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Positive solutions of a fuzzy nonlinear difference equations

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Abstract

In this paper, we study the boundedness and asymptotic behavior of fuzzy nonlinear difference equation of the form

$$x_{n+1} = \sum_{i=0}^{k} \frac{A_i x_n + x_{n-i}}{B_i x_{n-i}^{p_i}},$$

where $A_i, B_i \quad i \in \{0, 1, \cdots, k\}$, are the positive fuzzy numbers, p_i, $i \in \{0, 1, \cdots, k\}$ are positive constants and x_i, $i \in \{0, -1, \cdots, -k, -k+1\}$ are positive fuzzy numbers.

Keywords: Fuzzy Difference Equations, Nonlinear, Boundedness, Fuzzy number, α-cuts

1 Introduction

Fuzzy difference equation is a difference equation where constants and the initial values are fuzzy numbers and its solutions are sequence of fuzzy numbers. Fuzzy difference equations are important for studying and solving large proportions of problems in many topics in applied mathematics and it have been appeared in the field of physics, geography, medicine, biology etc. Fuzzy environment in parameters, variables and initial conditions to overcome imprecision or uncertainties in place of exact ones, by changing general difference equations into fuzzy difference equations.

In recent years, the study of existence and uniqueness of solutions and stability solution of the fractional difference equation are of great interest [1], [2], [3]. Also, the study of fuzzy difference equations have been gaining interest for many years.

The boundedness and asymptotic behavior of fuzzy difference equations have been studied in [6] - [10], [11] and [13].

In this paper, the boundedness and asymptotic behavior of fuzzy nonlinear difference equation has been studied. It has the form,

$$x_{n+1} = \sum_{i=0}^{k} \frac{A_i x_n + x_{n-i}}{B_i x_{n-i}^{p_i}} \tag{1.1}$$

where A_i, B_i are the positive fuzzy numbers, $i \in \{0, 1, \cdots, k\}$, $p_i, i \in \{0, 1, \cdots, k\}$ are positive constants and $x_i, i \in \{0, -1, \cdots, -k, -k+1\}$ are positive fuzzy numbers.

Let A be the set, $\overline{A}$ be the closure of A. If the following conditions are hold, then $A : \mathbb{R}^+ \to (0, 1]$ is a fuzzy number.

(i) A is normal,

(ii) A is convex fuzzy set,

(iii) A is upper semi-continuous,

(iv) The support of A, supp $A = \overline{\cup_{\alpha \in (0,1]}} = \overline{\{x : A(x) > 0\}}$ is compact.

In section 2, we obtain the existence and boundedness solutions of fuzzy difference equations. In section 3, we present the stability solutions of fuzzy difference equations.

Lemma 1.1. *Let f be continuous function from $\mathbb{R}^+ \times \mathbb{R}^+ \times \ldots + \mathbb{R}^+$ into $\mathbb{R}^+$ and $A_0, A_1, \ldots, A_k$ be fuzzy numbers. Then*

$$[f(A_0, A_1, \ldots, A_k)]_\alpha = f([A_0]_\alpha, f[A_1]_\alpha, \ldots, f[A_k]_\alpha), \quad \alpha \in (0, 1].$$

2 Solutions of Existence and Boundedness

In this section, we obtain the existence and boundedness solutions of fuzzy difference equation of (1.1).

Theorem 2.1. *Consider the equation (1.1) where A_i, B_i are positive fuzzy numbers, then for any positive fuzzy number $x_0, x_{-1}, \cdots, x_{-k}, x_{-k+1}$ there exists a unique positive solution x_n of (1.1) with initial conditions $x_0, x_{-1}, \cdots, x_{-k}, x_{-k+1}$.*

Proof. Let (x_n) be the sequence of fuzzy numbers satisfying (1.1) with the initial condition $x_0, x_{-1}, \cdots, x_{-k}, x_{-k+1}$. Let us take the α cuts, $\alpha \in (0, 1]$, then

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \; [A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}], \; [B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]. \tag{2.1}$$

It follows we get,

$$[x_i]_\alpha = [L_{i,\alpha}, R_{i,\alpha}], \quad \alpha \in (0,1]$$
$$[A_i]_\alpha = [A_{i,l,\alpha}, R_{i,r,\alpha}],$$
$$[B_i]_\alpha = [B_{i,l,\alpha}, R_{i,r,\alpha}], \quad i \in \{0,1,\ldots,k\}$$

From (1.1), (2.1) and lemma 1.1, we get

$$\begin{aligned}
[x_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] \\
&= [\sum_{i=0}^{k} \frac{A_i x_n + x_{n-i}}{B_i x_{n-i}^{p_i}}]_\alpha \\
&= \sum_{i=0}^{k} \frac{[A_i]_\alpha [x_n]_\alpha + [x_{n-i}]_\alpha}{[B_i]_\alpha [x_{n-i}^{p_i}]_\alpha} \\
&= \sum_{i=0}^{k} \frac{[A_{i,l,\alpha}][L_{n,\alpha}, R_{n,\alpha}] + [L_{n-i,\alpha}, R_{n-i,\alpha}]}{[B_{i,l,\alpha}][L_{n-i,\alpha}^{p_i}, R_{n-i,\alpha}^{p_i}]} \\
&= \sum_{i=0}^{k} \frac{[A_{i,l,\alpha} L_{n,\alpha}, A_{i,l,\alpha} R_{n,\alpha}] + [L_{n-i,\alpha}, R_{n-i,\alpha}]}{[B_{i,l,\ alpha}][L_{n-i,\alpha}^{p_i}, R_{n-i,\alpha}^{p_i}]} \\
[L_{n+1,\alpha}, R_{n+1,\alpha}] &= \sum_{i=0}^{k} \frac{A_{i,l,\alpha} L_{n,\alpha} + L_{n-i,\alpha}}{B_{i,l,\alpha} R_{n-i,\alpha}^{p_i}}, \frac{A_{i,l,\alpha} R_{n,\alpha} + R_{n-i,\alpha}}{B_{i,l,\alpha} R_{n-i,\alpha}^{p_i}}
\end{aligned}$$

$$\begin{aligned}
L_{n+1,\alpha} &= \sum_{i=0}^{k} \frac{A_{i,l,\alpha} L_{n,\alpha} + L_{n-i,\alpha}}{B_{i,l,\alpha} R_{n-i,\alpha}^{p_i}} \\
R_{n+1,\alpha} &= \sum_{i=0}^{k} \frac{A_{i,l,\alpha} R_{n,\alpha} + R_{n-i,\alpha}}{B_{i,l,\alpha} R_{n-i,\alpha}^{p_i}}
\end{aligned} \tag{2.2}$$

From preposition 1 in [10], $A_{i,l,\alpha}, B_{i,l,\alpha}$ are left continuous.

That is enough to prove that supp of x_n, that is $\overline{\cup_{\alpha \in (0,1]}[L_{n,\alpha} R_{n,\alpha}]}$ is bounded.

$$\begin{aligned}
L_{1,\alpha} &= \sum_{i=0}^{k} \frac{A_{l,\alpha} L_{0,\alpha} + L_{-1,\alpha}}{B_{r,\alpha} + R_{-1,\alpha}} \\
R_{1,\alpha} &= \sum_{i=0}^{k} \frac{A_{r,\alpha} R_{0,\alpha} + R_{-1,\alpha}}{B_{l,\alpha} + L_{-1,\alpha}}
\end{aligned} \tag{2.3}$$

There exists positive constants $M_L > 0$, $N_L > 0$, $M_j > 0$, $N_j > 0$, $M_{-i} > 0$, $N_{-i} > 0, M_0 > 0, N_0 > 0$ such that for all $a \in (0,1]$

$$\begin{aligned}
[A_{i,l,\alpha}, A_{i,r,\alpha}] &\subset \overline{\cup_{\alpha \in (0,1]}[A_{i,l,\alpha}, A_{i,r,\alpha}]} \subset [M_L, N_L] \\
[B_{i,l,\alpha}, B_{i,r,\alpha}] &\subset \overline{\cup_{\alpha \in (0,1]}[B_{i,l,\alpha}, B_{i,r,\alpha}]} \subset [M_j, N_j] \\
[L_{-l,\alpha}, R_{-i,\alpha}] &\subset \overline{\cup_{\alpha \in (0,1]}[L_{-l,\alpha}, R_{-i,\alpha}]} \subset [M_{-i}, N_{-i}] \\
[L_{0,\alpha}, R_{0,\alpha}] &\subset \overline{\cup_{\alpha \in (0,1]}[L_{0,\alpha}, R_{0,\alpha}]} \subset [M_0, N_0]
\end{aligned} \tag{2.4}$$

From (2.2) and (2.3), we get

$$\cup_{\alpha\in(0,1]}[L_{0,\alpha}, R_{0,\alpha}] \subset \Big[\frac{M_L M_0 + M_{-i}}{N_j N_{-i}}, \frac{N_L N_0 + N_{-i}}{M_j M_{-i}}\Big]$$

From this equation $\overline{\cup_{\alpha\in(0,1]}[L_{0,\alpha}, R_{0,\alpha}]}$ is compact. Proceeding this, we can prove that $\overline{\cup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]}$ is compact. □

Theorem 2.2. *Consider fuzzy difference equation of (1.1), where A_i, B_i, $i \in \{0, 1, \cdots, k\}$ and x_i, $i \in \{0, -1, \cdots, -k, -k+1\}$ are positive fuzzy numbers, if for every $\alpha \in (0, 1]$, $p_i.p_{\lambda_i} < 1$ then every solution of (1.1) is bounded and persists.*

Proof. If x_n is the unique positive solution of (1.1) with initial values $x_0, x_{-1}, \ldots, x_{-k}, x_{-k+1}$ such that $[x_n]_\alpha = [L_n, R_n]_\alpha$ holds, we consider the system,

$$\begin{aligned} S_{n+1} &= \frac{M_L S_n + S_{n-i}}{N_j S_{n-i}^{p_i}}, \\ T_{n+1} &= \frac{M_L T_n + T_{n-i}}{N_j T_{n-i}^{p_i}}. \end{aligned} \tag{2.5}$$

Let (S_n, T_n) be a solution of (1.1) with initial condition

$$S_i = M_i, \quad T_i = N_i, \quad i = 0, -1, \ldots, -k, -k+1. \tag{2.6}$$

From (2.1), (2.3), (2.4) and (2.5), we have

$$\begin{aligned} S_0 &= \frac{M_L S_0 + S_{-i}}{N_j S_{-i}^{p_i}} \leq L_{1,\alpha}, \\ R_{1,\alpha} &\leq \frac{N_L T_0 + T_{-i}}{M_j T_{-i}^{p_i}} = T_1 \end{aligned} \tag{2.7}$$

where N_L, M_L, M_j, N_j are defined in (2.1). From (2.1), (2.3), (2.5) and (2.6), we have

$$\begin{aligned} S_2 &= \frac{M_L H_1 + H_{1-i}}{N_j H_{1-i}^{p_i}} \leq L_{2,\alpha}, \\ R_{2,\alpha} &\leq \frac{N_L E_1 + E_{1-i}}{M_j E_{1-i}^{p_i}} = T_2. \end{aligned} \tag{2.8}$$

Inductively we can prove that,

$$H_n \leq L_{n,\alpha}, \quad R_{n,\alpha} \leq E_n, \quad n = 1, 2, \ldots \tag{2.9}$$

and preposition 2 of [10], we get

$$\overline{\cup_{\alpha\in(0,1]}[L_{0,\alpha}, R_{0,\alpha}]} \subset \Big[\sum_{i=0}^{k}(\frac{M_i + h}{N_i} h^{p_i}, h)\Big] \tag{2.10}$$

where $h = max\{1, \gamma, C_i\}$, $C_i = max\{R_i, \alpha, \alpha \in (0, 1]\}$ $i = 0, 1, \cdots - k, -k+1$.

Similarly if A_i, $i = 0, \ldots, k$ are positive real numbers.

$$\overline{\cup_{\alpha\in(0,1]}[L_{0,\alpha}, R_{0,\alpha}]} \subset \Big[w, \frac{1}{w}, \sum_{i=0}^{k} A_i\Big], \tag{2.11}$$

where $w = min\{L_{i,\alpha}, wherei = 0, 1, \cdots - k, -k+1\}$ From (2.10) and (2.11), every positive solution of (1.1) is bounded and presists. □

3 Stability solutions of fuzzy difference equations

In this section, we study the asymptotic stability of equilibrium point and obtain the stability solutions of fuzzy difference equation of (1.1).

Theorem 3.1. *Consider the system of difference equations*

$$\begin{aligned} y_{n+1} &= \sum_{i=0}^{k} \frac{C_i y_n + y_{n-i}}{F_i z_{n-i}^{p_i}} \\ z_{n+1} &= \sum_{i=0}^{k} \frac{E_i z_n + z_{n-i}}{D_i y_{n-i}^{p_i}} \end{aligned} \tag{3.1}$$

where C_i, D_i, E_i, F_i *are positive real numbers and the initial values* $y_i, z_i \quad i = 0, 1, \ldots, -k, -k+1$ *are positive real numbers. Then the following statements are true.*
(i) For any $u_1 > C_i + i - F_i$ *and* $u_2 = E_i + i - D_i \quad i = 0, 1, \ldots, n$ *then (3.1) has no solution that is eventually in* $[u, \infty) \times [u_2, \infty)$ *that is, eventually positive.*
(ii) For any $l_1 < C_i + i - F_i$ *and* $l_2 < E_i + i - D_i$ *the conditions* $C_i + i > F_i, E_i + i > D_i$ *hold then (2.11)) has no solution that is eventually in* $[0, l_1] \times [0, l_2]$.

Proof. Assume that there exists a solution $(y_n, z_n)_{n=-1}^{\infty}$ of (2.11) such that

$$y_n \le u_2, \quad z_n \le u_1, \quad n \ge N \tag{3.2}$$

for some $u_1 > C_i + i - F_i$ and $u_2 > E_i + i - D_i$
Let $N = 0$, then (3.2) becomes,

$$\begin{aligned} y_{n+1} &= \sum_{i=0}^{k} \frac{C_i y_n + y_{n-i}}{F_i u_1^{p_i}} = \frac{C_i}{F_i u_1^{p_i}} y_n + \frac{1}{F_i u_1^{p_i}} y_{n-i} \\ z_{n+1} &= \sum_{i=0}^{k} \frac{E_i z_n + z_{n-i}}{D_i u_2^{p_i}} = \frac{E_i}{D_i u_2^{p_i}} z_n + \frac{1}{D_i u_2^{p_i}} z_{n-i}. \end{aligned} \tag{3.3}$$

Using the results of difference inequalities in [5], then (3.3) becomes,

$$y_n \le a_n, \; z_n \le b_n, \; y_0 = a_0,$$

$$z_0 = b_0, \; y_{-1} = a_{-1}, \; z_{-1} = b_{-1},$$

where (a_n) and (b_n) satisfy

$$\begin{aligned} a_{n+1} &= \frac{C_i}{F_i u_1^{p_i}} a_n + \frac{1}{F_i u_1^{p_i}} a_{n-i}, \\ b_{n+1} &= \frac{C_i}{F_i u_2^{p_i}} b_n + \frac{1}{F_i u_2^{p_i}} b_{n-i}, \end{aligned} \tag{3.4}$$

and the characteristic equations of this linear homogeneous equations are

$$\lambda^2 - \frac{C_i}{F_i u_1^{p_i}}\lambda - \frac{C_i}{F_i u_1^{p_i}} = 0,$$
$$\lambda^2 - \frac{E_i}{D_i u_2^{p_i}}\lambda - \frac{E_i}{D_i u_2^{p_i}} = 0.$$

Then

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{n\to\infty} b_n = 0$$

for every solutions of (3.3).

Hence $\lim_{n\to\infty} y_n = 0, \quad \lim_{n\to\infty} z_n = 0.$

which is contradiction to (3.2).

Assume that there exists a solution $(y_n, z_n)_{n=-1}^{\infty}$ of (3.1) such that

$$y_n \geq l_2, \quad z_n \geq l_1, \quad n \geq N \tag{3.5}$$

for some $l_1 < C_i + i - F_i$ and $l_2 < E_i + i - D_i$.

Let $N = 0$, then (3.2) becomes,

$$\begin{aligned} y_{n+1} &= \sum_{i=0}^{k} \frac{C_i y_n + y_{n-i}}{F_i l_1^{p_i}} = \frac{C_i}{F_i l_1^{p_i}} y_n + \frac{1}{F_i l_1^{p_i}} y_{n-i}, \\ z_{n+1} &= \sum_{i=0}^{k} \frac{E_i z_n + z_{n-i}}{D_i l_2^{p_i}} = \frac{E_i}{D_i l_2^{p_i}} z_n + \frac{1}{D_i l_2^{p_i}} z_{n-i}. \end{aligned} \tag{3.6}$$

Using the results of difference inequalities in [5], then (3.3) becomes

$$y_n \geq c_n, \; z_n \geq d_n, \; y_0 = c_0,$$

$$z_0 = d_0, \; y_{-1} = c_{-1}, \; z_{-1} = d_{-1}$$

where (a_n) and (b_n) satisfy

$$\begin{aligned} c_{n+1} &= \frac{C_i}{F_i l_1^{p_i}} c_n + \frac{1}{F_i l_1^{p_i}} c_{n-i}, \\ d_{n+1} &= \frac{C_i}{F_i l_2^{p_i}} d_n + \frac{1}{F_i l_2^{p_i}} d_{n-i}, \end{aligned} \tag{3.7}$$

and the characteristic equations of this linear homogeneous equations are

$$\lambda^2 - \frac{C_i}{F_i u_1^{p_i}}\lambda - \frac{C_i}{F_i u_1^{p_i}} = 0,$$
$$\lambda^2 - \frac{E_i}{D_i u_2^{p_i}}\lambda - \frac{E_i}{D_i u_2^{p_i}} = 0.$$

Then

$$\lim_{n\to\infty} c_n = \infty, \quad \lim_{n\to\infty} d_n = \infty$$

for every solutions of (3.3).

Hence $\lim_{n\to\infty} y_n = \infty \quad \lim_{n\to\infty} z_n = \infty.$

which is contradiction to (3.2). $\square$

Remark 3.2. *If the conditions $C_i + i > F_i, \quad E_i + i > D_i$ hold, then*

(i) The equilibrium (0,0) is asymptotically stable.

(ii) Then (3.1) has a unique equilibrium $(C_i + i - F_i, E_i + i - D_i)$.

If the equation (2.1) hold, there exists the unique equilibrium $\overline{x}$ of (1.1), then we have,

$$L_\alpha = \sum_{i=0}^{k} \frac{A_{i,l,\alpha} L_\alpha + L_\alpha}{B_{i,r,\alpha} + R_\alpha},$$

$$R_\alpha = \sum_{i=0}^{k} \frac{A_{i,r,\alpha} L_\alpha + L_\alpha}{B_{i,l,\alpha} + R_\alpha}.$$

Then it follows, we get

$$L_\alpha = \sum_{i=0}^{k} A_{i,l,\alpha} + i - B_{i,r,\alpha},$$

$$R_\alpha = \sum_{i=0}^{k} A_{i,r,\alpha} + i - B_{i,l,\alpha}.$$

Let $\overline{x}$ be positive equilibrium point if and only if $A_{l,\alpha} = A_{r,\alpha}$ and $B_{l,\alpha} = B_{r,\alpha}$, where A and B are positive fuzzy numbers. Therefore

$$L_\alpha = R_\alpha = A + i - B \tag{3.8}$$

Theorem 3.3. *Let us consider the equation (1.1), and the condition $B < A + i, \quad i = 0, 1, \ldots, k$ hold, then the unique equilibrium $\overline{x}$ of (1.1) is stable, where A and B are positive fuzzy numbers.*

Proof. Assume that $\overline{x}$ is a positive equilibrium of equation (1.1) and ϵ be a positive real number such that

$$D(x_{-i}, \overline{x}) \leq \delta < \epsilon, i = 0, \ldots, k. \tag{3.9}$$

From (3.9) we get

$$\begin{aligned} &|L_{-i,\alpha} - L_\alpha| \leq \delta, \\ &|R_{-i,\alpha} - R_\alpha| \leq \delta, \quad i = 0, 1, \ldots, k, \quad \alpha \in (0, 1] \end{aligned} \tag{3.10}$$

$$\begin{aligned} L_\alpha &= \sum_{i=0}^{k} \left(A_{i,l,\alpha} + i - B_{i,r,\alpha}\right) \\ R_\alpha &= \sum_{i=0}^{k} \left(A_{i,r,\alpha} + i - B_{i,l,\alpha}\right) \end{aligned} \tag{3.11}$$

From (2.2), (3.9) and (3.10), we have

$$L_{i,\alpha} - L_\alpha = \frac{A_i L_{0,\alpha} + L_{-i,\alpha}}{B_i + R_{-i,\alpha}}$$
$$\leq \frac{A_i(L_\alpha + \delta) + i(L_\alpha + \delta)}{B_i + R_{-i,\alpha} - \delta}$$
$$= \frac{(L_\alpha + \delta)(A_i + 1) - L_\alpha(B_i + R_{-i,\alpha})}{B_i + R_{-i,\alpha} - \delta}$$

$$= \frac{(L_\alpha + \delta)(A_i + 1) - L_\alpha(B_i + A_i + i - B_i - \delta)}{B_i + R_{-i,\alpha} - \delta}$$
$$= \frac{(L_\alpha + \delta)(A_i + 1) - L_\alpha(A_i + i) - L_\alpha \delta}{B_i + R_{-i,\alpha} - \delta}$$
$$= \delta \frac{A_i + L_\alpha + 1}{B_i + R_\alpha - \delta}$$

$$L_{i,\alpha} - L_\alpha \leq \delta \frac{A_i + L_\alpha + 1}{B_i + R_\alpha - \delta} \tag{3.12}$$

$$L_{i,\alpha} - L_\alpha = \frac{A_{i,L_0,\alpha} + L_{-i,\alpha}}{B_i + R_{-i,\alpha} - \delta}$$
$$\geq \frac{A_i(L_\alpha - \delta) + i(L_\alpha - \delta)}{B_i + R_{-i,\alpha} + \delta}$$
$$= -\delta \frac{A_i + L_\alpha + 1}{B_i + R_\alpha + \delta}$$

$$L_{i,\alpha} - L_\alpha \geq -\delta \frac{A_i + L_\alpha + 1}{B_i + R_\alpha + \delta}. \tag{3.13}$$

From (3.11) and (3.12), we get

$$|L_{i,\alpha} - L_\alpha| \leq \delta < \epsilon. \tag{3.14}$$

From (2.2), (3.9) and (3.10), we have

$$L_{i,\alpha} - L_\alpha = \frac{A_i L_{0,\alpha} + R_{-i,\alpha}}{B_i + L_{-i,\alpha}}$$
$$\leq \frac{A_i(R_\alpha + \delta) + i(R_\alpha + \delta)}{B_i + L_{-i,\alpha} - \delta}$$
$$= \frac{(R_\alpha + \delta)(A_i + 1) - R_\alpha(B_i + R_{-i,\alpha})}{B_i + L_{-i,\alpha} - \delta}$$

$$= \frac{(R_\alpha + \delta)(A_i + 1) - R_\alpha(B_i + A_i + i - B_i - \delta)}{B_i + L_{-i,\alpha} - \delta}$$
$$= \frac{(R_\alpha + \delta)(A_i + 1) - R_\alpha(A_i + i) - R_\alpha \delta}{B_i + L_{-i,\alpha} - \delta}$$
$$= \delta \frac{A_i + R_\alpha + 1}{B_i + L_\alpha - \delta}.$$

That is,

$$L_{i,\alpha} - L_\alpha \leq \delta \frac{A_i + R_\alpha + 1}{B_i + L_\alpha - \delta} \tag{3.15}$$

$$\begin{aligned} R_{i,\alpha} - R_\alpha &= \frac{A_{i,R_0,\alpha} + R_{-i,\alpha}}{B_i + L_{-i,\alpha} - \delta} \\ &\geq \frac{A_i(R_\alpha - \delta) + i(R_\alpha - \delta)}{B_i + L_{-i,\alpha} + \delta} \\ &= -\delta \frac{A_i + R_\alpha + 1}{B_i + L_\alpha + \delta} \\ R_{i,\alpha} - R_\alpha &\geq -\delta \frac{A_i + L_\alpha + 1}{B_i + L_\alpha + \delta}. \end{aligned} \tag{3.16}$$

From (3.15) and (3.16), we get

$$|R_{i,\alpha} - R_\alpha| \leq \delta < \epsilon \tag{3.17}$$

From (3.14) and (3.17), we have

$$\begin{aligned} &|L_{n,\alpha} - R_\alpha| < \epsilon \\ &|R_{n,\alpha} - R_\alpha| < \epsilon \quad \alpha \in (0,1], \quad n = 0, 1, \ldots, \end{aligned} \tag{3.18}$$

Hence $D(x_n, \overline{x}) < \epsilon, \quad n \geq 0$.

This completes the proof of the theorem. □

References

[1] A.P. Lavanya and J. Daphy Louis Lovenia, Positive Solutions of Fractional difference Equations using Fixed Point Theorem, Adv. Inequal. Appl. (2016), 2016:9.

[2] A.P. Lavanya and J. Daphy Louis Lovenia, Stability of a class of Nonlinear Fractional Difference Equations, Int. J. Appl. Eng. Research (2016), Special issue Vol 11, No 1, pp. 50-57.

[3] A.P. Lavanya and J. Daphy Louis Lovenia, Oscillatory Solutions of a class of Nonlinear Fractional Difference Equations, Global Journal of Pure and Applied Mathematics (2016), special issue Vol 11, No 2, pp. 65-69.

[4] E.Y Deeba, A. De Korvin, Analysis by fuzzy difference equations of a model of CO 2 level in the blood, Appl. Math. Lett. 12 (1999) 3340.

[5] G. Stefanidou and G. Papaschinopoulos, Trichotomy, Stability and oscillation of a fuzzy difference equation, Advances in difference equations (2004), vol 4, pp. 337-357.

[6] G. Papaschinopoulos, C. J. Schinas, On the dynamics of two exponential type systems of difference equations Comput. Math. Appl.. (2012), PP. 2326-2334.

[7] G. Papaschinopoulos, B.K. Papadopoulos, On the fuzzy difference equation $x_{n+1} = A + x_n/x_{n-m}$, Fuzzy Sets and Systems 129 (2002) 7381.

[8] G. Papaschinopoulos, B.K. Papadopoulos, On the fuzzy difference equation $x_{n+1} = A + B/x_n$, Soft Comput. 6 (2002) 456461.

[9] G. Papaschinopoulos, C.J. Schinas, On the difference equation $x_{n+1} = \sum_{i=0}^{k-1} A_i/x_{n-i}^{pi} + 1/x_{n-k}^{p_k}$, J. Difference Equations Appl. 6 (2000) 7589.

[10] G. Papschinopoulos and Stefanidou, Boundedness and asymptotic behavior of the solutions of a fuzzy difference equation", Fuzzy sets and systems (2003), vol 140, pp. 523-539.

[11] G. Papaschinopoulos, C.J. Schinas, On the system of two difference equations $x_{n+1} = \sum_{i=0}^{k} A_i/y_{n-i}^{p_i}, y_{n+1} = \sum_{i=0}^{k} Bi/x_{n-i}^{q_i}$, J. Math. Anal. Appl., to appear.

[12] M. R. S. Kulenvic, G. Ladas and N. R. Prokup, A rational difference equation, Comput. Math. Appl., 41 (2001), 671-678.

[13] Q. H Zhang, L. H. Yang and X. Liao, Behavior of solutions to a fuzzy nonlinear difference equation, Iranian journal of fuzzy systems (2012), vol 9, No 2, pp. 1-12.

[14] R. Devault, G. Ladas, S.W. Schultz, Necessary and sufficient conditions for the boundedness of $x_{n+1} = Ax_{p_n} + B/x_{q_{n+1}}$, J. Difference Equations Appl. 3 (1998) 259266.

[15] S. Abbasbandy and Alavi. M, A method for solving fuzzy linear system, Iranian Journal of Fuzzy Systems, 2 (2005), 37-43.

[16] Y. Li, Y. Chen, I. Podlubny, Stability of fractional-order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag-Leffler stability, Comput. Math. Appl. 59(2010) 1810-1821.