Tamsui Oxford Journal of Informational and Mathematical Sciences 31(2) (2017) 104-111
Aletheia University

AN EXTREME POINT THEOREM ON HYPERSPACE

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Received March 2, 2017 Accepted August 18, 2017

2000 Mathematics Subject Classification. Primary 54A05, 54A20. Secondary 54B20.
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Abstract. Let X be a locally convex topological vector space endowed with original topology τ as well as corresponding weak topology τ_W. Suppose $WCC(X)$ is the collection of all non-empty weakly compact (τ_W-compact) convex subsets of X. We shall introduce a certain weak topology $\mathcal{T}_W$ on $WCC(X)$ and prove an extreme point theorem which is an extension of the classical Krein-Milman Theorem.

1. Introduction

Suppose X is a Banach space equipped with the norm topology (denoted by $\|\cdot\|$) as well as the weak topology (denoted by τ_w). Let $CC(X) = \{A \subseteq X : A$ is a non-empty compact convex subset of $X\}$, $WCC(X) = \{A \subseteq X : A$ is a non-empty weakly compact, convex subset of $X\}$ and $BCC(X) = \{A \subseteq X : A$ is a non-empty bounded, closed, convex subset of $X\}$. Then $(CC(X), h)$, $(WCC(X), h)$ and $(BCC(X), h)$ are known as the hyperspaces over the underlying space $(X, \|\cdot\|)$. If $\overline{X} = \{\overline{x} = \{x\} : x \in X\}$, then $(\overline{X}, h)$ is isometrically isomorphic to the underlying space $(X, \|\cdot\|)$. Thus every theorem proved on the hyperspaces is a natural extension of its corresponding counterpart of the underlying space $(X, \|\cdot\|)$.

Blaschke [2] proved that every infinite sequence $\{A_n\}$ with $A_n \in \mathcal{K}$ where $\mathcal{K}$ is an h-bounded and h-closed subset of the hyperspace $(CC(\mathbb{R}^n), h)$ contains a convergent subsequence $\{A_{n_i}\}$ (*i.e.*, there exists a subsequence $\{A_{n_i}\} \subseteq \mathcal{K}$ and $A_0 \in \mathcal{K}$ such that $\lim_{i\to\infty} A_{n_i} \overset{h}{=} A_0$, or $h(A_{n_i}, A_0) \to 0$ as $i \to \infty$). Blaschke's Theorem is an extension of the classical Heine-Borel Theorem which states that every closed and bounded subset $K \subseteq \mathbb{R}^n$ is sequentially compact. Many mathematicians have studied convergence of convex sets on different spaces ([1], [11], [12]).

In 1986, De Blasi and Myjak ([4]) introduced the concept of weak sequential convergence on the hyperspace $WCC(X)$. Suppose $A_n, A \in WCC(X)$, they define A_n converges to A_0 weakly $\left(A_n \overset{w}{\to} A_0\right)$ if and only if $\sigma_{A_n}(x^*) \to \sigma_{A_0}(x^*) = \sup\{x^*(a) \mid a \in A_0\}$ and proved an infinite dimensional version of Blaschke's Theorem and other results. The notion of weak topology $\mathcal{T}_w$ has been introduced and investigated by Hu and company ([3], [7], [8], [9], [10]). They showed that Browder-Kirk's fixed point theorem can be extended to the hyperspace $WCC(X)$ equipped with Hausdorff metric h as well as a certain weak topology $\mathcal{T}_w$ and many other results. We remind the readers that many fundamental results that are valid on the underlying space X cannot be extended to hyperspace. For example, it is well-known that every $\|\cdot\|$-closed (originally closed, strongly closed) convex set is

Key words and phrases. Extreme point, Locally convex topological vector space, Weak topology, Hyperspace, Banach spaces.

also τ_w-closed (weakly closed). Also for every compact convex set $K \subseteq X$ and $x \notin K$, there exists some $x^* \in X^*$ such that $d(x^*(x), x^*(K)) = \inf_{k \in K} d(x^*(x), x^*(k)) = \inf_{k \in K} \{|x^*(x) - x^*(k)|\} = \delta > 0$. Examples have been given in [7] that these results cannot be extended to hyperspace.

Suppose now X is a locally convex topological vector space and X^* its dual space. Let X be equipped with the original topology τ as well as the weak topology τ_w, and $WCC(X) = \{A \subseteq X : A$ is a τ_w-compact convex subset of $X\}$ is the corresponding hyperspace. A topology $\mathcal{T}_w$ will be introduced on X and the main result of this paper is to show that every $\mathcal{T}_w$-compact, convex subset of $WCC(X)$ has an extreme point. This result is an extension of the classical Krein-Milman Theorem.

2. Notations and preliminaries

Let X be a Banach space and X^* its topological dual, and $BCC(X)$ is the collection of all non-empty bounded, closed, convex subsets of X. In general we have $CC(X) \subsetneqq WCC(X) \subsetneqq BCC(X)$. For reflexive Banach space X, we have $WCC(X) = BCC(X)$. If X is finite dimensional, then $CC(X) = WCC(X) = BCC(X)$. To avoid avoid confusion we shall use small letters $a, b, c, \cdots, z$ to denote elements of the underlying space X, capital letters $A, B, \cdots, Z$ to denote elements of the hyperspaces $CC(X), WCC(X)$ and $BCC(X)$ as well as subsets of X, *e.g.*, $A, B \subseteq X$ and $A, B \in BCC(X)$. We shall use script letters to denote subsets of the corresponding hyperspaces, *e.g.*, $\mathcal{K} \subseteq BCC(X), \mathcal{W} \subseteq BCC(X)$. For $A, B \in BCC(X)$, let $A + B = \{a + b : a \in A, b \in B\}$, $N(A, \varepsilon) = \{x \in X : d(x, a) = \|x - a\| < \varepsilon$ for some $a \in A\}$ and $h(A, B) = \inf\{\varepsilon > 0 : A \subseteq N(B, \varepsilon), B \subseteq N(A, \varepsilon)\}$, equivalently, $h(A, B) = \max\left\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\right\}$. The metric h just defined is known as the Hausdorff metric and $(BCC(X), h)$ is known to be a complete metric space. Since h is induced by the $\|\cdot\|$ of the underlying space X, h is closely related to the norm $(\|\cdot\|)$ as well as $x^* \in X^*$. The following lemmas give some elementary properties of the Hausdorff h and its relationship with them.

Lemma 1. *Suppose* $A, B, C, D \in WCC(X)$ *and* $\alpha \in \mathbb{C}$. *Then we have*

(i) $h(A, \{0\}) = \sup\{\|a\| : a \in A\}$,
(ii) $h(A + B, C + D) \leq h(A, C) + h(B, D)$,
(iii) $h(\alpha A, \alpha B) = |\alpha| h(A, B)$,
(iv) $h([a_1, a_2], [b_1, b_2]) = \max\{|b_1 - a_1|, |b_2 - a_2|\}$ *for* $[a_1, a_2], [b_1, b_2] \in (CC(\mathbb{R}), h)$.

Lemma 2. *Suppose* $A, B \in WCC(X)$ *and* $x^*, y^* \in X^*$. *Then*

(i) $x^*(A), x^*(B) \in (CC(\mathbb{C}), h)$,
(ii) $A = B$ *if and only if* $x^*(A) = x^*(B)$ *for each* $x^* \in X^*$,
(iii) $h(x^*(A), x^*(B)) \leq \|x^*\| h(A, B)$,
(iv) $h(x^*(A), y^*(A)) \leq \|x^* - y^*\| h(A, \{0\})$.

Proof. Since $x^* : (X, \tau_w) \to (\mathbb{C}, \|\cdot\|)$ is continuous and linear, it follows that $x^*(A), x^*(B)$ are compact, convex subsets of $\mathbb{C}$ and (i) is proved.

If $A = B$, then $x^*(A) = x^*(B)$ for each $x^* \in X^*$. Suppose $A \neq B$, without loss of generality, we may assume there exists some $b_0 \in B$ such that $b_0 \notin A$. It follows then from Hahn-Banach Theorem that there exists some $x^* \in X^*$ which separates

b_0 from A, *i.e.*, there exists $x^* \in X^*$ such that $\sup \{\operatorname{Re} x^*(a) : a \in A\} < \operatorname{Re} x^*(b_0)$. That is a contradiction and (ii) is proved.

(iii) and (iv) follow from that

$$\|x^*(a) - x^*(b)\| = \|x^*(a-b)\| \leq \|x^*\| \cdot \|a-b\|,$$
$$\|x^*(a) - y^*(a)\| \leq \|x^* - y^*\| \cdot \|a\|$$

and the definition of Hausdorff metric. □

Now, it follows from Lemma *2* (i) that x^* maps the space $WCC(X)$ into the space $CC(\mathbb{C})$ or $x^* : (WCC(X), h) \to (CC(\mathbb{C}), h)$. Also, by Lemma *2* (iii) that $x^* : (WCC(X), h) \to (CC(\mathbb{C}), h)$ is continuous. Note that both the domain and the range are now hyperspaces endowed with corresponding Hausdorff metric h. Now, recall that the weak topology τ_w on X is defined to be the weakest topology such that each $x^* : (X, \tau_w) \to (\mathbb{C}, |\cdot|)$ is continuous. Analogously, we may define the weak topology on $WCC(X)$ as follows:

Definition 1. *The weak topology $\mathcal{T}_w$ on $WCC(X)$ is defined to be the weakest topology on $WCC(X)$ such that each $x^* : (WCC(X), \mathcal{T}_w) \to (CC(\mathbb{C}), h)$ is continuous. Thus a typical $\mathcal{T}_w$-neighborhood of $A \in WCC(X)$ is denoted by $\mathcal{W}(A; x_1^*, \cdots, x_n^*, \varepsilon) = \{B \in WCC(X); h(x_i^*(B), x_i^*(A)) < \varepsilon \text{ for } i = 1, 2, \cdots, n, \varepsilon > 0\}$.*

As mentioned in the introduction, several results have been extended to the hyperspace $WCC(X)$. In the next section, we shall further extend the notion of hyperspace and its corresponding topology $\mathcal{T}_w$ where the underlying space X is a locally convex topological vector space instead of a Banach space and prove an extreme point theorem which is an extension of the classical Krein-Milman Theorem.

3. Main Results

In this section, X is assumed to be a locally convex topological vector space, X^* its dual space and X is endowed with original topology τ as well as weak topology τ_w. Let $WCC(X) = \{A \subseteq X : A$ is a non-empty weakly compact, convex subset of $X\}$. Since each x^* is also weakly continuous (*i.e.* $x^* : (X, \tau_w) \to (\mathbb{C}, |\cdot|)$ is continuous and linear), it follows that for each $A \in WCC(X)$, $x^*(A)$ is a compact convex subset of the complex plane $\mathbb{C}$. Thus each x^* is a mapping from the set $WCC(X)$ into the metric space $(CC(\mathbb{C}), h)$. Define $\mathcal{T}_w$ to be the weakest topology and $WCC(X)$ such that each $x^* : (WCC(X), \mathcal{T}_w) \to (CC(\mathbb{C}), h)$ is continuous. Denote $\mathfrak{B}(x^*(A), \varepsilon) = \{B \in CC(\mathbb{C}) : h(x^*(A), B) < \varepsilon\}$ for $\varepsilon > 0$.

Some basic properties of the weak topology $\mathcal{T}_w$ are stated in the following lemma.

Lemma 3. *(a) The collection $\{\mathcal{W}(A; x_1^*, x_2^*, \cdots, x_n^*, \varepsilon) : x_i \in X^* \text{ for } i = 1, 2, \cdots, n, \varepsilon > 0\}$ is a local base at $A \in WCC(X)$ where*

$$\begin{aligned} & \mathcal{W}(A; x_1^*, x_2^*, \cdots, x_n^*, \varepsilon) \\ = \; & \{B \in WCC(X); h(x_i^*(B), x_i^*(A)) < \varepsilon \text{ for } i = 1, 2, \cdots, n\} \\ = \; & \bigcap_{i=1}^{n} \mathcal{W}(A; x_i^*, \varepsilon) = \bigcap_{i=1}^{n} (x_i^*)^{-1}(\mathfrak{B}(x_i^*(A), \varepsilon)\}. \end{aligned}$$

(b) $\mathcal{T}_w$ is a Hausdorff topology on $WCC(X)$, i.e., distinct $A, B \in WCC(X)$ have disjoint neighborhoods containing them.

Proof. *(a)* Since $x_i^* : (WCC(X), \mathcal{T}_w) \to (CC(\mathbb{C}), h)$ is continuous, $(\mathfrak{B}(x_i^*(A), \varepsilon)$ is open in $(CC(\mathbb{C}), h)$, we have $(x_i^*)^{-1}(\mathfrak{B}(x_i^*(A), \varepsilon)$ is open in $(WCC(X), \mathcal{T}_w)$, *i.e.*, $\mathcal{W}(A; x_i^*, \varepsilon) = (x_i^*)^{-1}(\mathfrak{B}(x_i^*(A), \varepsilon) \in \mathcal{T}_w$ and $\mathcal{W}(A; x_1^*, x_2^*, \cdots, x_n^*, \varepsilon)$ being the finite intersection of open sets is also open.

(b) Let $A, B \in WCC(X)$ with $A \neq B$. We may assume without loss of generality that there exists some $a \in A$ such that $a \notin B$. Since B is a τ_w-compact subset of the locally convex topological vector space (X, τ_w), it follows from the Hahn-Banach Separation Theorem that there exists some $x^* \in X^*$ such that $\sup_{b \in B} \operatorname{Re} x^*(b) < \operatorname{Re} x^*(a)$. Let $\delta = \operatorname{Re} x^*(a) - \sup_{b \in B} \operatorname{Re} x^*(b) > 0$. We have $\operatorname{Re} x^*(a) - \operatorname{Re} x^*(b) \geq \delta$ for all $b \in B$ which in turn implies that

$$|x^*(a) - x^*(b)| \geq |\operatorname{Re} x^*(a) - \operatorname{Re} x^*(b)| \geq \delta. \tag{1}$$

Suppose $0 < \varepsilon < \delta$ is chosen, we have $|x^*(a) - x^*(b)| > \varepsilon$ for all $b \in B$. Claim that $\mathcal{W}(A; x^*, \frac{\varepsilon}{2}) \cap \mathcal{W}(B; x^*, \frac{\varepsilon}{2}) = \emptyset$. Otherwise, there exists some $D \in \mathcal{W}(A; x^*, \frac{\varepsilon}{2}) \cap \mathcal{W}(B; x^*, \frac{\varepsilon}{2})$ and we have $h(x^*(D), x^*(A)) < \frac{\varepsilon}{2}$, $h(x^*(D), x^*(B)) < \frac{\varepsilon}{2}$. Consequently $x^*(A) \subset N(x^*(D), \frac{\varepsilon}{2})$, $x^*(D) \subset N(x^*(B), \frac{\varepsilon}{2})$. Hence for the given $a \in A$, there exists some $d \in D$ such that $|x^*(a) - x^*(d)| < \frac{\varepsilon}{2}$, and for the $d \in D$, there exists some $b \in B$ such that $|x^*(d) - x^*(b)| < \frac{\varepsilon}{2}$ which in turn implies that

$$|x^*(a) - x^*(b)| \leq |x^*(a) - x^*(d)| + |x^*(d) - x^*(b)| < \varepsilon < \delta.$$

That is a contradiction to the inequality (1) and the proof is complete. □

Suppose $A, B \in WCC(X)$. Then A, B are weakly compact, convex subsets of X. Since addition and scalar multiplication are continuous operations on (X, τ_w), we have $A + B, \alpha A$ are weakly compact, convex subsets of X, *i.e.*, $A + B, \alpha A \in WCC(X)$. Thus we may define, algebraic line segments, convex sets, extremal subsets and extreme points on the hyperspace $WCC(X)$ analogous to their counterparts on the underlying space X.

Definition 2. *(a)* $[A, B] = \{\alpha A + (1 - \alpha)B : A, B \in WCC(X), 0 \leq \alpha \leq 1\}$ *is called the closed line segment joining* A *and* B.

(b) A subset $\mathcal{K} \subset WCC(X)$ *is said to be convex if and only if* $A_1, A_2, \cdots, A_n \in \mathcal{K}$ *implies* $\sum_{i=1}^{n} \alpha_i A_i \in \mathcal{K}$ *where* $\alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1$.

(c) A mapping $T : WCC(X) \to WCC(X)$ *is said to be affine if and only if* $T(\alpha A + (1 - \alpha)B) = \alpha T(A) + (1 - \alpha)T(B)$ *where* $0 \leq \alpha \leq 1$.

(d) Suppose $\mathcal{K}_1, \mathcal{K}_2 \subset (WCC(X), \mathcal{T}_w)$ *are closed* ($\mathcal{T}_w$*-closed), convex subsets. Then* $\mathcal{K}_1$ *is said to be an extremal subset of* $\mathcal{K}_2$ *if and only if* $A, B \in \mathcal{K}_2$ *and* $\alpha A + (1 - \alpha)B \in \mathcal{K}_1$ *for some* $0 < \alpha < 1$ *implies that* $A, B \in \mathcal{K}_1$.

(e) Suppose $\mathcal{K}$ *is a* $\mathcal{T}_w$*-closed, convex subset of* $WCC(X)$. *Then* P *is said to be an extreme point of* $\mathcal{K}$ *if and only if* $A, B \in \mathcal{K}$, $0 < \alpha < 1$, $\alpha A + (1 - \alpha)B = P$ *implies* $A = B = P$.

We state the following lemmas whose proofs are similar as in the underlying space X.

Lemma 4. *Suppose* $\mathcal{K}$ *is a* $\mathcal{T}_w$*-closed, convex subset of the hyperspace* $(WCC(X), \mathcal{T}_w)$. *Then* *(a) If $P \in \mathcal{K}$, then P is an extreme point of $\mathcal{K}$ if and only if $\{P\}$ is an extremal subset of $\mathcal{K}$.*

(b) If $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_3$ are $\mathcal{T}_w$-closed, convex subsets, $\mathcal{K}_1$ is an extremal subset of $\mathcal{K}_2$, and $\mathcal{K}_2$ is an extremal subset of $\mathcal{K}_3$, then $\mathcal{K}_1$ is an extremal subset of $\mathcal{K}_3$.

The next lemma is essential in the proof of our main theorem.

Lemma 5. *(a) Suppose $[a_1, a_2], [b_1, b_2] \in \left(CC\left(\mathbb{R}^1\right), h\right)$. Then $h([a_1, a_2], [b_1, b_2]) = \max\{|b_1 - a_1|, |b_2 - a_2|\}$, and the mapping $T : \left(CC\left(\mathbb{R}^1\right), h\right) \to (\mathbb{R}^1, |\cdot|)$ defined by $T([a_1, a_2]) = a_2$ is a continuous (in fact, nonexpansive), affine mapping.*

(b) Suppose $\overline{X} = \{\overline{x} = \{x\} : x \in X\} \subset WCC(X)$. Then the mapping $T : X \to \overline{X} \subset WCC(X)$ defined by $Tx = \overline{x}$ is an isomorphic homeomorphism of (X, τ_w) onto $(\overline{X}, \mathcal{T}_w)$.

Proof. *(a)* That $h([a_1, a_2], [b_1, b_2]) = \max\{|b_1 - a_1|, |b_2 - a_2|\}$ follows immediately from the definition of Hausdorff metric. Next, $[a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$, $\alpha[a_1, a_2] = [\alpha a_1, \alpha a_2]$ implies

$$\begin{aligned} & T(\alpha[a_1, a_2] + (1-\alpha)[b_1, b_2]) \\ = \; & T([\alpha a_1, \alpha a_2] + [(1-\alpha)b_1, (1-\alpha)b_2]) \\ = \; & T([\alpha a_1 + (1-\alpha)b_1, \alpha a_2 + (1-\alpha)b_2]) \\ = \; & \alpha a_2 + (1-\alpha)b_2 \\ = \; & \alpha T([a_1, a_2]) + (1-\alpha)T([b_1, b_2]) \end{aligned}$$

where $0 < \alpha < 1$. Consequently T is affine. Finally

$$\begin{aligned} & |T([a_1, a_2]) - T([b_1, b_2])| \\ = \; & |a_2 - b_2| \\ \leq \; & \max\{|a_1 - b_1|, |a_2 - b_2| = h([a_1, a_2], [b_1, b_2]) \end{aligned}$$

showing that T is nonexpansive.

(b) Obviously, $T : X \to \overline{X}$ is one to one and onto. Also $T(x+y) = \overline{x+y} = \{x+y\} = \{x\} + \{y\} = \overline{x} + \overline{y} = T(x) + T(y)$ and $T(\alpha x) = \overline{\alpha x} = \{\alpha x\} = \alpha\{x\} = \alpha\overline{x} = \alpha T(x)$ showing that T is linear.

$y \in w(x; x_1^*, \cdots, x_n^*, \varepsilon)$ implies that $|x_i^*(y) - x_i^*(x)| < \varepsilon$ for $i = 1, 2, \cdots, n$, which in turn implies that $h(x_i^*(\overline{y}), x_i^*(\overline{x})) = h(x_i^*(Ty), x_i^*(Tx) < \varepsilon$ for $i = 1, 2, \cdots, n$. Hence $Ty = \overline{y} \in \mathcal{W}(\overline{x}; x_1^*, \cdots, x_n^*, \varepsilon)$. Similarly $\overline{y} \in \mathcal{W}(\overline{x}; x_1^*, \cdots, x_n^*, \varepsilon)$ implies $y \in w(x; x_1^*, \cdots, x_n^*, \varepsilon)$. Consequently $T(w(x; x_1^*, \cdots, x_n^*, \varepsilon)) = \mathcal{W}(\overline{x} = Tx; x_1^*, \cdots, x_n^*, \varepsilon)$ and the proof is complete. □

Our main theorem is an extension of the classical Krein-Milman's Extreme Point Theorem to the hyperspace $(WCC(X), \mathcal{T}_w)$.

Theorem 1. *Suppose X is a locally convex topological vector space equipped with original topology τ as well as weak topology τ_w, and $(WCC(X), \mathcal{T}_w)$ is the corresponding hyperspace. Suppose $\mathcal{K}$ is a $\mathcal{T}_w$-compact, convex subset of $(WCC(X), \mathcal{T}_w)$. Then $\mathcal{K}$ has an extreme point in $\mathcal{K}$.*

Proof. Let Ω denote the collection of all non-empty, $\mathcal{T}_w$-closed, convex subsets of $\mathcal{K}$. $\Omega \neq \emptyset$ since $\mathcal{K} \in \Omega$. Define a partial order in Ω by inverse inclusion, *i.e.*, $\mathcal{K}_2 \leq \mathcal{K}_1$ if and only if $\mathcal{K}_1 \subset \mathcal{K}_2$. If $\{\mathcal{K}_i\}_{i \in I} \subset \Omega$ is a totally ordered subset, we shall show that $\mathcal{K}_0 = \bigcap\limits_{i \in I} \mathcal{K}_i$ is an upper bound of $\{\mathcal{K}_i\}_{i\in I}$. Each $\mathcal{K}_i$ is $\mathcal{T}_w$-compact, convex and $\{\mathcal{K}_i\}_{i\in I}$ has finite intersection implies that $\mathcal{K}_0$ is a non-empty $\mathcal{T}_w$-compact, convex set. Suppose we have $A, B \in \mathcal{K}, 0 < \alpha < 1$ and $\alpha A + (1-\alpha)B \in \mathcal{K}_0$. Since $\mathcal{K}_0 \subset \mathcal{K}_i$ for each i, we have $\alpha A + (1-\alpha)B \in \mathcal{K}_i$ which in turn implies that $A, B \in \mathcal{K}_i$ because $\mathcal{K}_i$ is an extremal subset of $\mathcal{K}$. Thus $A, B \in \mathcal{K}_0$ showing that $\mathcal{K}_0$ is an extremal subset of $\mathcal{K}$ and consequently $\mathcal{K}_0$ is an upper bound of $\{\mathcal{K}_i\}_{i\in I}$. It follows now from Zorn's Lemma that Ω has a maximal element, denoted by $\mathcal{K}_\infty$. We claim that $\mathcal{K}_\infty$ is a singleton. Otherwise, there exists $A_0, B_0 \in \mathcal{K}_\infty$ with $A_0 \neq B_0$, without loss of generality, assume there exists some $b_0 \in B_0$ such that $b_0 \notin A_0$. By Hahn-Banach Separation Theorem, there exists some $x^* \in X^*$ such that $\sup\limits_{a\in A_0} \operatorname{Re} x^*(a) < \operatorname{Re} x^*(b_0)$. Let $\operatorname{Re} x^*(A_0) = [a_1, a_2]$, $\operatorname{Re} x^*(B_0) = [b_1, b_2] \in CC(\mathbb{R})$, we have $a_2 = \sup\limits_{a\in A_0} \operatorname{Re} x^*(a) < \operatorname{Re} x^*(b_0) \leq b_2$. Define $\mathbb{G} : (CC(\mathbb{R}), h) \to (\mathbb{R}, |\cdot|)$ by $\mathbb{G}([a_1, a_2]) = a_2$. It follows from Lemma 5 (a) that $\mathbb{G}$ is a nonexpansive (hence continuous) affine mapping.

Next, let $\mathbb{F} : (WCC(X), \mathcal{T}_w) \to (\mathbb{R}, |\cdot|)$ be defined by $\mathbb{F}(A) = \mathbb{G}(\operatorname{Re} x^*(A))$. $\mathbb{F} : (\mathcal{K}_\infty, \mathcal{T}_w) \to (\mathbb{R}, |\cdot|)$ is continuous implies $\mathbb{F}$ attains its maximum on $\mathcal{K}_\infty$, *i.e.*, there exists $b_\infty \in \mathbb{R}$ and $B_\infty \in \mathcal{K}_\infty$ such that $\mathbb{F}(B_\infty) = b_\infty = \sup\limits_{A\in\mathcal{K}_\infty} \mathbb{F}(A) = \max\limits_{A\in\mathcal{K}_\infty} \mathbb{F}(A)$. Since $a_2 < b_2 \leq b_\infty$, $A_0 \notin \mathbb{F}^{-1}(b_\infty)$. Claim that $\mathbb{F}^{-1}(b_\infty)$ is an extremal subset of $\mathcal{K}_\infty$. For that purpose, we let $D, E \in \mathcal{K}_\infty$, $0 < \alpha < 1$ with $\alpha D + (1-\alpha)E \in \mathbb{F}^{-1}(b_\infty)$. Thus $\mathbb{F}(\alpha D + (1-\alpha)E) = b_\infty$ which implies that $\alpha\mathbb{F}(D) + (1-\alpha)\mathbb{F}(E) = b_\infty$. Also $D, E \in \mathcal{K}_\infty$ implies that $\mathbb{F}(D), \mathbb{F}(E) \leq b_\infty$ and consequently, $\alpha\mathbb{F}(D) + (1-\alpha)\mathbb{F}(E) \leq b_\infty$. Hence $\mathbb{F}(D), \mathbb{F}(E) = b_\infty$ that implies $D, E \in \mathbb{F}^{-1}(b_\infty)$. Otherwise, we would have $\alpha\mathbb{F}(D) + (1-\alpha)\mathbb{F}(E) < b_\infty$, contradicting that $\alpha\mathbb{F}(D) + (1-\alpha)\mathbb{F}(E) = b_\infty$. Now that $\mathbb{F}^{-1}(b_\infty) \subset \mathcal{K}_\infty$, and $\mathbb{F}^{-1}(b_\infty)$ is an extremal subset of $\mathcal{K}_\infty$ implies $\mathbb{F}^{-1}(b_\infty) \subset \mathcal{K}_\infty$. But $A_0 \notin \mathbb{F}^{-1}(b_\infty)$ implies $\mathbb{F}^{-1}(b_\infty) \subsetneqq \mathcal{K}_\infty$ contradicting that $\mathcal{K}_\infty$ is a maximal element. Hence $\mathcal{K}_\infty$ is a singleton, say $\mathcal{K}_\infty = \{P\}$ proving that P is an extreme point of $\mathcal{K}$ and the proof is complete. □

The following corollary is the classical Krein-Milman extreme point theorem.

Corollary 1. *Let K be a non-empty compact, convex subset of a locally convex topological vector space X. Then K has an extreme point in K.*

Proof. Let X a locally convex topological vector space endowed with original topology τ as well as weak topology τ_w, and $(WCC(X), \mathcal{T}_w)$ is the corresponding hyperspace. K is τ-compact implies K is τ_w-compact. It follows from Lemma 5 (b) that $\overline{K} = \{\overline{x} = \{x\} : x \in K\}$ is a $\mathcal{T}_w$-compact, convex subset of $(WCC(X), \mathcal{T}_w)$ and hence has an extreme point $\overline{P} = \{p\}$ by Theorem 1. Consequently p is an extreme point of K and the proof is complete. □

Remark 1. *(a) Since Krein-Milman Theorem has numerous important applications in various branches of mathematics, we hope further investigation on the hyperspace $(WCC(X), \mathcal{T}_w)$ will lead to some useful applications.*

(b) The study of convex sets has always been interesting and useful. However, the traditional method has relied heavily on support fucntionals. With the $\mathcal{T}_w$-topology defined on $WCC(X)$, we hope it will provide an altermative way to study convex sets.

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