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On ω_θ-Continuity in the Product Space and Some Versions of Separation Axioms

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Abstract

In this paper, the concept of ω_θ-continuous function from an arbitrary topological space into the product space will be characterized. Moreover, some versions of separation axioms with respect to ω_θ-open set will be introduced and characterized.

Keywords: ω_θ-continuous, ω_θ-Hausdorff, ω_θ-regular, ω_θ-normal

1 Introduction

The first attempt to replace various concepts in topology with concepts possessing either of weaker or stronger properties was done by N. Levine [4] in 1963. In his work, Levine introduced the concept of semi-open set and used this to define other new concepts such as semi-closed set and semi-continuity of a function.

After this notable work of Levine on the concept of semi-open set, several mathematicians became interested in introducing other topological concepts which can replace the concept of open set. In 1968, N. Velicko [5] introduced the concept of super-continuity (or θ-continuity) between topological spaces. He also defined the concepts such as super-closure (or θ-closure) and super-interior (or θ-interior) of a subset of a topological space. In 2005, T.A. Al-Hawary [1] characterized super-continuity and gave relationships between super-continuity and the other well-known variations of continuity such as strong continuity, semi-continuity, and closure continuity.

Let $(X, \mathfrak{T})$ be a topological space and $A \subseteq X$. The *super-closure* and *super-interior* of A are, respectively, denoted and defined by

$$Cl_s(A) = \{x \in X : Cl(U) \cap A \neq \varnothing \text{ for every open set } U \text{ containing } x\}$$

and

$$Int_s(A) = \{x \in X : Cl(U) \subseteq A \text{ for some open set } U \text{ containing } x\},$$

where $Cl(U)$ is the closure of U in X. A subset A of X is *super-closed* if $Cl_s(A) = A$ and *super-open* if $Int_s(A) = A$. Equivalently, A is super-open if and only if $X \backslash A$ is super-closed.

In 2010, the authors in [2] introduced the concepts of ω_θ-open and ω_θ-closed sets on a topological space. They showed that the family of all ω_θ-open sets in a topological space X forms a topology on X. They also introduced the notions of ω_θ-interior and ω_θ-closure of a subset of a topological space.

A subset A of a topological space X is *ω_θ-open* in X if for every $x \in A$, there exists an open set O containing x such that $O \backslash Int_s(A)$ is countable. A subset B of X is *ω_θ-closed* if its complement $X \backslash B$ is ω_θ-open. The *ω_θ-closure* and *ω_θ-interior* of $A \subseteq X$ are, respectively, denoted and defined by

$$Cl_{\omega_\theta}(A) = \cap\{F : F \text{ is an } \omega_\theta\text{-closed set containing } A\}$$

and

$$Int_{\omega_\theta}(A) = \cup\{G : G \text{ is an } \omega_\theta\text{-open set contained in } A\}.$$

It is worth noting that $A \subseteq Cl_{\omega_\theta}(A)$ and $Int_{\omega_\theta}(A) \subseteq A$. Let $\mathcal{T}_{\omega_\theta}$ be the family of all ω_θ-open subsets of a topological space X. Since $\mathcal{T}_{\omega_\theta}$ is a topology on X, for any set $A \subseteq X$, $Int_{\omega_\theta}(A)$ is ω_θ-open and the largest ω_θ-open set contained in A. Moreover, for any set $A \subseteq X$, $Cl_{\omega_\theta}(A)$ is ω_θ-closed and the smallest ω_θ-closed set containing A. The topological space X is said to be

(i) *ω_θ-Hausdorff* if given any pair of distinct points $p, q \in X$, there exist disjoint ω_θ-open sets U and V such that $p \in U$ and $q \in V$;

(ii) *ω_θ-regular* if for each closed set F and each point $x \notin F$, there exist disjoint ω_θ-open sets U and V such that $x \in U$ and $F \subseteq V$; and

(iii) *ω_θ-normal* if for every pair of disjoint closed sets E and F of X, there exist disjoint ω_θ-open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

Let X and Y be topological spaces. A function $f : X \to Y$ is said to be *ω_θ-continuous* if for every $x \in X$ and every open set V of Y containing $f(x)$, there exists an ω_θ-open set U containing x such that $f(U) \subseteq V$.

Let $\mathcal{A}$ be an indexing set and $\{Y_\alpha : \alpha \in \mathcal{A}\}$ be a family of topological spaces. For each $\alpha \in \mathcal{A}$, let $\mathcal{T}_\alpha$ be the topology on Y_α. The *Tychonoff topology* on $\Pi\{Y_\alpha : \alpha \in \mathcal{A}\}$ is the topology generated by a subbase consisting of all sets $p_\alpha^{-1}(U_\alpha)$, where the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$, U_α ranges over all members of $\mathcal{T}_\alpha$, and α ranges over all elements of $\mathcal{A}$. Corresponding to $U_\alpha \subseteq Y_\alpha$, denote $p_\alpha^{-1}(U_\alpha)$ by

$\langle U_\alpha \rangle$. Similarly, for finitely many indices $\alpha_1, \alpha_2, \ldots, \alpha_n$, and sets $U_{\alpha_1} \subseteq Y_{\alpha_1}$, $U_{\alpha_2} \subseteq Y_{\alpha_2}, \ldots, U_{\alpha_n} \subseteq Y_{\alpha_n}$, the subset

$$\langle U_{\alpha_1} \rangle \cap \langle U_{\alpha_2} \rangle \cap \cdots \cap \langle U_{\alpha_n} \rangle = p_{\alpha_1}^{-1}(U_{\alpha_1}) \cap p_{\alpha_2}^{-1}(U_{\alpha_2}) \cap \cdots \cap p_{\alpha_n}^{-1}(U_{\alpha_n})$$

is denoted by $\langle U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n} \rangle$. We note that for each open set U_α subset of Y_α, $\langle U_\alpha \rangle = p_\alpha^{-1}(U_\alpha) = U_\alpha \times \Pi_{\beta \neq \alpha} Y_\beta$. Hence, a basis for the Tychonoff topology consists of sets of the form $\langle B_{\alpha_1}, B_{\alpha_2}, ..., B_{\alpha_k} \rangle$, where B_{α_i} is open in Y_{α_i} for every $i \in K = \{1, 2, ..., k\}$.

Now, the projection map $p_\alpha : \Pi\{Y_\alpha : \alpha \in \mathcal{A}\} \to Y_\alpha$ is defined by $p_\alpha(\langle y_\beta \rangle) = y_\alpha$ for each $\alpha \in \mathcal{A}$. It is known that every projection map is a continuous open surjection. Also, it is well known that a function f from an arbitrary space X into the Cartesian product Y of the family of spaces $\{Y_\alpha : \alpha \in \mathcal{A}\}$ with the Tychonoff topology is continuous if and only if each coordinate function $p_\alpha \circ f$ is continuous, where p_α is the α-th coordinate projection map.

In this paper, we gave a necessary and sufficient condition for a function from an arbitrary topological space into the product space to be ω_θ-continuous. We also characterized the concepts of ω_θ-Hausdorff, ω_θ-regular, and ω_θ-normal topological spaces and subsequently examined the relationships of these concepts and to the well-known separation axioms.

2 ω_θ-Continuity of Functions in the Product Space

This section gives a characterization of an ω_θ-continuous function from an arbitrary topological space into the product space.

We shall be using the following known result later.

Lemma 2.1 [2, p.296] *Let X be a topological space and $A \subseteq X$. Then*

(i) $Cl_{\omega_\theta}(A)$ *is* ω_θ*-closed in* X.

(ii) $x \in Cl_{\omega_\theta}(A)$ *if and only if* $A \cap G \neq \varnothing$ *for all* ω_θ*-open set* G *containing* x.

(iii) A *is* ω_θ*-closed if and only if* $A = Cl_{\omega_\theta}(A)$.

Next, we characterize the concept of ω_θ-continuous function.

Theorem 2.2 *Let $f : X \to Y$ be a function. Then the following statements are equivalent.*

(i) f *is* ω_θ*-continuous on* X.

(ii) $f^{-1}(A)$ *is* ω_θ*-open in* X *for each open subset* A *of* Y.

(iii) $f^{-1}(F)$ *is* ω_θ*-closed in* X *for each closed subset* F *of* Y.

(iv) $f^{-1}(B)$ *is* ω_θ*-open in* X *for each (subbasic) basic open set* B *in* Y*.*

(v) $f(Cl_{\omega_\theta}(A)) \subseteq Cl(f(A))$ *for each* $A \subseteq X$*.*

(vi) $Cl_{\omega_\theta}(f^{-1}(B)) \subseteq f^{-1}(Cl(B))$ *for each* $B \subseteq Y$*.*

Proof. By [2, Theorem 23], statements (i), (ii) and (iii) are equivalent.

$(ii) \Rightarrow (iv)$: This is immediate since (subbasic) basic open sets are open sets.

$(iv) \Rightarrow (ii)$: Suppose that $f^{-1}(B)$ is ω_θ-open in X for each $B \in \mathcal{B}$ where $\mathcal{B}$ is a basis for the topology in Y. Let G be an open set in Y. Then $G = \cup\{B : B \in \mathcal{B}^*\}$, where $\mathcal{B}^* \subseteq \mathcal{B}$. It follows that $f^{-1}(G) = \cup\left\{f^{-1}(B) : B \in \mathcal{B}^*\right\}$. Since the collection of all ω_θ-open sets forms a topology, $f^{-1}(G)$ is ω_θ-open in X.

$(i) \Rightarrow (v)$: Let $A \subseteq X$ and $p \in Cl_{\omega_\theta}(A)$. Let G be an open subset of Y containing $f(p)$. Since f is ω_θ-continuous on X, there exists an ω_θ-open subset O of X containing p such that $f(O) \subseteq G$. Since $p \in Cl_{\omega_\theta}(A)$, $O \cap A \neq \varnothing$. It follows that $\varnothing \neq f(O \cap A) \subseteq f(O) \cap f(A) \subseteq G \cap f(A)$. This implies that $f(p) \in Cl(f(A))$. Hence, $f(Cl_{\omega_\theta}(A)) \subseteq Cl(f(A))$.

$(v) \Rightarrow (vi)$: Let $B \subseteq Y$ and let $A = f^{-1}(B) \subseteq X$. By assumption, $f(Cl_{\omega_\theta}(A)) \subseteq Cl(f(A))$. Hence, $Cl_{\omega_\theta}(f^{-1}(B)) \subseteq f^{-1}(f(Cl_{\omega_\theta}(A))) \subseteq f^{-1}(Cl(f(A))) \subseteq f^{-1}(Cl(B))$.

$(vi) \Rightarrow (iii)$: Let F be a closed subset of Y. By assumption,

$$Cl_{\omega_\theta}(f^{-1}(F)) \subseteq f^{-1}(Cl(F)) = f^{-1}(F).$$

Hence, $f^{-1}(F) \subseteq Cl_{\omega_\theta}(f^{-1}(F))$. Then $Cl_{\omega_\theta}(f^{-1}(F)) = f^{-1}(F)$, which means that $f^{-1}(F)$ is ω_θ-closed. □

Theorem 2.3 *Let* $Y = \prod_{i=1}^{n} Y_{\alpha_i}$ *be a product space and* $A_{\alpha_i} \subseteq Y_{\alpha_i}$ *for each* $i \in \{1, 2, \ldots, n\}$*. Then* $Int_s\left(\prod_{i=1}^{n} A_{\alpha_i}\right) = \prod_{i=1}^{n} Int_s(A_{\alpha_i})$*.*

Proof. Let $x = \langle a_\alpha \rangle \in Int_s\left(\prod_{i=1}^{n} A_{\alpha_i}\right)$. Then there exists a basic open set $\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle$ containing x such that $x \in Cl(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle \subseteq \prod_{i=1}^{n} A_{\alpha_i}$. Moreover,

$$Cl(\langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle) = \langle Cl(U_{\alpha_1}), \ldots, Cl(U_{\alpha_n}) \rangle \subseteq \langle A_{\alpha_1}, \ldots, A_{\alpha_n} \rangle .$$

Hence, each $a_{\alpha_i} \in Cl(U_{\alpha_i}) \subseteq A_{\alpha_i}$, and so each $a_{\alpha_i} \in Int_s(A_{\alpha_i})$. It follows that $x = \langle a_\alpha \rangle \in \prod_{i=1}^{n} Int_s(A_{\alpha_i})$. The converse is proved similarly. □

Theorem 2.4 *Let X be a topological space and $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ a product space. A function $f : X \to Y$ is ω_θ-continuous on X if and only if each coordinate function $p_\alpha \circ f$ is ω_θ-continuous on X.*

Proof. Suppose that f is ω_θ-continuous on X. Let $\alpha \in \mathcal{A}$, and U_α be open in Y_α. Since p_α is continuous, $p_\alpha^{-1}(U_\alpha)$ is open in Y. Hence,

$$f^{-1}(p_\alpha^{-1}(U_\alpha)) = (p_\alpha \circ f)^{-1}(U_\alpha)$$

is an ω_θ-open set in X. Thus, $p_\alpha \circ f$ is ω_θ-continuous for every $\alpha \in \mathcal{A}$.

Conversely, suppose that each coordinate function $p_\alpha \circ f$ is ω_θ-continuous. Let G_α be open in Y_α. Then $\langle G_\alpha \rangle$ is a subbasic open set in Y and $(p_\alpha \circ f)^{-1}(G_\alpha) = f^{-1}(p_\alpha^{-1}(G_\alpha)) = f^{-1}(\langle G_\alpha \rangle)$ is an ω_θ-open set in X. Therefore, f is ω_θ-continuous on X. □

Corollary 2.5 *Let X be a topological space, $Y = \prod\{Y_\alpha : \alpha \in \mathcal{A}\}$ a product space, and $f_\alpha : X \to Y_\alpha$ a function for each $\alpha \in \mathcal{A}$. Let $f : X \to Y$ be the function defined by $f(x) = \langle f_\alpha(x) \rangle$. Then f is ω_θ-continuous on X if and only if each f_α is ω_θ-continuous for each $\alpha \in \mathcal{A}$.*

Proof. For each $\alpha \in \mathcal{A}$ and each $x \in X$, we have

$$(p_\alpha \circ f)(x) = p_\alpha(f(x)) = p_\alpha(\langle f_\beta(x) \rangle) = f_\alpha(x).$$

Thus, $p_\alpha \circ f = f_\alpha$ for every $\alpha \in \mathcal{A}$. The result now follows from Theorem 2.4. □

Theorem 2.6 *Let $Y = \prod_{i=1}^{n} Y_{\alpha_i}$ be a product space. If a nonempty set $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$ is ω_θ-open in Y, then each O_{α_i} is ω_θ-open in Y_{α_i}.*

Proof. Suppose that $O = \langle O_{\alpha_1}, \ldots, O_{\alpha_n} \rangle$ is ω_θ-open in Y. Then $p_{\alpha_i}(O) = O_{\alpha_i}$. Let $a_{\alpha_i} \in O_{\alpha_i}$. Then there exists $x = \langle a_{\alpha_i} \rangle \in O$ such that $p_{\alpha_i}(x) = a_{\alpha_i}$. Since O is ω_θ-open, there exists a basic open set $U = \langle U_{\alpha_1}, \ldots, U_{\alpha_n} \rangle$ containing x such that $U \setminus Int_s(O)$ is countable. Note that $p_{\alpha_i}(U) \setminus p_{\alpha_i}(Int_s(O)) \subseteq p_{\alpha_i}(U \setminus Int_s(O))$ and since $U \setminus Int_s(O)$ is countable, $p_{\alpha_i}(U \setminus Int_s(O))$ is countable. It follows that $p_{\alpha_i}(U) \setminus p_{\alpha_i}(Int_s(O))$ is also countable. Hence $p_{\alpha_i}(U) \setminus p_{\alpha_i}(Int_s(O)) = U_{\alpha_i} \setminus Int_s(O_{\alpha_i})$ is countable. Therefore, each O_{α_i} is ω_θ-open in Y_{α_i}. □

Theorem 2.7 *Let $X = \prod_{i=1}^{n} X_{\alpha_i}$ and $Y = \prod_{i=1}^{n} Y_{\alpha_i}$ be product spaces, and for each $i \in \{1, 2, \ldots, n\}$, let $f_{\alpha_i} : X_{\alpha_i} \to Y_{\alpha_i}$ be a function. If $f : X \to Y$ defined by $f(\langle x_{\alpha_i} \rangle) = \langle f_{\alpha_i}(x_{\alpha_i}) \rangle$, is ω_θ-continuous on X, then each f_{α_i} is ω_θ-continuous on X_{α_i}.*

Proof. Assume that $f : X \to Y$ is ω_θ-continuous. Let O_{α_i} be an open set in Y_{α_i}. For each $i \in \{1, 2, \ldots, n\}$, let $a_{\alpha_i} \in f_{\alpha_i}^{-1}(O_{\alpha_i}) := G_{\alpha_i}$. Then

$$x := \langle a_{\alpha_1}, \ldots, a_{\alpha_n}\rangle \in \langle G_{\alpha_1}, \ldots, G_{\alpha_n}\rangle = \left\langle f_{\alpha_1}^{-1}(O_{\alpha_1}), \ldots, f_{\alpha_n}^{-1}(O_{\alpha_n})\right\rangle$$
$$= f^{-1}(\langle O_{\alpha_1}, \ldots, O_{\alpha_n}\rangle).$$

Since each O_{α_i} is open in Y_{α_i}, $O := \langle O_{\alpha_1}, \ldots, O_{\alpha_n}\rangle$ is open in Y. Since f is ω_θ-continuous, $f^{-1}(O) = \langle G_{\alpha_1}, \ldots G_{\alpha_n}\rangle$ is ω_θ-open in X. Since $x \in \langle G_{\alpha_1}, \ldots G_{\alpha_n}\rangle$, there exists a basic open set $U = \langle U_{\alpha_1}, \ldots U_{\alpha_n}\rangle$ containing x such that $U \setminus Int_s(\langle G_{\alpha_1}, \ldots, G_{\alpha_n}\rangle)$ is countable. Note that

$$p_{\alpha_i}(U) \setminus p_{\alpha_i}(Int_s(\langle G_{\alpha_1}, \ldots, G_{\alpha_n}\rangle)) \subseteq p_{\alpha_i}(U \setminus Int_s(\langle G_{\alpha_1}, \ldots, G_{\alpha_n}\rangle)).$$

Since $U \setminus Int_s(\langle G_{\alpha_1}, \ldots, G_{\alpha_n}\rangle)$ is countable, $p_{\alpha_i}(U \setminus Int_s(\langle G_{\alpha_1}, \ldots, G_{\alpha_n}\rangle))$ is also countable. It follows that $p_{\alpha_i}(U) \setminus p_{\alpha_i}(Int_s(\langle G_{\alpha_1}, \ldots, G_{\alpha_n}\rangle))$ is countable. Hence $U_{\alpha_i} \setminus Int_s(G_{\alpha_i})$ is countable. This means that each $G_{\alpha_i} = f_{\alpha_i}^{-1}(O_{\alpha_i})$ is ω_θ-open in X_{α_i}. Thus, each f_{α_i} is ω_θ-continuous on X_{α_i}. □

3 Some Versions of Separation Axioms

This section provides some characterizations of ω_θ-Hausdorff, ω_θ-regular, and ω_θ-normal topological spaces.

Theorem 3.1 *Let X be a topological space. Then the following are equivalent:*

(i) *X is ω_θ-Hausdorff.*

(ii) *Let $p \in X$. For each $q \neq p$, there exists an ω_θ-open set U with $p \in U$ such that $q \notin Cl_{\omega_\theta}(U)$.*

(iii) *For each $p \in X$,*

$$C = \cap\{Cl_{\omega_\theta}(U) : U \text{ is an } \omega_\theta\text{-open set with } p \in U\} = \{p\}$$

Proof. $(i) \Rightarrow (ii)$: Let X be ω_θ-Hausdorff. Let $p \in X$ and $q \neq p$. Since X is an ω_θ-Hausdorff, there exists ω_θ-open sets U and V such that $p \in U$, $q \in V$, and $U \cap V = \varnothing$. By Lemma 2.1, $q \notin Cl_{\omega_\theta}(U)$.

$(ii) \Rightarrow (iii)$: Assume that (ii) holds. Let $p \in X$ and $q \neq p$. Note that $p \in C$. By (ii), there exists an ω_θ-open set U with $p \in U$ such that $q \notin Cl_{\omega_\theta}(U)$. This means that $q \notin C$. Since q is arbitrary, $C = \{p\}$.

$(iii) \Rightarrow (ii)$: Let $p \in X$ and $q \neq p$. By hypothesis, there exists an ω_θ-open set U such that $p \in U$ and $q \notin Cl_{\omega_\theta}(U)$, which implies that there exists an ω_θ-open set V with $q \in V$ such that $U \cap V = \varnothing$. Hence, X is ω_θ-Hausdorff. □

Remark 3.2 There is a topological space X that is an ω_θ-Hausdorff but not Hausdorff.

Consider $X = \{a, b, c\}$ with topology $\tau = \{X, \varnothing, \{a\}\}$. X is not Hausdorff since the open sets that contains a and b intersects, but X is ω_θ-Hausdorff since the disjoint ω_θ-open sets $\{a\}$, $\{b\}$, and $\{c\}$ contains a, b, and c, respectively.

Theorem 3.3 *Let X be a topological space. Then the following are equivalent:*

(i) X is ω_θ-regular.

(ii) For each $x \in X$ and open set U with $x \in U$, there exists an ω_θ-open set V with $x \in V$ such that $x \in V \subseteq Cl_{\omega_\theta}(V) \subseteq U$.

(iii) For each $x \in X$ and closed set F with $x \notin F$, there exists an ω_θ-open set V with $x \in V$ such that $Cl_{\omega_\theta}(V) \cap F = \varnothing$.

Proof. $(i) \Rightarrow (ii)$: Let $x \in X$ and U an open set with $x \in U$. Then $X \setminus U$ is closed and $x \notin X \setminus U$. By hypothesis, there exist ω_θ-open sets V and W such that $x \in V$, $X \setminus U \subseteq W$, and $V \cap W = \varnothing$. Thus, $V \subseteq X \setminus W$ so that $Cl_{\omega_\theta}(V) \subseteq Cl_{\omega_\theta}(X \setminus W) = X \setminus W$. Also, $Cl_{\omega_\theta}(V) \cap (X \setminus U) \subseteq Cl_{\omega_\theta}(V) \cap W = \varnothing$. Hence, $Cl_{\omega_\theta}(V) \subseteq U$ and so $x \in V \subseteq Cl_{\omega_\theta}(V) \subseteq U$.

$(ii) \Rightarrow (iii)$: Let $x \in X$ and F a closed set with $x \notin F$. Then $X \setminus F$ is open and $x \in X \setminus F$. By (ii), there exists an ω_θ-open set V with $x \in V$ such that $x \in V \subseteq Cl_{\omega_\theta}(V) \subseteq X \setminus F$. Hence, $Cl_{\omega_\theta}(V) \cap F = \varnothing$.

$(iii) \Rightarrow (i)$: Let F be closed and $x \notin F$. By (iii), there exists an ω_θ-open set V with $x \in V$ such that $Cl_{\omega_\theta}(V) \cap F = \varnothing$. Note that $X \setminus Cl_{\omega_\theta}(V)$ is ω_θ-open and $F \subseteq X \setminus Cl_{\omega_\theta}(V)$. Furthermore, $V \cap X \setminus Cl_{\omega_\theta}(V) = \varnothing$. Hence, X is ω_θ-regular. □

Remark 3.4 There is a topological space X that is an ω_θ-regular but not regular.

Consider $X = \{a, b, c\}$ with topology $\tau = \{X, \varnothing, \{a\}\}$. X is not regular since every open set that contains the closed set $\{b, c\}$ intersects with $\{a\}$. But X is ω_θ-regular since $\{b, c\}$ and $\{a\}$ are disjoint ω_θ-open sets containing $\{b, c\}$ and $\{a\}$, respectively.

Recall that a topological space X is said to be T_1 [3, p.138] if for each $p, q \in X$ with $p \neq q$, there exists open sets U and V such that $p \in U$ and $q \notin U$, and $q \in V$ and $p \notin V$.

Theorem 3.5 *Let X be a T_1-space. If X is ω_θ-regular, then X is ω_θ-Hausdorff.*

Proof. Let $x, y \in X$ with $x \neq y$. Note that X is a T_1-space, so there exist open sets U and V with $x \in U$ and $y \in V$ such that $x \notin V$ and $y \notin U$. Then $x \notin X \setminus U$ where $X \setminus U$ is closed. Since X is ω_θ-regular, there exist disjoint ω_θ-open sets E and F such that $x \in E$ and $X \setminus U \subseteq F$. Observe that $y \in X \setminus U$ and so $y \in F$. Hence, X is ω_θ-Hausdorff. $\square$

Theorem 3.6 *Let X be a topological space. Then the following are equivalent:*

(i) *X is ω_θ-normal.*

(ii) *For each closed set A and for each open set U containing A, there exists ω_θ-open set V containing A such that $Cl_{\omega_\theta}(V) \subseteq U$.*

(iii) *For each pair of disjoint closed sets A and B, there exists ω_θ-open set U containing A such that $Cl_{\omega_\theta}(U) \cap B = \varnothing$.*

Proof. $(i) \Rightarrow (ii)$: Let A be closed and U be an open set containing A. Then, $A \cap X \setminus U = \varnothing$ and so they are disjoint closed sets in X. Since X is ω_θ-normal, there exist disjoint ω_θ-open sets V and W such that $A \subseteq V$ and $X \setminus U \subseteq W$ (or $X \setminus W \subseteq U$). Now, $V \cap W = \varnothing$ which implies that $V \subseteq X \setminus W$. Hence, $Cl_{\omega_\theta}(V) \subseteq Cl_{\omega_\theta}(X \setminus W) = X \setminus W$. Therefore, $A \subseteq V \subseteq Cl_{\omega_\theta}(V) \subseteq X \setminus W \subseteq U$.

$(ii) \Rightarrow (iii)$: Let A and B be disjoint closed sets in X. Then, $A \subseteq X \setminus B$ and $X \setminus B$ is an open set containing A. By (ii), there exists ω_θ-open set V containing A such that $Cl_{\omega_\theta}(V) \subseteq X \setminus B$, which implies that $Cl_{\omega_\theta}(V) \cap B = \varnothing$.

$(iii) \Rightarrow (i)$: Let A and B be an disjoint closed sets in X. By (iii), there exists ω_θ-open set U with $A \subseteq U$ such that $Cl_{\omega_\theta}(U) \cap B = \varnothing$, which implies that $B \subseteq X \setminus Cl_{\omega_\theta}(U)$. Now, U and $X \setminus Cl_{\omega_\theta}(U)$ are disjoint ω_θ-open sets such that $A \subseteq U$ and $B \subseteq X \setminus Cl_{\omega_\theta}(U)$. Hence, X is ω_θ-normal. $\square$

Remark 3.7 There is a topological space X that is ω_θ-normal but not normal.

Consider $X = \{a, b, c\}$ with topology $\tau = \{X, \varnothing, \{a\}, \{a, b\}, \{a, c\}\}$. X is not normal since all open sets containing $\{a\}$ and $\{b\}$, respectively, intersect. However, X is ω_θ-normal since each subset of X is ω_θ-open, and so if A and B are any disjoint closed sets in X, then A and B are also disjoint ω_θ-open sets in X.

Theorem 3.8 *Let X be a T_1-space. If X is ω_θ-normal, then X is ω_θ-regular.*

Proof. Let $x \in X$ and F be a closed set with $x \notin F$. Since X is a T_1-space, $\{x\}$ is closed. Clearly, F and $\{x\}$ are disjoint closed sets. Since X is ω_θ-normal, there exist disjoint ω_θ-open sets U and V such that $F \subseteq U$ and $\{x\} \subseteq V$. Thus, $x \in V$. Hence, X is ω_θ-regular. $\square$

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