Tamsui Oxford Journal of Informational and Mathematical Sciences 31(2) (2017) 93-103
Aletheia University

Characterization on $N(k)$-Mixed Quasi-Einstein Manifold

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Received May 26, 2017 Accepted December 14, 2017

Mathematics Subject Classification 2010: 53C25.

Abstract

In the present paper we study characterizations of odd and even dimensional $N(k)$-mixed quasi-Einstein manifold and finally prove that a $N(k)$-mixed quasi-Einstein manifold is a semi mixed quasi-Einstein manifold under a certain condi-tion.

Key words : N(k)-mixed quasi Einstein manifold, semi mixed-Einstein manifold.

1. **Introduction**

A Riemannian manifold (M, g) with dimension $(n \geq 2)$ is said to be an Einstein manifold if the Ricci tensor satisfies the condition $S(X,Y) = \frac{r}{n} g(X,Y)$, holds on M, here S and r denote the Ricci tensor and the scalar curvature of (M, g) respectively. According to [8] the above equation is called the Einstein metric condition. Einstein manifolds play an important role in Riemannian Geometry, as well as in general theory of relativity. The notion of quasi-Einstien manifold were defined in ([3],[7],[10],[14],[16],[17]). A non-flat Riemannian manifold $(M,\ g)$, $(n \geq 2)$ is said to be an quasi Einstein manifold if its Ricci tensor S of type $(0,2)$ satisfies the condition

$$S(X,\ Y) = ag(X,Y) + bA(X)A(Y) \tag{1.1}$$

where a and b are scalars of which $b \neq 0$ and A is non-zero 1-form such that $g(X, U) = A(X)$ for all vector field X and U is a unit vector field.

The notion of quasi Einstein manifold was introduced in a paper [10] by M.C.Chaki and R.K.Maity. According to them a non-flat Riemannian manifold (M^n, g),$(n \geq 3)$ is defined to be a quasi Einstein manifold if its Ricci tensor S of type $(0, 2)$ satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) \tag{1.2}$$

and is not identically zero, where a, b are scalars $b \neq 0$ and A is a non-zero 1-form such that

$$g(X, U) = A(X), \ \forall X \ \in TM. \tag{1.3}$$

U being a unit vector field.
In such a case a, b are called the associated scalars. A is called the associated 1-form and U is called the generator of the manifold. Such an n-dimensional manifold is denoted by the symbol$(QE)_n$.
Again, U.C.De and G.C.Ghosh defined generalized quasi Einstein manifold [12]. A non-flat Riemannian manifold is called a generalized quasi Einstein manifold ([2], [4], [5], [6], [9], [15], [18])if its Ricci-tensor S of type $(0, 2)$ is non-zero and satisfies the condition

$$S(X, Y) = ag(X, Y) + bA(X)A(Y) + cB(X)B(Y) \tag{1.4}$$

where $a, b, c,$ are non-zero scalars and A, B are two 1-forms such that

$$g(X, U) = A(X) \ and \ g(X, V) = B(X) \tag{1.5}$$

U, V being unit vectors which are orthogonal, i.e,

$$g(U, V) = 0. \tag{1.6}$$

The vector fields U and V are called the generators of the manifold. This type of manifold are denoted by $G(QE)_n$.
The k-nullity distribution [29] of a Riemannian manifold M is defined by

$$N(k) : p \to N_p(k) = \{Z \in T_pM \setminus R(X, Y)Z = k(g(Y, Z)X - g(X, Z)Y)\}. \tag{1.7}$$

for all $X, Y \in TM$ and k is a smooth function. M.M.Tripathi and Jeong jik kim [28] introduced the notion of $N(k)$-quasi Einstein manifold which defined as follows: If the generator U belongs to the k-nullity distribution $N(k)$, then a quasi

Einstein manifold (M^n, g) is called $N(k)$-quasi Einstein manifold.
In [26], H.G. Nagaraja introduced the concept of $N(k)$-mixed quasi Einstein manifold and mixed quasi constant curvature.A non flat Riemannian manifold (M^n, g) is called a $N(k)$-mixed quasi Einstein manifold [11] if its Ricci tensor of type $(0, 2)$ is non zero and satisfies the condition

$$S(X,Y) = ag(X,Y) + bA(X)B(Y) + cB(X)A(Y), \tag{1.8}$$

where a, b, c, are smooth functions and A, B are non zero 1-forms such that

$$g(X,U) = A(X) \ and \ g(X,V) = B(X) \ \forall \ X, \tag{1.9}$$

U, V being the orthogonal unit vector fields called generators of the manifold belong to $N(k)$.Such aa manifold is denoted by the symbol $N(k) - (MQE)_n$.
Again a Riemannian manifold (M^n, g) is called of mixed quasi constant curvature [11] if it is conformally flat and curvature tensor $\acute{R}$ of type $(0, 4)$ satisfies the condition

$$\begin{aligned}\acute{R}(X,Y,Z,W) = p[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]\\ +q[g(X,W)A(Y)B(Z) - g(X,Z)A(Y)B(W)\\ +g(X,W)A(Z)B(Y) - g(X,Z)A(W)B(Y)]\\ +s[g(Y,Z)A(W)B(X) - g(Y,W)A(Z)B(X)\\ +g(Y,Z)A(X)B(W) - g(Y,W)A(X)B(Z)].\end{aligned} \tag{1.10}$$

Let M be an m-dimensional, $m \geq 3$, Riemannian manifold and $p \in M$. Denote by $K(\pi)$ or $K(U \wedge V)$ the sectional curvature of M associated with a plane section $\pi \subseteq T_pM$, where $\{U, V\}$ is an orthonormal basis of π. For a n-dimensional subspace $L \subseteq T_pM$, $2 \leq n \leq m$, its scalar cuvrvature $\tau(L)$ is denoted by $\tau(L) = \sum_{1 \leq i < j \leq n} K(e_i \wedge e_j)$, where $\{e_1, e_2, \ldots, e_n\}$ is any orthonormal basis of L([14]).
In [13] the result for odd dimensional Einstein spaces was obtained by Dumitru. Also in [7] Bejan generalized these results (both odd and even dimensions)to quasi Einstein manifold. Also characterization of super quasi-Einstein manifold for both of odd and even dimensions was studied in [20]. From above studies, we have given characterization of $N(k)$-mixed quasi-Einstein manifold for both of odd and even dimensions. Next we obtain that a $N(k)$-mixed quasi-Einstein manifold is semi mixed quasi-Einstein manifold if either of generators is parallel vector field.

Geodesic mappings of Einstein spaces were studied in ([19],[21],[22],[23],[24],[25]).

2. Characterization of $N(k)$-mixed quasi-Einstein manifold manifold

In this section we establish the characterization of odd and even dimensional $N(k)-(MQE)_n$.

Theorem 2.1. *A Riemannian manifold of dimension $(2n+1)$ with $n \geq 2$ is $N(k)$-mixed quasi-Einstein manifold if and only if the Ricci operator Q has eigen vector fields U and V such that at any point $p \in M$, there exist three real numbers α and β satisfying*

$$\tau(P) + \alpha = \tau(P^{\perp});\ U,\ V \in T_pP^{\perp},$$

$$\tau(N) + \alpha = \tau(N^{\perp});\ U \in T_pN,\ V \in T_pN^{\perp},$$

$$\tau(R) + \beta = \tau(R^{\perp});\ U \in T_pR,\ V \in T_pR^{\perp},$$

for any n-plane sections P, N and $(n+1)$-plane section R where $P^{\perp}, N^{\perp}$ and $R^{\perp}$ denote the orthogonal complements of P, N and R in T_pM respectively and

$$\alpha = a/2,\ \beta = -c/2,$$

where a, b, c are scalars.

Proof. First suppose that M is a $(2n+1)$ dimensional $N(k)$-mixed quasi-Einstein manifold, so

$$S(X,Y) = ag(X,Y) + bA(X)B(Y) + cA(Y)B(X), \tag{2.1}$$

where a, b, c are scalars such that b, c are nonzero and A, B are two nonzero 1-forms such that $g(X,U) = A(X)$ and $g(X,V) = B(X)$, $\forall X \in \chi(M)$, U, V being unit vectors which are orthogonal, i.e., $g(U,V) = 0$.

Let $P \subseteq T_pM$ be an n-dimensional plane orthogonal to U, V and let $\{e_1, e_2, \ldots, e_n\}$ be orthonormal basis of it. Since U and V are orthogonal to P, we can take orthonormal basis $\{e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ of $P^{\perp}$ such that $e_{2n} = U$ and $e_{2n+1} = V$. Thus $\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}, \ldots, e_{2n+1}\}$ is an orthonormal basis of T_pM.Then we can take $X = Y = e_i$ in (2.1), we have

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} a, & \text{for} \quad 1 \leq i \leq 2n-1 \\ a, & \text{for} \quad i = 2n \\ a, & \text{for} \quad i = 2n+1 \end{cases}$$

By use of (2.1) for any $1 \leq i \leq 2n+1$, we can write

$$S(e_1, e_1) = K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \cdots + K(e_1 \wedge e_{2n-1}) + K(e_1 \wedge U) + K(e_1 \wedge V) = a,$$

$$S(e_2, e_2) = K(e_2 \wedge e_1) + K(e_2 \wedge e_3) + \cdots + K(e_2 \wedge e_{2n-1}) + K(e_2 \wedge U) + K(e_2 \wedge V) = a,$$

....

$$S(e_{2n-1}, e_{2n-1}) = K(e_{2n-1} \wedge e_1) + K(e_{2n-1} \wedge e_2) + K(e_{2n-1} \wedge e_3) + \cdots + K(e_{2n-1} \wedge V) = a,$$

$$S(U,U) = K(U \wedge e_1) + K(U \wedge e_2) + \cdots + K(U \wedge e_{2n-1}) + K(U \wedge V) = a,$$

$$S(V,V) = K(V \wedge e_1) + K(V \wedge e_2) + \cdots + K(V \wedge e_{2n-1}) + K(V \wedge U) = a.$$

Adding first n-equations, we get

$$2\tau(P)+\sum_{1\leq i\leq n<j\leq 2n+1} K(e_i\wedge e_j)=na. \tag{2.2}$$

Then adding the last $(n+1)$-equations, we have

$$2\tau(P^{\perp})+\sum_{1\leq j\leq n<i\leq 2n+1} K(e_i\wedge e_j)=(n+1)a. \tag{2.3}$$

Then, by substracting the equation (2.2) and (2.3), we obtain

$$\tau(P^{\perp})-\tau(P)=a/2.$$

Therefore $\tau(P)+\alpha=\tau(P^{\perp})$, where,

$$\alpha=a/2.$$

Similarly, Let $N\subseteq T_pM$ be an n-dimensional plane orthogonal to V and let $\{e_1,e_2,\ldots,e_n\}$ be orthonormal basis of it. Since V is orthogonal to N, we can take an orthonormal basis $\{e_{n+1},e_{n+2},\ldots,e_{2n+1}\}$ of $N^{\perp}$ orthogonal to U, such that $e_n=U$ and $e_{2n+1}=V$, respectively. Thus, $\{e_1,e_2,\ldots,e_n,e_{n+1},e_{n+2},\ldots,e_{2n+1}\}$ is an orthonormal basis of T_pM. Then we can take $X=Y=e_i$ in (2.1) to have

$$S(e_i,e_i)=\sum_{j=1}^{2n+1} R(e_j,e_i,e_i,e_j)=\begin{cases} a, & 1\leq i\leq n-1\\ a, & i=n\\ a, & n+1\leq i\leq 2n\\ a, & i=2n+1\end{cases}$$

Adding first n-equations, we get

$$2\tau(N)+\sum_{1\leq i\leq n<j\leq 2n+1} K(e_i\wedge e_j)=na, \tag{2.4}$$

and adding the last $(n+1)$-equations, we have

$$2\tau(N^{\perp})+\sum_{1\leq j\leq n<i\leq 2n+1} K(e_i\wedge e_j)=(n+1)a. \tag{2.5}$$

Then, by substracting the equation (2.4) and (2.5), we obtain

$$\tau(N^{\perp})-\tau(N)=a/2.$$

Therefore $\tau(N)+\alpha=\tau(N^{\perp})$, where,

$$\alpha=a/2.$$

Analogously, Let $R\subseteq T_pM$ be an $(n+1)$-plane orthogonal to V and let $\{e_1,e_2,\ldots,e_{n+1}\}$ be orthonormal basis of it. Since V is orthogonal to R, we can take an orthonormal basis $\{e_{n+2},e_{n+3},\ldots,e_{2n},e_{2n+1}\}$ of $R^{\perp}$ orthogonal to U, such that $e_{n+1}=U$

and $e_{2n+1} = V$. Thus, $\{e_1, e_2, \dots, e_n, e_{n+1}, e_{n+2}, \dots, e_{2n+1}\}$ is an orthonormal basis of T_pM. Then we can take $X = Y = e_i$ in (2.1) to have

$$S(e_i, e_i) = \sum_{j=1}^{2n+1} R(e_j, e_i, e_i, e_j) = \begin{cases} a, & 1 \le i \le n \\ a, & i = n+1 \\ a, & n+2 \le i \le 2n \\ a, & i = 2n+1 \end{cases}$$

Adding the first $(n+1)$-equations, we get

$$2\tau(R) + \sum_{1 \le i \le n+1 < j \le 2n+1} K(e_i \wedge e_j) = (n+1)a, \tag{2.6}$$

and adding the last n-equations, we have

$$2\tau(R^\perp) + \sum_{1 \le j \le n+1 < i \le 2n+1} K(e_i \wedge e_j) = na. \tag{2.7}$$

Then, by substracting the equation (2.6) and (2.7), we obtain

$$\tau(R^\perp) - \tau(R) = -a/2.$$

Therefore $\tau(R) + \beta = \tau(R^\perp)$, where,

$$\beta = -a/2.$$

Conversely, let V_1 be an arbitrary unit vector of T_pM, at $p \in M$, orthogonal to U and V. We take an orthonormal basis $\{e_1, e_2, \dots, e_n, e_{n+1}, e_{n+2}, \dots, e_{2n+1}\}$ of T_pM such that $V_1 = e_1, e_{n+1} = U$ and $e_{2n+1} = V$. We consider n-plane section N and $(n+1)$-plane section R in T_pM as follows $N =$ span $\{e_2, \dots, e_n, e_{n+1}\}$ and $R =$ span $\{e_1, e_2, \dots, e_n, e_{n+1}\}$ respectively.Then we have $N^\perp =$ span $\{e_1, e_{n+2}, \dots, e_{2n}, e_{2n+1}\}$ and $R^\perp =$ span $\{e_{n+2}. \dots, e_{2n}\}$ respectively. Now

$$\begin{aligned} S(V_1, V_1) &= [K(e_1 \wedge e_2) + K(e_1 \wedge e_3) + \cdots + K(e_1 \wedge e_{n+1})] \\ &\quad + [K(e_1 \wedge e_{n+2}) + \vdots + K(e_1 \wedge e_{2n}) + K(e_1 \wedge e_{2n+1})] \\ &= [\tau(R) - \sum_{2 \le i < j \le n+1} K(e_i \wedge e_j)] + [\tau(N^\perp) - \sum_{n+2 \le i < j \le 2n+1} K(e_i \wedge e_j)] \end{aligned}$$

$$= \tau(R) - \tau(N) + \tau(N^\perp) - \tau(R^\perp) = [\tau(R) - \tau(N)] + [\tau(N) + \alpha - \tau(R) - \beta] = \alpha - \beta.$$

Therefore, $S(V_1, V_1) = \alpha - \beta$, for any unit vector $V_1 \in T_pM$, orthogonal to U and V. Then we can write for any $1 \le i \le 2n+1$, $S(e_i, e_i) = \alpha - \beta$, since $S(V_1, V_1) = (\alpha - \beta)g(V_1, V_1)$. It follows that $S(X, X) = (\alpha - \beta)g(X, X)$ and $S(Y, Y) = (\alpha - \beta)g(Y, Y) + K_1 A(Y)B(Y) + K_2 B(Y)A(Y)$ for any $X \in [\text{span}\{U\}]^\perp$ and $Y \in [\text{span}\{V\}]^\perp$, where A, B are the dual forms of U and V with respect to g, respectively and K_1, K_2 are scalars, such that $K_1 \neq 0, K_2 \neq 0$.

Now from the above equations, we get from symmetry that S with tensors $(\alpha - \beta)g$ and $(\alpha - \beta) + K_1(A \otimes B) + K_2(A \otimes B)$ must coincide on the complement of U and V, respectively, that is $S(X,Y) = (\alpha-\beta)g(X,Y) + K_1A(X)B(Y) + K_2B(X)A(Y)$, for any $X, Y \in [\text{span}\{U,V\}]^{\perp}$. Since U and V are eigenvector fields of Q, we also have $S(X,U) = 0$ and $S(Y,V) = 0$ for any $X, Y \in T_pM$ orthogonal to U and V. Thus, we can extend the above equation to

$$S(X,Z) = (\alpha-\beta)g(X,Z) + K_1A(X)B(Z) + K_2A(Z)B(X) \tag{2.8}$$

, for any $X \in [\text{span}\{U,V\}]^{\perp}$ and $Z \in T_pM$, where K_1, K_2, are scalars and $K_1 \neq 0, K_2 \neq 0$,. Now, let us consider the n-plane section P and $(n+1)$-plane section R in T_pM as follows $P = \text{span}\ \{e_1, e_2, \ldots, e_n\}$ and $R = \text{span}\ \{e_1, e_2, \ldots, e_n, U\}$. Then we have $P^{\perp} = \text{span}\ \{U, e_{n+2}, \ldots, e_{2n+1}\}$ and $R^{\perp} = \text{span}\ \{e_{n+2}, \ldots, e_{2n}, e_{2n+1}\}$ respectively. Now

$$S(U,U) = [K(U\wedge e_1) + K(U\wedge e_2) + \cdots + K(U\wedge e_n)]$$
$$+[K(U\wedge e_{n+2}) + \cdots + K(U\wedge e_{2n}) + K(e_1\wedge e_{2n+1})]$$

$$= [\tau(R) - \sum_{1\leq i<j\leq n} K(e_i\wedge e_j)] + [\tau(P^{\perp}) - \sum_{n+2\leq i<j\leq 2n+1} K(e_i\wedge e_j)]$$

$$= \tau(R)-\tau(P)+\tau(P^{\perp})-\tau(R^{\perp}) = [\tau(R)-\tau(P)]+[\alpha+\tau(P)-\beta-\tau(R)] = \alpha-\beta.$$

Therefore we can write

$$S(U,U) = (\alpha-\beta)g(U,U). \tag{2.9}$$

Analogously, let us consider the n-plane section P and $N \in T_pM$ as follows $P = \text{span}\ \{e_1, e_2, \ldots, e_n\}$ and $N = \text{span}\ \{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ respectively. Then we have $P^{\perp} = \text{span}\ \{e_{n+1}, e_{n+2}, \ldots, e_{2n}, V\}$ and $N^{\perp} = \text{span}\ \{e_1, \ldots, e_n, V\}$ respectively. Now, we have

$$S(V,V) = [K(V\wedge e_1) + K(V\wedge e_2) + \cdots + K(V\wedge e_n)]$$
$$+[K(V\wedge e_{n+1}) + K(V\wedge e_{n+2}) + \cdots + K(e_2\wedge e_{2n})]$$
$$= [\tau(N^{\perp}) - \sum_{1\leq i<j\leq n} K(e_i\wedge e_j)] + [\tau(P^{\perp}) - \sum_{n+1\leq i<j\leq 2n} K(e_i\wedge e_j)]$$

$$= \tau(N^{\perp})-\tau(P)+\tau(P^{\perp})-\tau(N) = [\tau(N)+\alpha-\tau(P)]+[\alpha+\tau(P)-\tau(N)] = 2\alpha.$$

Then, we get

$$S(V,V) = 2\alpha g(V,V) + K_1A(V)B(V) + K_2A(V)B(V). \tag{2.10}$$

Now from (2.8), (2.9) and (2.10) we can write the Ricci tensor by

$$S(X,\ Y) = \ \lambda_1 g(X,\ Y) + K_1A(X)B(Y) + K_2B(X)A(Y), \tag{2.11}$$

for any $X, Y \in T_pM$. From (2.11) it follows that M is a $N(k)$-mixed quasi-Einstein manifold, where λ_1, K_1, K_2, are scalars and $K_1 \neq 0, K_2 \neq 0$,. Hence the theorem is proved.

Theorem 2.2. *A Riemannian manifold of dimension $2n$ with $n \geq 2$ is $N(k)$-mixed quasi-Einstein manifold if and only if the Ricci operator Q has eigen vector fields U and V such that at any point $p \in M$, satisfying*

$$\tau(P) = \tau(P^{\perp});\ U, V \in T_pP^{\perp},$$

$$\tau(N) = \tau(N^{\perp});\ U \in T_pN, V \in T_pN^{\perp},$$

$$\tau(R) = \tau(R^{\perp});\ U \in T_pR, V \in T_pR^{\perp},$$

for any n-plane section P, N and $(n+1)$-plane section R where $P^{\perp}, N^{\perp}$ and $R^{\perp}$ denote the orthogonal complements of P, N and R in T_pM respectively.

Proof. Let P and R be n-plane sections and N be an $(n-1)$-plane section such that, $P = \text{span}\{e_1, e_2, \ldots, e_n\}$, $R = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$ and $N = \text{span}\{e_2, e_3, \ldots, e_n\}$ respectively. Therefore the orthogonal complements of these sections can be written as $P^{\perp} = \text{span}\{e_{n+1}, e_{n+2}, \ldots, e_{2n}\}$, $R^{\perp} = \text{span}\{e_1, e_2, \ldots, e_n\}$ and $N^{\perp} = \text{span}\{e_1, e_{n+1}, \ldots, e_{2n}\}$.
Then rest of the proof is similar to the proof of Theorem 2.1.

3. $N(k) - MQE_n$ with the parallel vector field generators

Theorem 3.1. *A $N(k)$- mixed quasi-Einstein manifold is semi-mixed quasi-Einstein manifold if either of generators is parallel vector field.*

Proof. By the definition of the Riemannian curvature tensor, if U is parallel vector field, then we find that

$$R(X, Y)U = \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U = 0,$$

and consequently we get

$$S(X, U) = 0. \tag{3.1}$$

Again, putting $Y = U$ in the equation (1.8) and applying (1.9) , we have

$$S(X, U) = ag(X, U) + cg(X, V).$$

So, if U is a parallel vector field, by (3.1), we get

$$ag(X, U) + cg(X, V) = 0. \tag{3.2}$$

Now, putting $X = V$ in the equation (3.2) and using (1.9) we get $c = 0$. So, if U is parallel vector field in a mixed-quasi-Einstein manifold, then the manifold is semi-mixed quasi Einstein manifold.

Again, if V is parallel vector field, then $R(X, Y)V = 0$. Contracting, we get

$$S(Y, V) = 0. \tag{3.3}$$

Putting $X = V$ in the equation (1.8) and applying (1.9), we obtain

$$S(Y, V) = ag(Y, V) + bg(Y, U).$$

If, V is a parallel vector field, by (3.3), we get

$$ag(Y, V) + bg(Y, U) = 0. \tag{3.4}$$

Putting $Y = U$ and using (3.4), (1.9), we obtain $b = 0$, i.e., the manifold is semi mixed quasi-Einstein manifold.

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