\frac{d}{1-\alpha} > 1$ whenever $\operatorname{Re} d > 1-\alpha$ and $\alpha < 1$, we obtain

$$\frac{|t+1|}{\left|t + \frac{d}{1-\alpha}\right|} < 1, \ \forall t \geq 0,$$

hence

$$\varphi_2(t) < 1, \ t \geq 0. \tag{3.15}$$

Using the inequalities (3.14) and (3.15), from (3.13) we deduce that

$$\frac{|L(z;t)|}{|a_1(t)|} < \frac{r}{(1-r)^2} + \left|\frac{\alpha - 1 + d}{d}\right| \frac{r}{1-r}, \ |z| \leq r, \ \forall t \geq 0,$$

hence the second assumption of Lemma 2.1 holds, and according to this lemma we conclude that the function $L(z;t)$ is a subordination chain.

Now, by using Lemma 2.5, we will show that $F(z) \prec G(z)$. Without loss of generality, we can assume that ϕ and G are analytic and univalent in $\overline{\mathrm{U}}$ and $G'(\zeta) \neq 0$ for $|\zeta| = 1$. If not, then we could replace ϕ with $\phi_\rho(z) = \phi(\rho z)$ and G with $G_\rho(z) = G(\rho z)$, where $\rho \in (0,1)$. These new functions will have the desired properties and we would prove our result using part (iii) of Lemma 2.5.

With our above assumption, we will use part (i) of the Lemma 2.5. If we denote by $\psi(G(z), zG'(z)) = \phi(z)$, we only need to show that $\psi \in \Psi[\phi, G]$, i.e. ψ is an admissible function. Because

$$\psi(G(\zeta), m\zeta G'(\zeta)) = \frac{\alpha - 1 + d}{d} G(z) + \frac{(1-\alpha)(1+t)}{d} zG'(z) = L(\zeta;t),$$

where $m = 1+t$, $t \geq 0$, since $L(z;t)$ is a subordination chain and $\phi(z) = L(z;0)$, it follows that

$$\psi(G(\zeta), m\zeta G'(\zeta)) \notin \phi(\mathrm{U}).$$

According to the Remark 2.1 we have $\psi \in \Psi[\phi, G]$, and using Lemma 2.5 we obtain that $F(z) \prec G(z)$ and, moreover, G is the best dominant. □

Remark 3.1. It is easy to check that the values of δ given by (3.3) satisfies the inequality $0 < \delta \leq \dfrac{1}{2}$, whenever $\alpha < 1$ and $\operatorname{Re} d > 1-\alpha$.

For the special case $d = 1$, $s = -1$ and $p = 1$, taking $\beta := 1-\alpha$, Theorem 3.1 reduces to:

Corollary 3.1. *Let $0 < \beta < 1$ and for a given function $g \in \Sigma$ suppose that the inequality* (3.1) *holds, where*

$$\phi(z) = z^2 \left[\beta z g'(z) + (1+\beta)\, g(z)\right], \tag{3.16}$$

and

$$\delta = \delta(\beta; 1) = \frac{\beta^2 + (1-\beta)^2 - |\beta^2 - (1-\beta)^2|}{4\beta(1-\beta)} = \begin{cases} \dfrac{\beta}{2(1-\beta)}, & \text{if } \ 0 < \beta \le 1/2, \\ \dfrac{1-\beta}{2\beta}, & \text{if } \ 1/2 \le \beta < 1. \end{cases} \tag{3.17}$$

If $f \in \Sigma$ such that

$$z^2 \left[\beta z f'(z) + (1+\beta) f(z)\right] \prec z^2 \left[\beta z g'(z) + (1+\beta) g(z)\right],$$

then

$$z^2 f(z) \prec z^2 g(z),$$

and the function g is the best dominant.

Now we will prove a dual of Theorem 3.1, in the sense that the subordinations are replaced by superordinations.

Theorem 3.2. *Let $\alpha < 1$ and $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$, with $\operatorname{Re} d > 1 - \alpha$. For a given function $g \in \Sigma_p$, suppose that the function ϕ defined by* (3.2) *satisfies the condition* (3.1)*, with δ given by* (3.3)*.*

Let $f \in \Sigma_p$ such that $z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} f(z) + \alpha \mathcal{L}_{p,d}^{s+1} f(z)\right]$ is univalent in U *and $z^{p+1}\mathcal{L}_{p,d}^{s+1} f(z) \in \mathcal{Q}$. Then,*

$$z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} g(z) + \alpha \mathcal{L}_{p,d}^{s+1} g(z)\right] \prec z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} f(z) + \alpha \mathcal{L}_{p,d}^{s+1} f(z)\right],$$

implies

$$z^{p+1}\mathcal{L}_{p,d}^{s+1} g(z) \prec z^{p+1}\mathcal{L}_{p,d}^{s+1} f(z),$$

and the function $z^{p+1}\mathcal{L}_{p,d}^{s+1} g(z)$ is the best subordinant.

Proof. Denoting

$$\varphi(z) = z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} f(z) + \alpha \mathcal{L}_{p,d}^{s+1} f(z)\right]$$

and

$$F(z) = z^{p+1}\mathcal{L}_{p,d}^{s+1} f(z), \quad G(z) = z^{p+1}\mathcal{L}_{p,d}^{s+1} g(z), \tag{3.18}$$

then we need to prove that $\phi(z) \prec \varphi(z)$ implies $G(z) \prec F(z)$.

If we differentiate the second part of the relation (3.18), using the identity (1.3) we obtain

$$z^{p+1}\mathcal{L}_{p,d}^{s} g(z) = \frac{1}{d}\left[z G'(z) + (d-1) G(z)\right].$$

Replacing the left-hand side of the above relation in (3.2) we have

$$\phi(z) = \frac{\alpha - 1 + d}{d}\, G(z) + \frac{1-\alpha}{d}\, z G'(z). \tag{3.19}$$

If we let $q(z) = 1 + \dfrac{zG''(z)}{G'(z)}$, like in the proof of Theorem 3.1 it follows that the inequality (3.7) holds, i.e. $\operatorname{Re} q(z) > 0$ for all $z \in$ U.

Letting

$$L(z;t) = \frac{\alpha - 1 + d}{d}\, G(z) + \frac{(1-\alpha)t}{d}\, zG'(z), \tag{3.20}$$

from (3.19) we have $L(z;1) = \phi(z)$. Thus, $L(z;t) = a_1(t)z + \dots$, and then

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = \frac{\alpha - 1 + d + (1-\alpha)t}{d}\, G'(0) = \frac{\alpha - 1 + d + (1-\alpha)t}{d},$$

hence $\lim\limits_{t\to+\infty} |a_1(t)| = +\infty$, and because $\alpha < 1$ and $\operatorname{Re} d > 1 - \alpha$ we obtain $a_1(t) \neq 0$, $\forall t \geq 0$.

From (3.20), a simple computation shows that

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = \operatorname{Re}\left[\frac{\alpha - 1 + d}{1-\alpha} + t\left(1 + \frac{zG''(z)}{G'(z)}\right)\right] = \frac{\alpha - 1 + \operatorname{Re} d}{1-\alpha} + t \operatorname{Re} q(z).$$

Since we already mentioned that the inequality (3.7) holds, combining with the assumptions $\alpha < 1$ and $\operatorname{Re} b > -\alpha$, the above relation implies that

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0,\ \forall z \in \mathrm{U},\ \forall t \geq 0.$$

Also, for all $t \geq 0$ we have

$$\frac{|L(z;t)|}{|a_1(t)|} \leq \frac{|\alpha - 1 + d|\,|G(z)| + |1-\alpha||t|\,|zG'(z)|}{|\alpha - 1 + d + (1-\alpha)t|}. \tag{3.21}$$

and from the right-hand sides of these inequalities in (3.12), we obtain that

$$\frac{|L(z;t)|}{|a_1(t)|} \leq \frac{|\alpha - 1 + d|}{|1-\alpha|}\frac{r}{1-r}\,\varphi_1(t) + \frac{r}{(1-r)^2}\,\varphi_2(t),\ |z| \leq r,\ \forall t \geq 0, \tag{3.22}$$

where

$$\varphi_1(t) = \frac{1}{\left|t + \frac{\alpha-1+d}{1-\alpha}\right|} \quad \text{and} \quad \varphi_2(t) = \frac{|t|}{\left|t + \frac{\alpha-1+d}{1-\alpha}\right|}.$$

Since $\operatorname{Re}\frac{d}{1-\alpha} > 0$ for $\operatorname{Re} d > 1 - \alpha$ and $\alpha < 1$, it follows

$$\left|t + \frac{\alpha - 1 + d}{1-\alpha}\right| \geq \left|\frac{\alpha - 1 + d}{1-\alpha}\right| \quad \text{and} \quad |t| < \left|t + \frac{\alpha - 1 + d}{1-\alpha}\right|,\ \forall t \geq 0,$$

and thus

$$\varphi_1(t) \leq \left|\frac{1-\alpha}{\alpha - 1 + d}\right|, \quad \varphi_2(t) < 1,\ t \geq 0.$$

Using the above inequalities together with (3.21) we deduce that

$$\frac{|L(z;t)|}{|a_1(t)|} < \frac{r}{1-r} + \frac{r}{(1-r)^2},\ |z| \leq r,\ \forall t \geq 0,$$

hence the second assumption of Lemma 2.1 holds. Now, from this lemma we obtain that the function $L(z;t)$ is a subordination chain.

Using the fact that (3.7) holds, since $G \in A$, we have that G is convex (univalent) in U. Thus, if we denote by $\chi(G(z), zG'(z)) = \phi(z)$, then $L(z;t) = \chi(q(z), tzq'(z))$, and the differential equation $\chi(G(z), zG'(z)) = \phi(z)$ has the univalent solution G.

According to Lemma 2.4, we conclude that $\phi(z) \prec \varphi(z)$ implies $G(z) \prec F(z)$, and furthermore, since G is a univalent solution of the differential equation $\chi(G(z), zG'(z)) = \phi(z)$, it follows that it is the best subordinant of the given differential superordination. $\square$

Taking $d = 1$, $s = -1$ and $p = 1$ in Theorem 3.2, denoting $\beta := 1 - \alpha$, we obtain the next special case:

Corollary 3.2. *Let $0 < \beta < 1$ and for a given function $g \in \Sigma$ suppose that the function ϕ defined by* (3.16) *satisfies the condition* (3.1)*, with δ given by* (3.17)*.*

Let $f \in \Sigma$ such that $z^2 [\beta z f'(z) + (1+\beta) f(z)]$ is univalent in U *and $z^2 f(z) \in \mathcal{Q}$. Then,*

$$z^2 [\beta z g'(z) + (1+\beta) g(z)] \prec z^2 [\beta z f'(z) + (1+\beta) f(z)]$$

implies

$$z^2 g(z) \prec z^2 f(z),$$

and the function g is the best subordinant.

Combining the Theorem 3.2 with Theorem 3.1, we obtain the following *sandwich-type theorem*:

Theorem 3.3. *Let $\alpha < 1$ and $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$, with $\operatorname{Re} d > 1 - \alpha$. For the two given functions g_1, $g_2 \in \Sigma_p$, suppose that*

$$\operatorname{Re}\left[1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right] > -\delta, \ z \in \mathrm{U}, \quad (k = 1, 2),$$

where

$$\phi_k(z) = z^{p+1} \left[(1-\alpha)\mathcal{L}_{p,d}^{s} g_k(z) + \alpha \mathcal{L}_{p,d}^{s+1} g_k(z)\right], \quad (k = 1, 2), \tag{3.23}$$

and δ is given by (3.3)*.*

Let $f \in \Sigma_p$ such that $z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} f(z) + \alpha \mathcal{L}_{p,d}^{s+1} f(z)\right]$ is univalent in U *and $z^{p+1}\mathcal{L}_{p,d}^{s+1} f(z) \in \mathcal{Q}$. Then,*

$$z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} g_1(z) + \alpha \mathcal{L}_{p,d}^{s+1} g_1(z)\right] \prec z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} f(z) + \alpha \mathcal{L}_{p,d}^{s+1} f(z)\right] \prec$$
$$z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} g_2(z) + \alpha \mathcal{L}_{p,d}^{s+1} g_2(z)\right]$$

implies

$$z^{p+1}\mathcal{L}_{p,d}^{s+1} g_1(z) \prec z^{p+1}\mathcal{L}_{p,d}^{s+1} f(z) \prec z^{p+1}\mathcal{L}_{p,d}^{s+1} g_2(z).$$

Moreover, the functions $z^{p+1}\mathcal{L}_{p,d}^{s+1} g_1(z)$ and $z^{p+1}\mathcal{L}_{p,d}^{s+1} g_2(z)$ are respectively the best subordinant and the best dominant.

The assumptions that the functions

$$\phi_3(z) = z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} f(z) + \alpha \mathcal{L}_{p,d}^{s+1} f(z)\right] \tag{3.24}$$

and

$$\Phi(z) = z^{p+1}\mathcal{L}_{p,d}^{s+1} f(z) \tag{3.25}$$

need to be univalent in U are difficult to be checked. Thus, in the following *sandwich-type result* we will replace these assumptions by another sufficient conditions, that are more easy to be verified.

Corollary 3.3. *Let $\alpha < 1$ and $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$, with $\operatorname{Re} d > 1 - \alpha$. For the given functions f, g_1, $g_2 \in \Sigma_p$, suppose that*

$$\operatorname{Re}\left[1 + \frac{z\phi_k''(z)}{\phi_k'(z)}\right] > -\delta,\ z \in \mathrm{U}, \quad (k = 1, 2, 3), \tag{3.26}$$

where ϕ_1, ϕ_2 and ϕ_3 are defined by (3.23) *and* (3.24) *respectively, and δ is given by* (3.3). *Then,*

$$z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s}\, g_1(z) + \alpha\mathcal{L}_{p,d}^{s+1} g_1(z)\right] \prec z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s} f(z) + \alpha\mathcal{L}_{p,d}^{s+1} f(z)\right] \prec$$
$$z^{p+1}\left[(1-\alpha)\mathcal{L}_{p,d}^{s}\, g_2(z) + \alpha\mathcal{L}_{p,d}^{s+1} g_2(z)\right]$$

implies

$$z^{p+1}\mathcal{L}_{p,d}^{s+1} g_1(z) \prec z^{p+1}\mathcal{L}_{p,d}^{s+1} f(z) \prec z^{p+1}\mathcal{L}_{p,d}^{s+1} g_2(z).$$

Moreover, the functions $z^{p+1}\mathcal{L}_{p,d}^{s+1} g_1(z)$ and $z^{p+1}\mathcal{L}_{p,d}^{s+1} g_2(z)$ are respectively the best subordinant and the best dominant.

Proof. In order to prove our corollary, we have to show that the condition (3.26) for $k = 3$ implies the univalence of the functions ϕ_3 and Φ defined by (3.24) and (3.25).

Since $0 < \delta \leq \frac{1}{2}$ from Remark 3.1, the condition (3.26) for $k = 3$ means that $\phi_3 \in K(-\delta) \subseteq K\left(-\frac{1}{2}\right)$, and from [9] it follows that ϕ_3 is a close-to-convex function in U, hence it is univalent in U. Furthermore, by using the same techniques as in the proof of Theorem 3.1 we can prove the convexity (univalence) of Φ and so the details may be omitted. Therefore, by applying Theorem 3.3 we obtain the desired result. □

The following special case of Corollary 3.3 is obtained for $d = 1$, $s = -1$ and $p = 1$, with $\beta := 1 - \alpha$:

Corollary 3.4. *Let $0 < \beta < 1$ and for the given functions f, g_1, $g_2 \in \Sigma$, suppose that the inequalities* (3.26) *hold, where*

$$\phi_1(z) = z^2\left[\beta z g_1'(z) + (1+\beta) g_1(z)\right], \quad \phi_2(z) = z^2\left[\beta z g_2'(z) + (1+\beta) g_2(z)\right],$$
$$\phi_3(z) = z^2\left[\beta z f'(z) + (1+\beta) f(z)\right],$$

and δ is given by (3.17). *Then,*

$$z^2\left[\beta z g_1'(z) + (1+\beta) g_1(z)\right] \prec z^2\left[\beta z f'(z) + (1+\beta) f(z)\right] \prec z^2\left[\beta z g_2'(z) + (1+\beta) g_2(z)\right]$$

implies

$$z^2 g_1(z) \prec z^2 f(z) \prec z^2 g_2(z).$$

Moreover, the functions g_1 and g_2 are respectively the best subordinant and the best dominant.

Next, we will give an interesting special case of our main results, obtained for an appropriate choice of the function g and the corresponding parameters.

Thus, for $\alpha < 1$ and $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$, with $\operatorname{Re} d > 1 - \alpha$, let consider the function $g \in \Sigma$ defined by

$$g(z) = z^{-1} + \sum_{n=0}^{\infty} a_n z^n,\ z \in \dot{\mathrm{U}},$$

with

$$a_n = \frac{1}{n+2}\frac{n+1+d}{(1-\alpha)(1+n)+d}\left(\frac{n+1+d}{d}\right)^s\binom{-2(\delta+1)}{n+1},\ n \geq 0,$$

where δ is given by (3.3), and

$$\binom{\tau}{n} = \frac{\tau(\tau-1)\dots(\tau-n+1)}{n!},\ \tau \in \mathbb{C},\ n \in \mathbb{N}.$$

If the function ϕ is defined by (3.2) with $p = 1$, then

$$\phi(z) = \frac{1-(1+z)^{-(2\delta+1)}}{2\delta+1},\ z \in \mathrm{U},$$

where the power is the principal one, i.e.

$$(1+z)^{-(2\delta+1)}\Big|_{z=0} = 1.$$

A simple computation shows that

$$\mathrm{Re}\left[1+\frac{z\phi''(z)}{\phi'(z)}\right] = \mathrm{Re}\,\frac{1-(2\delta+1)z}{1+z} > -\delta,\ z \in \mathrm{U},$$

and from Theorem 3.1 and Theorem 3.2 we obtain:

Example 3.1. Let $\alpha < 1$ and $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$, with $\mathrm{Re}\, d > 1-\alpha$, and let δ be given by (3.3).

1. If $f \in \Sigma$ such that

$$z^2\left[(1-\alpha)\mathcal{L}_{1,d}^s f(z) + \alpha\mathcal{L}_{1,d}^{s+1} f(z)\right] \prec \frac{1-(1+z)^{-(2\delta+1)}}{2\delta+1},$$

then

$$z^2\mathcal{L}_{1,d}^{s+1} f(z) \prec z + \sum_{n=0}^{\infty}\frac{1}{n+2}\frac{d}{(1-\alpha)(1+n)+d}\binom{-2(\delta+1)}{n+1}z^{n+2},$$

and the right-hand side function is the best dominant (the power is the principal one).

2. If $f \in \Sigma$ such that $z^2\left[(1-\alpha)\mathcal{L}_{1,d}^s f(z) + \alpha\mathcal{L}_{1,d}^{s+1} f(z)\right]$ is univalent in U and $z^2\mathcal{L}_{1,d}^{s+1} f(z) \in \mathcal{Q}$, then

$$\frac{1-(1+z)^{-(2\delta+1)}}{2\delta+1} \prec z^2\left[(1-\alpha)\mathcal{L}_{1,d}^s f(z) + \alpha\mathcal{L}_{1,d}^{s+1} f(z)\right]$$

implies

$$z + \sum_{n=0}^{\infty}\frac{1}{n+2}\frac{d}{(1-\alpha)(1+n)+b}\binom{-2(\delta+1)}{n+1}z^{n+2} \prec z^2\mathcal{L}_{1,d}^{s+1} f(z),$$

and the right-hand side function is the best subordinant (the power is the principal one).

By similar reasons, for the above mentioned choice of the function g, the Theorem 3.3 reduces to the following *sandwich-type results*:

Example 3.2. Let $\alpha < 1$ and $d \in \mathbb{C} \setminus \mathbb{Z}_0^-$, with $\operatorname{Re} d > 1 - \alpha$, and let δ_1, $\delta_2 \leq \delta$ where δ is given by (3.3).

If $f \in \Sigma$ such that $z^2 \left[(1-\alpha)\mathcal{L}_{1,d}^s f(z) + \alpha \mathcal{L}_{1,d}^{s+1} f(z)\right]$ is univalent in U and $z^2 \mathcal{L}_{1,d}^{s+1} f(z) \in \mathcal{Q}$, then

$$\frac{1-(1+z)^{-(2\delta_1+1)}}{2\delta_1+1} \prec z^2 \left[(1-\alpha)\mathcal{L}_{1,d}^s f(z) + \alpha \mathcal{L}_{1,d}^{s+1} f(z)\right] \prec \frac{1-(1+z)^{-(2\delta_2+1)}}{2\delta_2+1},$$

implies

$$z + \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{d}{(1-\alpha)(1+n)+d} \binom{-2(\delta_1+1)}{n+1} z^{n+2} \prec z^2 \mathcal{L}_{1,d}^{s+1} f(z) \prec$$
$$z + \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{d}{(1-\alpha)(1+n)+d} \binom{-2(\delta_2+1)}{n+1} z^{n+2}.$$

Moreover, the left-hand side functions and the right-hand side are, respectively, the best subordinant and the best dominant (the powers are the principal ones).

For $d = 1$, $s = -1$ and $p = 1$, for $\beta := 1 - \alpha$ the Example 3.2 gives us the next result:

Example 3.3. Let $0 < \beta < 1$ and let δ_1, $\delta_2 \leq \delta$ where δ is given by (3.17).

If $f \in \Sigma$ such that $z^2 \left[\beta z f'(z) + (1+\beta) f(z)\right]$ is univalent in U and $z^2 f(z) \in \mathcal{Q}$, then

$$\frac{1-(1+z)^{-(2\delta_1+1)}}{2\delta_1+1} \prec z^2 \left[\beta z f'(z) + (1+\beta) f(z)\right] \prec \frac{1-(1+z)^{-(2\delta_2+1)}}{2\delta_2+1},$$

implies

$$z + \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{1}{\beta(1+n)+1} \binom{-2(\delta_1+1)}{n+1} z^{n+2} \prec z^2 f(z) \prec$$
$$z + \sum_{n=0}^{\infty} \frac{1}{n+2} \frac{1}{\beta(1+n)+1} \binom{-2(\delta_2+1)}{n+1} z^{n+2}.$$

Moreover, the left-hand side functions and the right-hand side are, respectively, the best subordinant and the best dominant (the powers are the principal ones).

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