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Applications of Differential Subordination on The Erdelyi-Kober Type Integral

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Abstract

In this paper, we defined subclasses of $p-$valent meromorphic functions defined by using integral operator. We investigate several interesting subordination properties for this subclasses of multivalent meromorphic functions. Relevant connections of the results which are presented in this paper with various known results are also considered.

Keywords: Multivalent meromorphic functions, differential subordination, linear operator.

1 Introduction

Let $\Sigma_{p,n}$ denote the class of multivalent meromorphic functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=n}^{\infty} a_k z^k \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\};\ n > -p), \tag{1}$$

which are analytic in the punctured unit disk $\mathbb{U}^* = \{z : z \in \mathbb{C},\ \text{and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$. For convenience, we write $\Sigma_{p,-p+1} = \Sigma_p$.

Definition 1.1 *For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, written $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$ which is analytic in $\mathbb{U}$, satisfying the following conditions:*

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1; \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(\omega(z)); \qquad (z \in \mathbb{U}).$$

Indeed it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, we have the following equivalence (see [10], [15], [16]):

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For $\mu > 0$, a, $c \in \mathbb{R}$ be such that $(c-a) \geq 0, a \geq \mu p \, (p \in \mathbb{N})$ and $f(z) \in \Sigma_{p,n}$ given by (1), the modified an Erdelyi-Kober type integral operator $J_{p,\mu}^{a,c} : \Sigma_{p,n} \to \Sigma_{p,n}$ (see [9]).

- For $(c-a) > 0$ by

$$J_{p,\mu}^{a,c} f(z) = \frac{\Gamma(c-\mu p)}{\Gamma(a-\mu p)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} f(zt^{\mu}) dt; \tag{2}$$

- For $a = c$ by

$$J_{p,\mu}^{a,a} f(z) = f(z). \tag{3}$$

Using (2) and (3), the operator $J_{p,\mu}^{a,c} f(z)$ can be expressed as follows

$$J_{p,\mu}^{a,c} f(z) = z^{-p} + \frac{\Gamma(c-\mu p)}{\Gamma(a-\mu p)} \sum_{k=n}^{\infty} \frac{\Gamma(a+\mu k)}{\Gamma(c+\mu k)} a_k z^k, \tag{4}$$

where $\mu > 0$, a, $c \in \mathbb{R}, (c-a) \geq 0, a \geq \mu p \, (p \in \mathbb{N})$.

It is readily verified from (4) that

$$z(J_{p,\mu}^{a,c} f(z))' = \frac{c-\mu p-1}{\mu} J_{p,\mu}^{a,c-1} f(z) - \frac{c-1}{\mu} J_{p,\mu}^{a,c} f(z), \ ((c-a-1) > 0), \tag{5}$$

$$z(J_{p,\mu}^{a,c} f(z))'' = \frac{c-\mu p-1}{\mu} (J_{p,\mu}^{a,c-1} f(z))' - \frac{c+\mu-1}{\mu} (J_{p,\mu}^{a,c} f(z))' \ ((c-a-1) > 0). \tag{6}$$

We also note that the operator $J_{p,\mu}^{a,c} f(z)$ generalizes several previously studied familiar operators, and we will show some of the interesting particular cases as follows

(i) $J_{1,\mu}^{a,c} f(z) = I_{\mu}(a,c) f(z) (a, c \in \mathbb{C}, \mu > 0, Re(a) > \mu, Re(c-a) \geq 0)$(see [6]),

(ii) $J_{p,1}^{a+p,c+p} f(z) = \ell_p(a,c) f(z) (a \in \mathbb{R}, c \in \mathbb{R} \backslash \mathbb{Z}_0^-, \mathbb{Z}_0^- = \{0, 1, 2, ...\}, p \in \mathbb{N})$(see [13]),

(iii) $J_{p,1}^{n+2p,p+1} f(z) = D^{n+p-1} f(z)$ (n is an integer, $n > -p, p \in \mathbb{N}$) (see [1], [3], [23]).

(iV) $J_{p,1}^{a,a+1} f(z) = J_p^a f(z)$ (Re$(a) > p$, $p \in \mathbb{N}$) (see [11]).

Now, we introduce a new subclasses of functions in $\Sigma_{p,n}$, by making use of the linear operator $J_{p,\mu}^{a,c}$ as follows.

Definition 1.2 *A function $f \in \Sigma_{p,n}$ is said to be in the class $\Sigma S^{*a,c}_{p,\mu}(\alpha; A, B)$, if it satisfies*

$$\frac{-1}{p-\alpha}\left(\frac{z(J^{a,c}_{p,\mu}f(z))'}{J^{a,c}_{p,\mu}f(z)}+\alpha\right) \prec \frac{1+Az}{1+Bz},$$
$$(a, c \in \mathbb{R};\ \mu > 0; (c-a) > 0;\ a > \mu p;\ 0 \le \alpha < p;\ -1 \le B < A \le 1;\ z \in \mathbb{U}).$$

In particular, for $A = 1$ and $B = -1$, then

$$\Sigma S^{*a,c}_{p,\mu}(\alpha; 1, -1) = \left\{ f \in \Sigma_{p,n} : -Re\left(\frac{z(J^{a,c}_{p,\mu}f(z))'}{J^{a,c}_{p,\mu}f(z)}\right) > \alpha,\ z \in \mathbb{U}\right\}.$$

Definition 1.3 *For fixed parameters A and B $(-1 \le B < A \le 1)$, we say that a function $f \in \Sigma_{p,n}$ is in the class $\Sigma C^{a,c}_{p,\mu}(A, B)$ if it satisfies the following subordination condition:*

$$-\frac{z^{p+1}}{p}(J^{a,c}_{p,\mu}f(z))' \prec \frac{1+Az}{1+Bz} \tag{7}$$
$$(a, c \in \mathbb{R};\ \mu > 0; (c-a) > 0;\ a > \mu p;\ 0 \le \alpha < p;\ z \in \mathbb{U}).$$

In view of the definition of differential subordination, (7) is equivalent to the following condition:

$$\left|\frac{z^{p+1}(J^{a,c}_{p,\mu}f(z))' + p}{Bz^{p+1}(J^{a,c}_{p,\mu}f(z))' + pA}\right| < 1.$$

We note that $\Sigma C^{a,c}_{p,\mu}\left(1-\frac{2\alpha}{p}, -1\right) = \Sigma C^{a,c}_{p,\mu}(\alpha)$, $(0 \le \alpha < p)$.

Meromorphically multivalent functions have been extensively studied by Liu and Srivastava [13], Cho and Kim [4], Cho et al. [5], El-Ashwah et al. [8] and others (see [2], [20], [21]).

In this paper, we investigate some properties of subclasses of multivalent meromorphic functions which are defined by the linear operator $J^{a,c}_{p,\mu}$.

2 Preliminary lemmas

To prove our results, we need the following lemmas.

Lemma 2.1 *([15]) Let a function h be analytic and convex (univalent) in U with $h(0) = 1$. Suppose that the function ϕ given by*

$$\phi(z) = 1 + b_{n+p}z^{n+p} + b_{n+p+1}z^{n+p+1} + \ldots \tag{8}$$

is analytic in $\mathbb{U}$. If

$$\phi(z) + \frac{z\phi'(z)}{\nu} \prec h(z) \quad (Re\{\nu\} \ge 0,\ \nu \ne 0), \tag{9}$$

then

$$\phi(z) \prec \psi(z) = \frac{\nu}{n+p}z^{-\frac{\nu}{n+p}}\int_0^z t^{\frac{\nu}{n+p}-1}h(t)dt \prec h(z),$$

where ψ is the best dominant of (9).

Lemma 2.2 *([19]) Let the function ϕ given by (8) be in the class $P(\gamma)$. Then*

$$Re\{\phi(z)\} \geq 2\gamma - 1 + \frac{2(1-\gamma)}{1+|z|} \quad (0 \leq \gamma < 1, z \in \mathbb{U}).$$

Lemma 2.3 *([22]) For $0 \leq \gamma_1 < \gamma_2 < 1$,*

$$P(\gamma_1) * P(\gamma_2) \subset P(\gamma_3), \text{ where } \gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2).$$

The result is the best possible.

For any real or complex numbers a, b and c ($c \neq \mathbb{Z}_0^- := \{0, -1, -2, ...\}$), the Gaussian hypergeometric function is defined by

$${}_2F_1(a, b; c; z) = 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots .$$

Lemma 2.4 *([24]) For any real or complex numbers a, b, $c\,(c \neq \mathbb{Z}_0^-)$, we have*

$$\begin{aligned}
\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt &= \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a, b; c; z), \quad (Re(c) > Re(b) > 0), \\
{}_2F_1(a, b; c; z) &= (1-z)^{-a}\,{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \\
{}_2F_1(a, b; c; z) &= {}_2F_1(a, b-1; c; z) + \frac{az}{c}\,{}_2F_1(a+1, b; c+1; z), \\
{}_2F_1\left(a, b; \frac{a+b+1}{2}; \frac{1}{2}\right) &= \frac{\sqrt{\pi}\,\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}, \\
{}_2F_1\left(1, 1; 2; \frac{z}{z+1}\right) &= \frac{z}{z+1}\ln(1+z).
\end{aligned}$$

Lemma 2.5 *[17] Let ϕ be analytic in $\mathbb{U}$ with $\phi(0) = 1$ and $\phi(z) \neq 0$ for $0 < |z| < 1$ and let $A, B \in \mathbb{C}$ with $A \neq B$, $|B| \leq 1$.*

(i) *Let $B \neq 0$ and $\tau \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ satisfy either $\left|\frac{\tau(A-B)}{B} - 1\right| \leq 1$ or $\left|\frac{\tau(A-B)}{B} + 1\right| \leq 1$. If ϕ satisfies*

$$1 + \frac{z\phi'(z)}{\tau\phi(z)} \prec \frac{1+Az}{1+Bz}, \tag{10}$$

then

$$\phi(z) \prec (1+Bz)^{\tau\left(\frac{A-B}{B}\right)},$$

and this is best dominant.

(ii) *Let $B = 0$ and $\tau \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be such that $|\tau A| < \pi$. If ϕ satisfies (10), then*

$$\phi(z) \prec e^{\tau A z}$$

and this is the best dominant.

Lemma 2.6 *([12]) Let $\lambda \neq 0$ be a real number, $\frac{a}{\lambda} > 0$ and $0 \leq \beta < 1$. Let $g(z) = 1 + c_n z^n + c_{n+1} z^{n+1} + \ldots$, be analytic in $\mathbb{U}$ and*

$$g(z) \prec 1 + \frac{aMz}{n\lambda + a} \quad (n \in \mathbb{N}),$$

where

$$M = \frac{(1-\beta)|\lambda|(1 + \frac{n\lambda}{a})}{|1 - \lambda + \lambda\beta| + \sqrt{1 + (1 + \frac{n\lambda}{a})^2}}.$$

If $P(z) = 1 + d_n z^n + d_{n+1} z^{n+1} + \ldots$ is analytic in $\mathbb{U}$ and satisfies the subordination relation

$$g(z)\{1 - \lambda + \lambda[(1-\beta)P(z) + \beta]\} \prec 1 + Mz,$$

then $Re(P(z)) > 0$ for $z \in \mathbb{U}$.

3 Subordination properties of $\mathbf{J}_{p,\mu}^{a,c}$

Unless otherwise mentioned, we assume throughout this paper that $\beta > 0$, $a, c \in \mathbb{R}$, $a > \mu p$, $\mu > 0$, $(c - 1 - a) > 0$, $-1 \leq B < A \leq 1$, $p \in \mathbb{N}$.

Theorem 3.1 *Let $-1 \leq B_j < A_j \leq 1$, $j = 1, 2$. If $f_j \in \Sigma_{p,n}$ satisfy the following subordination condition*

$$z^p\{(1-\beta)J_{p,\mu}^{a,c} f_j(z) + \beta J_{p,\mu}^{a,c-1} f_j(z)\} \prec \frac{1 + A_j z}{1 + B_j z}, \tag{11}$$

then

$$z^p\{(1-\beta)J_{p,\mu}^{a,c} F(z) + \beta J_{p,\mu}^{a,c-1} F(z)\} \prec \frac{1 + (1-2\delta)z}{1 - z},$$

where $F = J_{p,\mu}^{a,c}(f_1 * f_2)$ and

$$\delta = 1 - \frac{4(A_1 - B_1)(A_2 - B_2)}{(1 - B_1)(1 - B_2)} \left(1 - \frac{1}{2} {}_2F_1(1, 1; \frac{c - \mu p - 1}{\beta\mu} + 1; \frac{1}{2})\right). \tag{12}$$

The result is the best possible when $B_1 = B_2 = -1$.

Proof. Let $f_j \in \Sigma_{p,n}$, $j = 1, 2$, satisfy the subordination condition (11). Then, by setting

$$\phi_j(z) = z^p\{(1-\beta)J_{p,\mu}^{a,c} f_j(z) + \beta J_{p,\mu}^{a,c-1} f_j(z)\} \prec \frac{1 + A_j z}{1 + B_j z}, \ j = 1, 2, \tag{13}$$

we have

$$\phi_j \in P(\gamma_j), \ \gamma_j = \frac{1 - A_j}{1 - B_j}, \ j = 1, 2.$$

By making use of (5) and (13), we obtain

$$J_{p,\mu}^{a,c} f_j(z) = \frac{c - \mu p - 1}{\beta\mu} z^{-p - \frac{c - \mu p - 1}{\beta\mu}} \int_0^z t^{\frac{c - \mu p - 1}{\beta\mu} - 1} \phi_j(z) dt, j = 1, 2. \tag{14}$$

Now, if we let $F = J^{a,c}_{p,\mu}(f_1 * f_2)$, then by using (14) and the fact that

$$J^{a,c}_{p,\mu}F(z) = J^{a,c}_{p,\mu}f_1(z) * J^{a,c}_{p,\mu}f_2(z),$$

a simple computation shows that

$$J^{a,c}_{p,\mu}F(z) = \frac{c-\mu p-1}{\beta\mu} z^{-p-\frac{c-\mu p-1}{\beta\mu}} \int_0^z t^{\frac{c-\mu p-1}{\beta\mu}-1}\phi_0(t)dt,$$

where

$$\begin{aligned} \phi_0(z) &= z^p\{(1-\beta)J^{a,c}_{p,\mu}F(z) + \beta J^{a,c-1}_{p,\mu}F(z)\} \\ &= \frac{c-\mu p-1}{\beta\mu} z^{-\frac{c-\mu p-1}{\beta\mu}} \int_0^z t^{\frac{c-\mu p-1}{\beta\mu}-1}(\phi_1 * \phi_2)(t)dt. \end{aligned} \tag{15}$$

Since $\phi_j \in P(\gamma_j)$, $j = 1, 2$, it follows from Lemma 2.3 that

$$\phi_1 * \phi_2 \in P(\gamma_3), \text{ where } \gamma_3 = 1 - 2(1-\gamma_1)(1-\gamma_2),$$

and the bound γ_3 is the best possible. Using Lemma 2.2 in (15), we obtain that

$$\begin{aligned} Re\{\phi_0(z)\} &= \frac{c-\mu p-1}{\beta\mu}\int_0^1 u^{\frac{c-\mu p-1}{\beta\mu}-1} Re(\phi_1 * \phi_2)(uz)du \\ &\geq \frac{c-\mu p-1}{\beta\mu}\int_0^1 u^{\frac{c-\mu p-1}{\beta\mu}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u|z|}\right)du \\ &= \frac{c-\mu p-1}{\beta\mu}\int_0^1 u^{\frac{c-\mu p-1}{\beta\mu}-1}\left(2\gamma_3 - 1 + \frac{2(1-\gamma_3)}{1+u}\right)du \\ &= 1 - \frac{4(A_1-B_1)(A_2-B_2)}{(1-B_1)(1-B_2)}\left(1 - \frac{c-\mu p-1}{\beta\mu}\int_0^1 \frac{u^{\frac{c-\mu p-1}{\beta\mu}-1}}{1+u}du\right) = \delta, \end{aligned}$$

where δ is given by (12).

When $B_1 = B_2 = -1$, we consider the functions $f_j \in \Sigma_{p,n}\,(j = 1, 2)$ which satisfy the hypothesis (11) and are given by

$$J^{a,c}_{p,\mu}f_j(z) = \frac{c-\mu p-1}{\beta\mu} z^{-p-\frac{c-\mu p-1}{\beta\mu}} \int_0^z t^{\frac{c-\mu p-1}{\beta\mu}-1}\left(\frac{1+A_j t}{1-t}\right)dt, \;\; j = 1, 2.$$

Since

$$\left(\frac{1+A_1 z}{1-z}\right) * \left(\frac{1+A_2 z}{1-z}\right) = 1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-z},$$

it follows from (15) that

$$\begin{aligned} \phi_0(z) &= \frac{c-\mu p-1}{\beta\mu}\int_0^1 u^{\frac{c-\mu p-1}{\beta\mu}-1}\left(1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-uz}\right)du \\ &= 1 - (1+A_1)(1+A_2) + \frac{(1+A_1)(1+A_2)}{1-z} \times {}_2F_1(1,1;\frac{c-\mu p-1}{\beta\mu}+1;\frac{z}{z-1}). \end{aligned}$$

Therefore

$$\phi_0(z) \to 1-(1+A_1)(1+A_2)+\frac{1}{2}(1+A_1)(1+A_2){}_2F_1(1,1;\frac{c-\mu p-1}{\beta\mu}+1;\frac{1}{2})$$

as $z \to -1$, which evidently completes the proof of Theorem 3.1. ■

In its special case when $A_j = 1-2\gamma_j$ $(0 \le \gamma_j < 1)$, $B_j = -1$, $j = 1,2$, Theorem 3.1 yields the following corollary.

Corollary 3.1 *If $f_j \in \Sigma_{p,n}$ satisfy the following condition:*

$$z^p\{(1-\beta)J_{p,\mu}^{a,c}f_j(z)+\beta J_{p,\mu}^{a,c-1}f_j(z)\} \prec \frac{1+(1-2\gamma_j)z}{1-z} \quad (j=1,2,\ z\in\mathbb{U}),$$

then

$$z^p\{(1-\beta)J_{p,\mu}^{a,c}F(z)+\beta J_{p,\mu}^{a,c-1}F(z)\} \prec \frac{1+(1-2\delta)z}{1-z}$$

where $F = J_{p,\mu}^{a,c}(f_1 * f_2)$, $\delta = 1-2(1-\gamma_1)(1-\gamma_2)\times\left[2-{}_2F_1(1,1;\frac{c-1-\mu p}{\beta\mu}+1;\frac{1}{2})\right]$, $(z\in\mathbb{U})$.

Letting $A_j = 1-2\zeta_j$ $(0 \le \zeta_j < p)$, $B_j = -1$, $(j=1,2)$, $a = c$ and $\lambda = \frac{\beta\mu}{c-p-1}$ in Theorem 3.1, yeilds the following corollary.

Corollary 3.2 *If $f_j(z) \in \Sigma_{p,n}$ satisfies*

$$Re\left(z^p\left\{(1+p\lambda)f_j(z)+\lambda z(f_j(z))'\right\}\right) > \zeta_j \quad (j=1,2),$$

then

$$Re\left(z^p\left\{(1+p\lambda)(f_1*f_2)(z)+\lambda z(f_1*f_2)'(z)\right\}\right) > \eta_1 \quad (j=1,2),$$

where

$$\eta_1 = 1-4\left(1-\frac{\zeta_1}{p}\right)\left(1-\frac{\zeta_2}{p}\right)\left[1-\frac{1}{2}{}_2F_1\left(1,1;\frac{\mu}{\lambda}+1;\frac{1}{2}\right)\right]$$

In the next theorem, we have determined the sufficient condition for the functions $z^pJ_{p,\mu}^{a,c}f(z)$, to be a member of the class $P(\rho)$.

Theorem 3.2 *If $f \in \Sigma_{p,n}$ satisfy the following subordination condition*

$$z^p\{(1-\beta)J_{p,\mu}^{a,c}f(z)+\beta J_{p,\mu}^{a,c-1}f(z)\} \prec \frac{1+Az}{1+Bz}, \tag{16}$$

then

$$Re\left(z^pJ_{p,\mu}^{a,c}f(z)\right) > \rho \quad (z\in\mathbb{U}),$$

where

$$\rho = \begin{cases} \frac{A}{B}+(1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1;\frac{c-\mu p-1}{\beta\mu(n+p)}+1;\frac{B}{B-1}), & \text{if } B\neq 0,\\ 1-\frac{c-\mu p-1}{c-1-\mu p+(n+p)\mu\beta}A, & \text{if } B=0.\end{cases} \tag{17}$$

The result is the best possible.

Proof. Set

$$h(z) = z^p J_{p,\mu}^{a,c} f(z) \text{ for } f \in \Sigma_{p,n}, \tag{18}$$

then the function h is of the form (8). Differentiating (18) with respect to z and using the identity (5), we obtain

$$z^p J_{p,\mu}^{a,c-1} f(z) = h(z) + \frac{\mu}{c - \mu p - 1} z h'(z) \tag{19}$$

By using (16), (18) and (19), we obtain

$$h(z) + \frac{\beta\mu}{c - \mu p - 1} z h'(z) \prec \frac{1 + Az}{1 + Bz}.$$

Now, applying Lemma 2.1, we get

$$h(z) \prec g(z) = \frac{c - \mu p - 1}{\beta\mu(n+p)} z^{-\frac{c-\mu p-1}{\beta\mu(n+p)}} \int_0^z t^{\frac{c-\mu p-1}{\beta\mu(n+p)}-1} \left(\frac{1 + At}{1 + Bt}\right) dt.$$

Applying Lemma 2.4, we get

$$g(z) = \begin{cases} \frac{A}{B} + (1 - \frac{A}{B})(1 + Bz)^{-1} {}_2F_1(1, 1; \frac{c-\mu p-1}{\beta\mu(n+p)} + 1; \frac{Bz}{Bz+1}), & \text{if } B \neq 0, \\ 1 + \frac{c-\mu p-1}{c-1-\mu p+(n+p)\mu\beta} Az, & \text{if } B = 0. \end{cases} \tag{20}$$

Now, we will show that

$$\inf\{Re(g(z)) : |z| < 1\} = g(-1). \tag{21}$$

We have $Re\{\frac{1+Az}{1+Bz}\} \geq \frac{1-Ar}{1-Br}$, $|z| = r < 1$, and setting

$$h(s, z) = \frac{1 + Azs}{1 + Bzs} \quad (0 \leq s \leq 1) \text{ and } d(s) = \frac{c - \mu p - 1}{\beta\mu(n+p)} s^{\frac{c-\mu p-1}{\beta\mu(n+p)}-1} ds,$$

which is a positive measure on the closed interval $[0, 1]$, we get

$$g(z) = \int_0^1 h(s, z) d(s),$$

so that

$$\text{Re}(g(z)) \geq \int_0^1 \frac{1 - Asr}{1 - Bsr} d(s) = g(-r) \quad |z| = r < 1. \tag{22}$$

As $r \to 1^-$ in (22), we obtain (21).

Now, by using (20) and (21), we get

$$Re\{z^p J_{p,\mu}^{a,c} f(z)\} > \rho,$$

where ρ is given by (17).

To show the estimate (17) is the best possible, we consider the following function $f \in \Sigma_{p,n}$ defined by

$$z^p J_{p,\mu}^{a,c} f(z) = \frac{c - \mu p - 1}{\beta\mu(n+p)} \int_0^1 u^{\frac{c-\mu p-1}{\beta\mu(n+p)}-1} \left(\frac{1 + Auz}{1 + Buz}\right) du,$$

For the above function, we find that

$$z^p\{(1-\beta)J^{a,c}_{p,\mu}f(z)+\beta J^{a,c-1}_{p,\mu}f(z)\}=\frac{1+Az}{1+Bz}$$

and

$$z^pJ^{a,c}_{p,\mu}f(z)\longrightarrow\frac{c-\mu p-1}{\beta\mu(n+p)}\int_0^1 u^{\frac{c-\mu p-1}{\beta\mu(n+p)}-1}\left(\frac{1-Au}{1-Bu}\right)du$$

$$\rho=\begin{cases}\frac{A}{B}+(1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1;\frac{c-\mu p-1}{\beta\mu(n+p)}+1;\frac{B}{B-1}), & \text{if } B\neq 0,\\ 1-\frac{c-\mu p-1}{c-1-\mu p+(n+p)\mu\beta}A, & \text{if } B=0,\end{cases}$$

which completes the proof of Theorem 3.2. ■

In its special case when $A=1-2\gamma$, $(0\le\gamma<1)$ $B=-1$ and $\beta=1$, Theorem 3.2 yields the following corollary.

Corollary 3.3 *If $f\in\Sigma_{p,n}$ satisfy the following condition:*

$$z^pJ^{a,c-1}_{p,\mu}f(z)\prec\frac{1+(1-2\gamma)z}{1-z}\quad(z\in\mathbb{U}),$$

then

$$Re(z^pJ^{a,c}_{p,\mu}f(z))>\gamma+(1-\gamma)\left[{}_2F_1(1,1;\frac{c-\mu p-1}{\mu(n+p)}+1;\frac{1}{2})-1\right]\quad(z\in\mathbb{U}).$$

The result is the best possible.

Theorem 3.3 *If $f\in\Sigma_{p,n}$ satisfies the following subordination:*

$$-\frac{z^{p+1}}{p}\left[(1-\beta)(J^{a,c}_{p,\mu}f(z))'+\beta(J^{a,c-1}_{p,\mu}f(z))'\right]\prec\frac{1+Az}{1+Bz},\quad(z\in\mathbb{U}),\tag{23}$$

then

$$-\frac{z^{p+1}}{p}(J^{a,c}_{p,\mu}f(z))'\prec\zeta(z),\tag{24}$$

where

$$\zeta(z)=\begin{cases}\frac{A}{B}+(1-\frac{A}{B})(1+Bz)^{-1}{}_2F_1(1,1;\frac{c-\mu p-1}{\mu\beta(n+p)}+1,\frac{Bz}{1+Bz}), & B\neq 0,\\ 1+\frac{c-\mu p-1}{c-1-\mu p+\mu\beta(n+p)}Az, & B=0,\end{cases}$$

is the best dominant of (24). Furthermore,

$$Re\left\{-\frac{z^{p+1}}{p}(J^{a,c}_{p,\mu}f(z))'\right\}>\zeta_1,\tag{25}$$

where

$$\zeta_1=\begin{cases}\frac{A}{B}+(1-\frac{A}{B})(1-B)^{-1}{}_2F_1(1,1;\frac{c-\mu p-1}{\mu\beta(n+p)}+1,\frac{B}{B-1}), & B\neq 0,\\ 1-\frac{c-\mu p-1}{c-1-\mu p+\mu\beta(n+p)}A, & B=0,\end{cases}$$

The result is best possible.

Proof. Set

$$h(z) = -\frac{z^{p+1}}{p}\left(J_{p,\mu}^{a,c}f(z)\right)', \qquad f \in \Sigma_{p,n}, \tag{26}$$

then the function $h(z)$ is of the form (8) and is analytic in $\mathbb{U}$. Differentiating (26) with respect to z and using the identity (6), we obtain

$$h(z) + \frac{\mu\beta}{c-\mu p-1}zh'(z) = -\frac{z^{p+1}}{p}\left[(1-\beta)(J_{p,\mu}^{a,c}f(z))' + \beta(J_{p,\mu}^{a,c-1}f(z))'\right] \prec \frac{1+Az}{1+Bz}.$$

Applying Lemma 2.1 and Lemma 2.4, we have

$$\begin{aligned} -\frac{z^{p+1}}{p}(J_{p,\mu}^{a,c}f(z))' &\prec \zeta(z) \\ &= \frac{c-\mu p-1}{\mu\beta(n+p)}z^{-\frac{c-\mu p-1}{\mu\beta(n+p)}}\int_0^z t^{\frac{c-\mu p-1}{\mu\beta(n+p)}-1}\left(\frac{1+At}{1+Bt}\right)dt \\ &= \begin{cases} \frac{A}{B} + (1-\frac{A}{B})(1+Bz)^{-1}{}_2F_1(1,1;\frac{c-\mu p-1}{\mu\beta(n+p)}+1,\frac{Bz}{1+Bz}) & B \neq 0, \\ 1 + \frac{c-\mu p-1}{c-1-\mu p+\mu\beta(n+p)}Az, & B = 0. \end{cases} \end{aligned}$$

This proves the assertion (24) of Theorem 3.3. Now, we will show that

$$\inf_{|z|<1}\{Re(\zeta(z))\} = \zeta(-1).$$

We have $Re\{\frac{1+Az}{1+Bz}\} \geq \frac{1-Ar}{1-Br}$, $|z| = r < 1$, and setting

$$g(s,z) = \frac{1+Azs}{1+Bzs} \quad (0 \leq s \leq 1) \text{ and } d\nu(s) = \frac{c-\mu p-1}{\mu\beta(n+p)}s^{\frac{c-\mu p-1}{\beta\mu(n+p)}-1}ds,$$

which is a positive measure on the closed interval $[0,1]$, we get

$$\zeta(z) = \int_0^1 g(s,z)d\nu(s),$$

so that

$$Re(\zeta(z)) \geq \int_0^1 \frac{1-Asr}{1-Bsr}d\nu(s) = \zeta(-r) \quad |z| = r < 1. \tag{27}$$

Letting $r \to 1^-$ in the above inequality, we obtain the assertion (25). Finally, the estimate (25) is the best possible as $\zeta(z)$ is the best dominant of (24), using arguments similar to those detailed above with Theorem 3.2. ■

Taking $\beta = \frac{\sigma(c-\mu p-1)}{\mu}$ in Theorem 3.3, we obtain the following corollary.

Corollary 3.4 *If $f \in \Sigma_{p,n}$ satisfies*

$$\frac{-z^{p+1}}{p}\left((1+\sigma(p+1))(J_{p,\mu}^{a,c}f(z))' + \sigma z(J_{p,\mu}^{a,c}f(z))''\right) \prec \frac{1+Az}{1+Bz},$$

then

$$\frac{-z^{p+1}}{p}(J_{p,\mu}^{a,c}f(z))' \prec g_2(z) \prec \frac{1+Az}{1+Bz}, \tag{28}$$

where the function $g_2(z)$ given by

$$g_2(z) = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1, 1; \frac{1+\sigma(n+p)}{\sigma(n+p)}; \frac{Bz}{1+Bz}\right) & (B \neq 0) \\ 1 + \frac{1}{1+\sigma(n+p)} Az & (B = 0), \end{cases}$$

is the best dominant of (28). Furthermore,

$$\frac{-z^{p+1}}{p}(J_{p,\mu}^{a,c} f(z))' > \zeta_2 \quad (z \in \mathbb{U}),$$

where

$$\zeta_2 = \begin{cases} \frac{A}{B} + \left(1 - \frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1, 1; \frac{1+\sigma(n+p)}{\sigma(n+p)}; \frac{B}{B-1}\right) & (B \neq 0) \\ 1 - \frac{1}{1+\sigma(n+p)} & (B = 0), \end{cases}$$

the result is the best possible.

Taking $A = 1 - \frac{2\delta}{p}$ $(0 \le \delta < p)$, $B = -1$, $\beta = \frac{c-\mu p-1}{\mu}$, $n = 2-p$ in Theorem 3.3 and using (6), we obtain the following corollary.

Corollary 3.5 *If $f \in \Sigma_{p,2-p}$ satisfies*

$$Re\left\{-z^{p+1}\left((2+p)(J_{p,\mu}^{a,c} f(z))' + z(J_{p,\mu}^{a,c} f(z))''\right)\right\} > \delta,$$

then

$$Re\left\{-z^{p+1}(J_{p,\mu}^{a,c} f(z))'\right\} > \delta + (p-\delta)(\frac{\pi}{2} - 1) \quad 0 \le \delta < p,$$

the result is the best possible.

Taking $\delta = -\frac{p(\pi-2)}{4-\pi}$ in Corollary 3.5, we obtain the following corollary.

Corollary 3.6 *If $f \in \Sigma_{p,2-p}$ satisfies*

$$Re\left\{-z^{p+1}\left((2+p)(J_{p,\mu}^{a,c} f(z))' + z(J_{p,\mu}^{a,c} f(z))''\right)\right\} > -\frac{p(\pi-2)}{4-\pi}, \ (0 \le \delta < p),$$

then

$$Re\left\{-z^{p+1}(J_{p,\mu}^{a,c} f(z))'\right\} > 0,$$

the result is the best possible.

Remark 3.1 *For $a = c$, Corollary 3.6 reduces to the result of Pap [18].*

Taking $A = 1 - \frac{2\delta}{p}$ $(0 \le \delta < p)$, $B = -1$, $\beta = \frac{c-\mu p-1}{\mu}$, $n = 1-p$ in Theorem 3.3 and using (6), we obtain the following corollary.

Corollary 3.7 *If $f \in \Sigma_{p,1-p}$ satisfies*

$$Re\left\{-z^{p+1}\left((2+p)(J_{p,\mu}^{a,c} f(z))' + z(J_{p,\mu}^{a,c} f(z))''\right)\right\} > \delta,$$

then

$$Re\left\{-z^{p+1}(J_{p,\mu}^{a,c} f(z))'\right\} > \delta + 2(p-\delta)(\ln 2 - 1) \quad 0 \le \delta < p,$$

the result is the best possible.

Theorem 3.4 *Let* $f \in \Sigma C^{a,c}_{p,\mu}(\alpha)$, $(0 \leq \alpha < p,)$ *then*

$$Re\left\{-z^{p+1}\left[(1-\beta)(J^{a,c}_{p,\mu}f(z))' + \beta(J^{a,c-1}_{p,\mu}f(z))'\right]\right\} > \alpha, \quad (|z| < R),$$

where

$$R = \left\{\sqrt{1+\left(\frac{\beta\mu}{c-\mu p-1}(p+n)\right)^2} - \frac{\beta\mu}{c-\mu p-1}(p+n)\right\}^{\frac{1}{p+n}}. \tag{29}$$

Proof. Let $f \in \Sigma C^{a,c}_{p,\mu}(\alpha)$, then we write

$$-z^{p+1}(J^{a,c}_{p,\mu}f(z))' = \alpha + (p-\alpha)q(z), \quad (z \in \mathbb{U}), \tag{30}$$

where $q(z)$ is of the form (8) and is analytic in $\mathbb{U}$. Differentiating (30) with respect to z, we have

$$-\frac{1}{p-\alpha}\left[(1-\beta)z^{p+1}(J^{a,c}_{p,\mu}f(z))' + \beta z^{p+1}(J^{a,c-1}_{p,\mu}f(z))' + \alpha\right] = q(z) + \frac{\beta\mu}{c-\mu p-1}zq'(z) \tag{31}$$

Applying the following well knowing estimate [14]:

$$\frac{|zq'(z)|}{Re(q(z))} \leq \frac{2(p+n)r^{p+n}}{1-r^{2(p+n)}}, \qquad (|z| = r < 1),$$

in (31), we have

$$\begin{aligned} -\frac{1}{p-\alpha}Re\left[(1-\beta)z^{p+1}(J^{a,c}_{p,\mu}f(z))' + \beta z^{p+1}(J^{a,c-1}_{p,\mu}f(z))' + \alpha\right] \\ \geq Re\{q(z)\}\left\{1 - \frac{2\beta\mu(p+n)r^{p+n}}{(c-\mu p-1)(1-r^{2(p+n)})}\right\}, \end{aligned} \tag{32}$$

such that the right hand side of (32) is positive if $r < R$, where R given by (29). In order to show that the bound R is best possible, we consider the function $f \in \Sigma_{p,n}$ defined by

$$-z^{p+1}(J^{a,c}_{p,\mu}f(z))' = \alpha + (p-\alpha)\frac{1+z^{p+n}}{1-z^{p+n}}, \quad (0 \leq \alpha < p;\ z \in \mathbb{U}).$$

Note that

$$\begin{aligned} -\frac{1}{p-\alpha}\left[(1-\beta)z^{p+1}(J^{a,c}_{p,\mu}f(z))' + \beta z^{p+1}(J^{a,c-1}_{p,\mu}f(z))' + \alpha\right] \\ = \frac{1-z^{2(p+n)} + 2\frac{\beta\mu}{c-\mu p-1}(p+n)z^{p+n}}{(1-z)^{2(p+n)}} = 0, \end{aligned}$$

for $z = Re^{\frac{i\pi}{p+n}}$, the proof of Theorem 3.4 is completed. ■

Putting $\beta = 1$ in Theorem 3.4, we deduce the following corollary.

Corollary 3.8 *If* $f(z) \in \Sigma C_{p,\mu}^{a,c}(\alpha)$ $(0 \leq \alpha < p)$, *then* $f(z) \in \Sigma C_{p,\mu}^{a,c-1}(\alpha)$ *for* $|z| < \check{R}$, *where*

$$\check{R} = \left\{ \sqrt{1 + \left(\frac{\mu}{c - \mu p - 1}(p+n) \right)^2} - \frac{\mu}{c - \mu p - 1}(p+n) \right\}^{\frac{1}{p+n}}$$

The result is the best possible.

Theorem 3.5 *Let* $\tau \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ *and* $A, B \in \mathbb{C}$ *with* $A \neq B$ *and* $|B| \leq 1$. *Suppose that*

$$\left| \frac{\tau(c - \mu p - 1)(A - B)}{\mu B} - 1 \right| \leq 1 \quad or \quad \left| \frac{\tau(c - \mu p - 1)(A - B)}{\mu B} + 1 \right| \leq 1 \quad if\ B \neq 0,$$

and

$$\left| \frac{\tau(c - \mu p - 1)}{\mu} A \right| \leq \pi, \quad if \quad B = 0.$$

If $f \in \Sigma_{p,n}$ *with* $J_{p,\mu}^{a,c} f(z) \neq 0$ *for all* $z \in \mathbb{U}$, *then*

$$\frac{J_{p,\mu}^{a,c-1} f(z)}{J_{p,\mu}^{a,c} f(z)} \prec \frac{1 + Az}{1 + Bz}$$

implies

$$(z^p J_{p,\mu}^{a,c} f(z))^\tau \prec h_1(z),$$

where

$$h_1(z) = \begin{cases} (1 + Bz)^{\frac{\tau}{\mu B}(c - \mu p - 1)(A - B)} & B \neq 0, \\ e^{\frac{\tau(c - \mu p - 1)}{\mu} Az} & B = 0, \end{cases}$$

is the best dominant.

Proof. Let

$$h(z) = (z^p J_{p,\mu}^{a,c} f(z))^\tau \quad (z \in \mathbb{U}), \tag{33}$$

then $h(z)$ is analytic in $\mathbb{U}$, $h(0) = 1$ and $h(z) \neq 0$. Taking the logarithmic differentiation on both sides of (33) and using the identity (5), we obtain

$$1 + \frac{\mu}{\tau(c - \mu p - 1)} \frac{zh'(z)}{h(z)} = \frac{J_{p,\mu}^{a,c-1} f(z)}{J_{p,\mu}^{a,c} f(z)} \prec \frac{1 + Az}{1 + Bz}.$$

Now the assertions of Theorem 3.5 follows from Lemma 2.5. ■

Taking $B = -1$ and $A = 1 - 2\delta$, $0 \leq \delta < 1$ in Theorem 3.5, we get the following corollary:

Corollary 3.9 *Let* $\tau \in \mathbb{C}^*$ *satisfies either*

$$\left| \frac{2\tau(c - \mu p - 1)(1 - \delta)}{\mu} - 1 \right| \leq 1 \quad or \quad \left| \frac{2\tau(c - \mu p - 1)(1 - \delta)}{\mu} + 1 \right| \leq 1.$$

If $f \in \Sigma_{p,n}$ *with* $J_{p,\mu}^{a,c} f(z) \neq 0$ *for all* $z \in \mathbb{U}$, *then*

$$Re \left\{ \frac{J_{p,\mu}^{a,c-1} f(z)}{J_{p,\mu}^{a,c} f(z)} \right\} \prec \frac{1 + (1 - 2\delta)z}{1 - z}$$

implies

$$(z^p J_{p,\mu}^{a,c} f(z))^\tau \prec h_1(z),$$

where

$$h_1(z) = (1-z)^{\frac{-2\tau}{\mu}(c-\mu p-1)(1-\delta)},$$

is the best dominant.

Remark 3.2 *Put $p=\mu=1$ and $a=c=p+2$ in Theorem 3.5 we obtain the result obtained by Obradovic and Owa [17] (see also [7]).*

4 Sufficient conditions for the class $\Sigma S_{p,\mu}^{*a,c}(\alpha)$

In this section, we obtain the sufficient condition for the function f to be a member of the class $\Sigma S_{p,\mu}^{*a,c}(\alpha)$.

Theorem 4.1 *If $f \in \Sigma_{p,n}$ satisfy the following subordination condition:*

$$z^p \left\{(1-\beta) J_{p,\mu}^{a,c} f(z) + \beta J_{p,\mu}^{a,c-1} f(z)\right\} \prec 1 + M_1 z, \tag{34}$$

where

$$M_1 = \frac{\zeta\eta}{|1-\eta| + \sqrt{1+\zeta^2}}$$

with $\zeta = \frac{c-1-\mu p+(n+p)\mu\beta}{c-\mu p-1}$ and $\eta = \frac{\mu\beta}{c-\mu p-1}(\alpha - p)$. Then $f \in \Sigma S_{p,\mu}^{*a,c}(\alpha)$.

Proof. Set

$$g(z) = z^p J_{p,\mu}^{a,c} f(z), \tag{35}$$

then g is of the form (8) and is analytic in $\mathbb{U}$. From Theorem 3.2 with $A = M_1$ and $B = 0$, we have

$$g(z) \prec 1 + \frac{c-\mu p-1}{c-1-\mu p+(n+p)\mu\beta} M_1 z,$$

which is equivalent to

$$|g(z) - 1| < \frac{M_1}{\zeta} = N < 1; \quad (z \in \mathbb{U}). \tag{36}$$

If we set

$$P(z) = \frac{-1}{p-\alpha}\left(\frac{z(J_{p,\mu}^{a,c} f(z))'}{J_{p,\mu}^{a,c} f(z)} + \alpha\right), \tag{37}$$

then by using the identity (5) followed by (35), we obtain

$$z^p(J_{p,\mu}^{a,c-1} f(z)) = \left(1 + \frac{\mu}{c-\mu p-1}(p-\alpha) - \frac{\mu}{c-\mu p-1}(p-\alpha)P(z)\right) g(z). \tag{38}$$

In view of (38), the hypothesis (34) can be written as follows:

$$|(1-\eta)g(z) + \eta P(z) g(z) - 1| < M_1 = \zeta N, \;\; (z \in \mathbb{U}). \tag{39}$$

We need to show that (39) yields

$$\mathrm{Re}(P(z)) > 0. \tag{40}$$

Suppose that this is false. Since $P(0) = 1$, there exists a point $z_0 \in \mathbb{U}$ such that $P(z_0) = ix$ for some $x \in \mathbb{R}$. To prove (40), it is sufficient to obtain a contradiction from the inequality

$$W = |(1-\eta)g(z_0) + \eta P(z_0)g(z_0) - 1| \geq M_1. \tag{41}$$

Let $g(z_0) = u + iv$, then by using (36) and the triangle inequality, we obtain that

$$\begin{aligned} W^2 &= |(1-\eta)g(z_0) + \eta P(z_0)g(z_0) - 1|^2 \\ &= (u^2+v^2)(\eta x)^2 + 2vx\eta + |(1-\eta)g(z_0) - 1|^2 \\ &\geq (u^2+v^2)\eta^2 x^2 + 2vx\eta + (\eta - |1-\eta|N)^2, \end{aligned}$$

then

$$\Phi(x) = W^2 - M_1^2 \geq (u^2+v^2)\eta^2x^2 + 2vx\eta + (\eta - |1-\eta|N)^2 - N^2\zeta^2,$$

then (41) holds true if $\Phi(x) \geq 0$, for any $x \in \mathbb{U}$. Since $(u^2+v^2)\eta^2 > 0$, The inequality $\Phi(x) \geq 0$, holds true if the discriminant $\Delta \leq 0$, that is

$$\Delta = 4\eta^2 \left\{v^2 - (u^2+v^2)[(\eta - |1-\eta|N)^2 - N^2\zeta^2]\right\} \leq 0,$$

which is equivalent to

$$v^2 \left\{1 - [(\eta - |1-\eta|N)^2 + N^2\zeta^2]\right\} \leq u^2 \left\{(\eta - |1-\eta|N)^2 - N^2\zeta^2\right\}.$$

Putting $\Phi(z_0) - 1 = \rho e^{i\theta}$ for some real $\theta \in \mathbb{R}$, we get

$$\frac{v^2}{u^2} = \frac{\rho^2 \sin^2\theta}{(1+\rho\cos\theta)^2},$$

since the above expression attains its maximum value at $\cos\theta = -\rho$, by using (36), we obtain

$$\begin{aligned} \frac{v^2}{u^2} &\leq \frac{\rho^2}{1-\rho^2} \leq \frac{N^2}{1-N^2} \\ &\leq \frac{(\eta - |1-\eta|N)^2 - N^2\zeta^2}{1 - (\eta - |1-\eta|N)^2 + N^2\zeta^2)}, \end{aligned}$$

which yields $\Delta \leq 0$. Therefore, $W \geq M_1$, which contradicts (39). Hence, $\mathrm{Re}(P(z)) > 0$. This proves that $f \in \Sigma S^{*a,c}_{p,\mu}$, which completes the proof of Theorem 4.1. ■
Taking $\beta = 1$ in Theorem 4.1, we obtain the following corollary:

Corollary 4.1

$$z^p \left\{J^{a,c-1}_{p,\mu} f(z)\right\} \prec 1 + M_0 z,$$

where

$$M_0 = \frac{\zeta\eta}{|1-\eta| + \sqrt{1+\zeta^2}}$$

with $\zeta = \frac{c+n\mu-1}{c-\mu p-1}$ *and* $\eta = \frac{\mu}{c-\mu p-1}(\alpha - p)$. *Then* $f \in \Sigma S^{*a,c}_{p,\mu}(\alpha)$.

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