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Existence and uniqueness of positive solutions for Sturm-Liouville BVPs of multi-term fractional differential equations *

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Abstract. In this article, we establish the existence and uniqueness results for the positive solutions to Sturm-Liouville boundary value problems of the nonlinear fractional differential equation. Our analysis rely on the well known fixed point theorems. An example is given to illustrate the efficiency of the main theorem.

Keywords. Positive solution; fractional differential equation; Sturm-Liouville boundary value problems; fixed-point theorem;

1 Introduction

Fractional differential equations have many applications in modeling of physical and chemical processes. In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors, see the text books [4,7,9], the survey paper [2] and papers [1,3,5,6,8,10,12,13,18] and the references therein.

In the literature, $D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0$ is known as a **single term** fractional differential equation. In certain cases, we find equations containing more than one fractional differential terms. These equations are called **multi-term fractional differential equations**. A classical

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example is the so-called **Bagley Torvik equation**

$$AD_{0+}^{2}y(x) + BD_{0+}^{\frac{3}{2}}y(x) + Cy(x) = f(x),$$

where A, B, C are constants and f is a given function. This equation arises from for example the modelling of motion of a rigid plate immersed in a Newtonian fluid. It was originally proposed in [16].

This paper is motivated by [11]. E. R. Kaufmann and E. Mboumi studied the following boundary value problem for the single term fractional differential equations

$$\begin{cases} D_{0+}^{\alpha}u(t) + a(t)f(u(t)) = 0, \quad 0 < t < 1, 1 < \alpha < 2, \\ u(0) = 0, u'(1) = 0\,, \end{cases} \tag{1}$$

by using the properties of the Green's function of the corresponding BVP, where $f : [0,1] \times [0,+\infty) \to [0,\infty)$ is continuous, $a \in L^{\infty}[0,1]$, there exists a constant $m > 0$ such that $a(t) \geq m$ a.e. $t \in [0,1]$. By using the Leggett-Williams fixed point theorem, the Krasnoselskii fixed point theorem, the authors in [11] proved that BVP(1) has at least one or three positive solutions.

In recent paper [17], the authors studied the existence of positive solutions of the following boundary value problem of fractional differential equation

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t), D_{0+}^{\alpha-1}u(t)) = 0, t \in (0,1), 1 < \alpha < 2, \\ a \lim\limits_{t\to 0} t^{2-\alpha}u(t) - b \lim\limits_{t\to 0} D_{0+}^{\alpha-1}u(t) = \int_0^1 g(t, u(t), D_{0+}^{\alpha-1}u(t)), \\ cD_{0+}^{\alpha-1} + du(1) = \int_0^1 h(t, u(t), D_{0+}^{\alpha-1}u(t))dt, \end{cases}$$

where $a, b, c, d \geq 0$ with $\delta = ad + bd\Gamma(\alpha) + ac\Gamma(\alpha) > 0$, f, g, h defined on $(0,1) \times [0,\infty) \times R$ are nonnegative Caratheodory functions that may be singular at $t = 0$ and $t = 1$, $f(t,0,0) \not\equiv 0$ on each subinterval of $[0,1]$.

In paper [19], the authors studied the existence of multiple positive solutions of the following boundary value problem of fractional differential equation

$$\begin{cases} {}^{c}D_{0+}^{\alpha}u(t) = f(t, u(t), {}^{c}D_{0+}^{\beta}u(t)), t \in (0,1), 1 < \alpha < 2, \\ u(0) + u'(0) = 0, \\ u(1) + u'(1) = 0 \end{cases}$$

where $f[0,1] \times [0,\infty) \times R \to [0,\infty)$ is continuous, ${}^{c}D_{0+}^{*}$ is the standard caputo fractional derivative of order $*$, $1 < \alpha < 2$ and $0 < \beta < \alpha - 1$.

Motivated by [17,19], in this paper, we discuss the existence and uniqueness of the positive solutions to the Sturm-Liouville boundary value problems of the nonlinear fractional differential equation of the form

$$\begin{cases} D_{0+}^{\alpha}u(t) + f(t, u(t), D_{0+}^{\beta}u(t)) = 0, \quad t \in (0,1), 1 < \alpha < 2, \\ a \lim\limits_{t\to 0} t^{2-\alpha}u(t) - b \lim\limits_{t\to 0} D_{0+}^{\alpha-1}u(t) = 0, \\ c\, D_{0+}^{\alpha-1}u(1) + du(1) = 0, \end{cases} \tag{2}$$

where $\alpha \in (1,2]$ and $\beta \in (\alpha-1,\alpha)$, $a,b,c,d \in [0,\infty)$, D_{0+}^{α} (or D_{0+}^{β}) is the Riemann-Liouville fractional derivative of order α (or β), and f is defined on $(0,1)\times[0,\infty)\times R$ and is nonnegative.

We obtain the results on the existence and uniqueness of the positive solutions of BVP(2) by using the fixed point theorems. An example is given to illustrate the efficiency of the main theorem.

To our knowledge, BVP(2) has not been studied in known papers since f is singular at $t=0$ and $t=1$ and $\beta \in (0,\alpha)$ is supposed. It is easy to see that BVP(2) becomes the BVP of the the following form

$$\begin{cases} u''(t)+f(t,u(t),u'(t))=0, & t\in(0,1), 1<\alpha<2,\\ au(t)-bu'(0)=0, \\ cu'(1)+du(1)=0, \end{cases}$$

if $\alpha=2$, where $f(t,u)$ is continuous and nonnegative on $[0,1]\times[0,\infty)$, $a\geq 0$, $b\geq 0$, $c\geq 0$ and $d\geq 0$ with $ab+cd+ab>0$. Such a problem is called Sturm-Liouville BVP [14,15] that comes from situation involving nonlinear elliptic problems in annular regions, one may see [14]. So our results generalize the theorems in [14,15,17].

The remainder of this paper is as follows: in section 2, we present preliminary results. In section 3, the main theorems are proved. An example is given in section 4 to illustrate the main results.

2 Preliminary results

For the convenience of the readers, we present here the necessary definitions from the fractional calculus theory. These definitions and results can be found in the literatures [3,4,6,7,9].

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $g:(0,\infty)\to R$ is given by

$$I_{0+}^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds,$$

provided that the right-hand side exists.

Definition 2.2. The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $g:(0,\infty)\to R$ is given by

$$D_{0+}^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t \frac{g(s)}{(t-s)^{\alpha-n+1}}ds,$$

where $n-1<\alpha\leq n$, provided that the right-hand side is point-wise defined on $(0,\infty)$.

Lemma 2.1. Let $n-1<\alpha\leq n$, $u\in C^0(0,\infty)\bigcap L^1(0,\infty)$. Then

$$I_{0+}^{\alpha}D_{0+}^{\alpha}u(t)=u(t)+C_1t^{\alpha-1}+C_2t^{\alpha-2}+\ldots+C_nt^{\alpha-n},$$

where $C_i\in R$, $i=1,2,\ldots n$.

Lemma 2.2. Suppose that $\delta=bd\Gamma(\alpha)+ad+ac\Gamma(\alpha)\neq 0$. Given a continuous function $h:(0,1)\to R$ satisfying that there exist $k>-1$ and $l\in(\max\{-1,\beta-\alpha\},0]$ such that

$|h(t)| \leq t^k(1-t)^l$ for all $t \in (0,1)$. Then the unique solution of

$$\begin{cases} D^{\alpha}_{0+}u(t) + h(t) = 0, 0 < t < 1, \\ a\lim_{t\to 0} t^{2-\alpha}u(t) - b\ \lim_{t\to 0}\ D^{\alpha-1}_{0+}u(t) = 0, \\ c\ D^{\alpha-1}_{0+}u(1) + du(1) = 0, \end{cases} \tag{3}$$

is

$$u(t) = \int_0^1 G(t,s)h(s)ds, \tag{4}$$

where

$$G(t,s) = \begin{cases} -\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{act^{\alpha-1}+bdt^{\alpha-2}(1-s)^{\alpha-1}+bc\Gamma(\alpha)t^{\alpha-2}}{\delta} + \frac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}, & s \leq t, \\ \frac{act^{\alpha-1}}{\delta} + \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta} + \frac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{bdt^{\alpha-2}(1-s)^{\alpha-1}}{\delta}, & t \leq s. \end{cases} \tag{5}$$

Proof. We may apply Lemma 2.1 to reduce BVP(3) to an equivalent integral equation

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2}, t \in (0,1)$$

for some $c_i \in R, \quad i = 1,2$. Since $\lim_{s\to 0}\Gamma(s) = \infty$, we get

$$t^{2-\alpha}u(t) = -t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + c_1t + c_2$$

and

$$D^{\alpha-1}u(t) = -\int_0^t h(s)ds + c_1\Gamma(\alpha).$$

It is easy to see from $k + l = 2 > 0$ that

$$t^{2-\alpha}\left|\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right| \leq t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(1-s)^l ds$$

$$\leq t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^k(t-s)^l ds$$

$$= t^{2-\alpha+\alpha+k+l}\int_0^1 \frac{(1-w)^{\alpha+l-1}}{\Gamma(\alpha)}w^k dw \to 0 \text{ as } t \to 0,$$

$$\left|\int_0^t h(s)ds\right| \leq \int_0^t s^k(1-s)^l ds \to 0 \text{ as } t \to 0.$$

From the boundary conditions in (3), we get

$$ac_2 - bc_1\Gamma(\alpha) = 0,$$
$$c\left(-\int_0^1 h(s)ds + c_1\Gamma(\alpha)\right) + d\left(-\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + c_1 + c_2\right) = 0.$$

It follows that

$$c_1 = \frac{ac}{\delta}\int_0^1 h(s)ds + \frac{ad}{\delta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds,$$

and

$$c_2 = \frac{bc\Gamma(\alpha)}{\delta}\int_0^1 h(s)ds + \frac{bd}{\delta}\int_0^1 (1-s)^{\alpha-1}h(s)ds.$$

Therefore, the unique solution of BVP(3) is

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds + \left[\frac{ac}{\delta}\int_0^1 h(s)ds + \frac{ad}{\delta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right]t^{\alpha-1} \\ &\quad + \left[\frac{bc\Gamma(\alpha)}{\delta}\int_0^1 h(s)ds + \frac{bd}{\delta}\int_0^1 (1-s)^{\alpha-1}h(s)ds\right]t^{\alpha-2} \\ &= \int_0^t \Bigg(\frac{act^{\alpha-1} + bdt^{\alpha-2}(1-s)^{\alpha-1} + bc\Gamma(\alpha)t^{\alpha-2}}{\delta} \\ &\qquad - \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}\Bigg)h(s)ds \\ &\quad + \int_t^1 \left(\frac{act^{\alpha-1}}{\delta} + \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta} + \frac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{bdt^{\alpha-2}(1-s)^{\alpha-1}}{\delta}\right)h(s)ds \\ &= \int_0^1 G(t,s)h(s)ds. \end{aligned}$$

Here G is defined by (5). Note $|\Gamma(0)| = \infty$ with $\frac{1}{\Gamma(0)} = 0$. Furthermore, we have for $\beta \neq \alpha - 1$ that

$$\begin{aligned} D_{0+}^{\beta}u(t) &= -\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}h(s)ds \\ &\quad + \left[\frac{ac}{\delta}\int_0^1 h(s)ds + \frac{ad}{\delta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right]\frac{\Gamma(\alpha)t^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} \\ &\quad + \left[\frac{bc\Gamma(\alpha)}{\delta}\int_0^1 h(s)ds + \frac{bd}{\delta}\int_0^1 (1-s)^{\alpha-1}h(s)ds\right]\frac{\Gamma(\alpha-1)t^{\alpha-\beta-2}}{\Gamma(\alpha-\beta-1)} \end{aligned}$$

and for $\beta = \alpha - 1$ that

$$D_{0+}^{\beta}u(t) = -\int_0^t h(s)ds + \left[\frac{ac}{\delta}\int_0^1 h(s)ds + \frac{ad}{\delta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}h(s)ds\right]\Gamma(\alpha).$$

Reciprocally, let u satisfy (4). Then

$$a\lim_{t\to 0} t^{2-\alpha}u(t) - b\lim_{t\to 0} D^{\alpha-1}u(t) = 0, c\, D_0^{\alpha-1}u(1) + du(1) = 0,$$

furthermore, we have $D_0^{\alpha}u(t) = -h(t)$. The proof is complete.

Lemma 2.3. Suppose that $\delta = bd\Gamma(\alpha) + ad + ac\Gamma(\alpha) > 0$. Then

$$t^{2-\alpha}G(t,s) \leq \frac{ad + ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}t + \frac{bc + bd + bc\Gamma(\alpha)}{\delta}, \text{ for all } s,t \in [0,1],$$

and

$$G(t,s) \geq 0 \text{ for all } t \in (0,1), s \in [0,1].$$

Proof. One sees from (5) that

$$\begin{aligned} t^{2-\alpha}G(t,s) &= \begin{cases} -\frac{t^{2-\alpha}(t-s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{act+bd(1-s)^{\alpha-1}+bc\Gamma(\alpha)}{\delta} + \frac{adt(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}, & s \leq t, \\ \frac{act}{\delta} + \frac{bc\Gamma(\alpha)}{\delta} + \frac{adt(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{bd(1-s)^{\alpha-1}}{\delta}, & t \leq s. \end{cases} \\ &\leq \begin{cases} \frac{act+bd(1-s)^{\alpha-1}+bc\Gamma(\alpha)}{\delta} + \frac{adt(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}, & s \leq t, \\ \frac{act}{\delta} + \frac{bc\Gamma(\alpha)}{\delta} + \frac{adt(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{bd(1-s)^{\alpha-1}}{\delta}, & t \leq s. \end{cases} \\ &\leq \frac{act}{\delta} + \frac{bc\Gamma(\alpha)}{\delta} + \frac{adt(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{(bc+bd)(1-s)^{\alpha-1}}{\delta} \\ &\leq \frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}t + \frac{bc+bd+bc\Gamma(\alpha)}{\delta}. \end{aligned}$$

On the other hand, we have from (5) that

$$\begin{aligned} G(t,s) &= \begin{cases} \frac{-\delta(t-s)^{\alpha-1}+ac\Gamma(\alpha)t^{\alpha-1}+bd\Gamma(\alpha)t^{\alpha-2}(1-s)^{\alpha-1}+adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta}, s \leq t, \\ \frac{act^{\alpha-1}}{\delta} + \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta} + \frac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{bdt^{\alpha-2}(1-s)^{\alpha-1}}{\delta}, t \leq s. \end{cases} \\ &= \begin{cases} \frac{ac\Gamma(\alpha)[t^{\alpha-1}-(t-s)^{\alpha-1}]+bd\Gamma(\alpha)[t^{\alpha-2}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]+ad[t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}]}{\delta\Gamma(\alpha)} \\ \qquad + \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta}, s \leq t, \\ \frac{act^{\alpha-1}}{\delta} + \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta} + \frac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{bdct^{\alpha-2}(1-s)^{\alpha-1}}{\delta}, t \leq s. \end{cases} \\ &\geq \begin{cases} \frac{-[bd\Gamma(\alpha)+ad+ac\Gamma(\alpha)](t-s)^{\alpha-1}+ac\Gamma(\alpha)t^{\alpha-1}+bd\Gamma(\alpha)(1-s)^{\alpha-1}+adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} \\ \qquad + \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta}, s \leq t, \\ \frac{act^{\alpha-1}}{\delta} + \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta} + \frac{adt^{\alpha-1}(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{bdt^{\alpha-2}(1-s)^{\alpha-1}}{\delta}, t \leq s. \end{cases} \\ &\geq \begin{cases} \frac{bc\Gamma(\alpha)t^{\alpha-2}}{\delta}, s \leq t, \\ 0, t \leq s. \end{cases} \\ &\geq 0. \end{aligned}$$

The proof is completed.

For our construction, we let

$$X = \left\{ x \in C(0,1] \quad : \quad \begin{array}{l} D_{0^+}^{\beta} x \in C(0,1] \\ \text{there exist the limits } \lim_{t\to 0} t^{2-\alpha}x(t), \\ \text{and } \lim_{t\to 0} t^{2+\beta-\alpha} D_{0^+}^{\beta} x(t) \end{array} \right\}.$$

For $x \in X$, let

$$\|x\| = \max\left\{ \sup_{0<t\leq 1} t^{2-\alpha}|x(t)|, \sup_{0<t\leq 1} t^{2+\beta-\alpha}|D_{0^+}^{\beta} x(t)| \right\}.$$

Then X is a Banach space. We seek solutions of BVP(2) that lie in the cone

$$P = \{u \in X : u(t) \geq 0,\ 0 < t \leq 1\}.$$

Suppose that $\delta = bd\Gamma(\alpha) + ad + ac\Gamma(\alpha) > 0$. Define the operator $T : P \to P$, by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds.$$

Lemma 2.4. Suppose that $\delta = bd\Gamma(\alpha) + ad + ac\Gamma(\alpha) > 0$ and $f : (0,1) \times [0,\infty) \times \to [0,\infty)$ satisfies

(i) $t \to f(t, t^{\alpha-2}u, t^{2+\beta-\alpha}v)$ is continuous,

(ii) $(u,v) \to f(t, t^{\alpha-2}u, t^{2+\beta-\alpha}v)$ is continuous,

(iii) for each $r > 0$ there exist $M_r > 0$, $k > -1$ and $l \in (\max\{-1, \beta-\alpha\}, 0]$ such that $|f(t, t^{\alpha-2}u, t^{2+\beta-\alpha}v)| \leq Mt^k(1-t)^l$ for all $t \in (0,1)$ and $|u|, |v| \leq r$.

Then T is completely continuous.

Proof. We divide the proof into three steps.

Step 1. T is continuous.

Let $\{y_n\}_{n=0}^{\infty}$ be a sequence such that $y_n \to y_0$ in X. Then

$$r = \sup_{n=0,1,2,\cdots} ||y_n|| < \infty.$$

So there exist $M_r > 0$, $k > -1$ and $l \in (\max\{-1, \beta-\alpha\}, 0]$ such that

$$|f(t, y_n(t), D_{0+}^{\beta}y_n(t))| = |f(t, t^{\alpha-2}t^{2-\alpha}y_n(t), t^{\alpha-\beta-2}t^{2+\beta-\alpha}D_{0+}^{\beta}y_n(t))| \leq Mt^k(1-t)^l, t \in (0,1).$$

Then for $t \in (0,1)$, we have

$$\begin{aligned}
& t^{2-\alpha}|(Ty_n)(t) - (Ty_0)(t)| \\
= & \left| \int_0^1 t^{2-\alpha}G(t,s)f(s,y_n(s),D_{0+}^{\beta}y_n(s))ds - \int_0^1 t^{2-\alpha}G(t,s)f(s,y_0(s),D_{0+}^{\beta}y_0(s))ds \right| \\
\leq & \int_0^1 t^{2-\alpha}G(t,s)|f(s,y_n(s),D_{0+}^{\beta}y_n(s)) - f(s,y_0(s),D_{0+}^{\beta}y_0(s))|ds \\
\leq & \left(\frac{ad + ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}t + \frac{bc + bd + bc\Gamma(\alpha)}{\delta} \right) \int_0^1 |f(s,y_n(s),D_{0+}^{\beta}y_n(s)) - f(s,y_0(s),D_{0+}^{\beta}y_0(s))|ds \\
\leq & \left(\frac{ad + ac\Gamma(\alpha)}{\delta\Gamma(\alpha)} + \frac{bc + bd + bc\Gamma(\alpha)}{\delta} \right) \int_0^1 |f(s,y_n(s),D_{0+}^{\beta}y_n(s)) - f(s,y_0(s),D_{0+}^{\beta}y_0(s))|ds \\
\leq & 2M_r \left(\frac{ad + ac\Gamma(\alpha)}{\delta\Gamma(\alpha)} + \frac{bc + bd + bc\Gamma(\alpha)}{\delta} \right) \int_0^1 s^k(1-s)^l ds,
\end{aligned}$$

and

$$\begin{aligned}
& t^{2+\beta-\alpha}|D_{0+}^{\beta}(Ty_n)(t) - D_{0+}^{\beta}(Ty_0)(t)| \\
\leq & t^{2+\beta-\alpha} \int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} |f(s,y_n(s),D_{0+}^{\beta}y_n(s)) - f(s,y_0(s),D_{0+}^{\beta}y_0(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&+\left[\frac{ac}{\delta}\int_0^1 |f(s,y_n(s),D^{\beta}_{0+}y_n(s))-f(s,y_0(s),D^{\beta}_{0+}y_0(s))|ds\right.\\
&\left.+\frac{ad}{\delta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y_n(s),D^{\beta}_{0+}y_n(s))-f(s,y_0(s),D^{\beta}_{0+}y_0(s))|ds\right]\frac{\Gamma(\alpha)t}{\Gamma(\alpha-\beta)}\\
&+\left[\frac{bc\Gamma(\alpha)}{\delta}\int_0^1 |f(s,y_n(s),D^{\beta}_{0+}y_n(s))-f(s,y_0(s),D^{\beta}_{0+}y_0(s))|ds\right.\\
&\left.+\frac{bd}{\delta}\int_0^1 (1-s)^{\alpha-1}|f(s,y_n(s),D^{\beta}_{0+}y_n(s))-f(s,y_0(s),D^{\beta}_{0+}y_0(s))|ds\right]\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-\beta-1)|}\\
\leq\ & 2M_r t^{2+\beta-\alpha}\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}s^k(1-s)^l ds\\
&+2M_r\left[\frac{ac}{\delta}\int_0^1 s^k(1-s)^l ds+\frac{ad}{\delta}\int_0^1\frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^k ds\right]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\\
&+2M_r\left[\frac{bc\Gamma(\alpha)}{\delta}\int_0^1 s^k(1-s)^l ds+\frac{bd}{\delta}\int_0^1 (1-s)^{\alpha+l-1}s^k ds\right]\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-\beta-1)|}\\
\leq\ & 2M_r t^{2+k+l}\int_0^1 \frac{(1-w)^{\alpha+l-\beta-1}}{\Gamma(\alpha-\beta)}w^k dw\\
&+2M_r\left[\frac{ac}{\delta}\int_0^1 s^k(1-s)^l ds+\frac{ad}{\delta}\int_0^1\frac{(1-s)^{\alpha+l-1}}{\Gamma(\alpha)}s^k ds\right]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\\
&+2M_r\left[\frac{bc\Gamma(\alpha)}{\delta}\int_0^1 s^k(1-s)^l ds+\frac{bd}{\delta}\int_0^1 (1-s)^{\alpha+l-1}s^k ds\right]\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-\beta-1)|}\\
&\leq 2M_r\frac{\mathbf{B}(\alpha+l-\beta,k+1)}{\Gamma(\alpha-\beta)}+2M_r\left[\frac{ac}{\delta}\mathbf{B}(l+1,k+1)+\frac{ad}{\delta}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\right]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}\\
&+2M_r\left[\frac{bc\Gamma(\alpha)}{\delta}\mathbf{B}(l+1,k+1)+\frac{bd}{\delta}\mathbf{B}(\alpha+l,k+1)\right]\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-\beta-1)|}.
\end{aligned}
$$

By the assumptions imposed on f and dominated convergence theorem, one sees that

$$\sup_{t\in(0,1)} t^{2-\alpha}|(Ty_n)(t)-(Ty_0)(t)|\to 0,\ \sup_{t\in(0,1)} t^{2+\beta-\alpha}|D^{\beta}_{0+}(Ty_n)(t)-D^{\beta}_{0+}(Ty_0)(t)|\to 0, n\to\infty.$$

Then we have $||Ty_n-Ty||\to 0$ as $n\to\infty$. So T is continuous.

Step 2. T maps bounded sets into bounded sets in X.

It suffices to show that for each $r>0$, there exists a positive number $L>0$ such that for each $x\in M=\{y\in X:||y||\leq r\}$, we have $||Ty||\leq L$. By the assumption, there exist $M_r>0$, $k>-1$ and $l\in(\max\{-1,\beta-\alpha\},0]$ such that

$$|f(t,y(t),D^{\beta}_{0+}y(t))|=f(t,t^{\alpha-2}t^{2-\alpha}y(t),t^{\alpha-\beta-2}t^{2+\beta-\alpha}D^{\beta}_{0+}y(t))|\leq M_r t^k(1-t)^l, t\in(0,1).$$

By the definition of T, we get

$$
\begin{aligned}
t^{2-\alpha}|(Ty)(t)| &= \int_0^1 t^{2-\alpha}G(t,s)f(s,y(s),D^{\beta}_{0+}y(s))ds\\
&\leq M_r\left(\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bc+bd+bc\Gamma(\alpha)}{\delta}\right)\int_0^1 s^k(1-s)^l ds
\end{aligned}
$$

$$\leq \ M_r\left(\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}+\frac{bc+bd+bc\Gamma(\alpha)}{\delta}\right)\mathbf{B}(l+1,k+1)=:\prod_1.$$

Similarly we get that

$$\begin{aligned}
& t^{2+\beta-\alpha}|D_{0+}^{\beta}(Ty)(t)| \\
\leq \ & t^{2+\beta-\alpha}\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,y(s),D_{0+}^{\beta}y(s))|ds \\
& +\left[\frac{ac}{\delta}\int_0^1|f(s,y(s),D_{0+}^{\beta}y(s))|ds\right. \\
& \left.+\frac{ad}{\delta}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s),D_{0+}^{\beta}y(s))|ds\right]\frac{\Gamma(\alpha)t}{\Gamma(\alpha-\beta)} \\
& +\left[\frac{bc\Gamma(\alpha)}{\delta}\int_0^1|f(s,y(s),D_{0+}^{\beta}y(s))|ds\right. \\
& \left.+\frac{bd}{\delta}\int_0^1(1-s)^{\alpha-1}|f(s,y(s),D_{0+}^{\beta}y(s))|ds\right]\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-\beta-1)|} \\
& \leq M_r\frac{\mathbf{B}(\alpha+l-\beta,k+1)}{\Gamma(\alpha-\beta)}+M_r\left[\frac{ac}{\delta}\mathbf{B}(l+1,k+1)+\frac{ad}{\delta}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\right]\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \\
& +M_r\left[\frac{bc\Gamma(\alpha)}{\delta}\mathbf{B}(l+1,k+1)+\frac{bd}{\delta}\mathbf{B}(\alpha+l,k+1)\right]\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-\beta-1)|}+:\prod_2.
\end{aligned}$$

It follows that

$$||Ty||\leq\max\{\prod_1,\prod_2\}$$

for each $y\in\{y\in X:||y||\leq r\}$. Then T maps bounded sets into bounded sets in X.

Step 3. Let $M=\{y\in X:||y||\leq r\}$. Prove that both $t^{2-\alpha}TM$ is equicontinuous on $[0,1]$ and $t^{2+\beta-\alpha}D_{0+}^{\beta}TM$ is equicontinuous on each subinterval $[a,b]\subset(0,1]$, equiconvergent at $t=0$.

Let $t_1,t_2\in[0,1]$ with $t_1<t_2$ and $y\in M=\{y\in X:||y||\leq r\}$. By the assumption, there exist $M_r\geq 0$, $k>-1$, $l>(\max\{-1,\beta-\alpha,0]$ such that

$$|f(t,y(t),D_{0+}^{\beta}y(t))|=f(t,t^{\alpha-2}t^{2-\alpha}y(t),t^{\alpha-\beta-2}t^{2+\beta-\alpha}D_{0+}^{\beta}y(t))|\leq M_rt^k(1-t)^l,t\in(0,1).$$

One sees, for $s\in[0,t_1]$, from (5) that

$$\begin{aligned}
t_1^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}G(t_2,s) \ &= \ \frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}-t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} \\
&\quad +\frac{ac[t_1-t_2]}{\delta}+\frac{ad[t_1-t_2](1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}.
\end{aligned}$$

For $s\in[t_1,t_2]$, we have

$$t_1^{2-\alpha}G(t_1,s)-t_2^{2-\alpha}G(t_2,s) \ = \ \frac{ac[t_1-t_2]}{\delta}+\frac{ad[t_1-t_2](1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}+\frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1}}{\Gamma(\alpha)}.$$

For $s \in [t_1, 1]$, we have

$$t_1^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}G(t_2,s) \quad = \quad \frac{ac[t_1-t_2]}{\delta} + \frac{ad[t_1-t_2](1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}.$$

Then

$$\begin{aligned}
& |t_1^{2-\alpha}(Ty)(t_1) - t_1^{2-\alpha}(Ty)(t_2)| \\
= & \left|\int_0^1 t_1^{2-\alpha}G(t_1,s)f(s,y(s),D_{0+}^{\beta}y(s))ds - \int_0^1 t_2^{2-\alpha}G(t_2,s)f(s,y(s),D_{0+}^{\beta}y(s))ds\right| \\
\leq & \int_0^{t_1} |t_1^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}G(t_2,s)|f(s,y(s),D_{0+}^{\beta}y(s))ds \\
& + \int_{t_1}^{t_2} |t_1^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}G(t_2,s)|f(s,y(s),D_{0+}^{\beta}y(s))ds \\
& + \int_{t_2}^{1} |t_1^{2-\alpha}G(t_1,s) - t_2^{2-\alpha}G(t_2,s)|f(s,y(s),D_{0+}^{\beta}y(s))ds \\
\leq & \ M_r \int_0^1 \frac{t_2^{2-\alpha}(t_2-s)^{\alpha-1} - t_1^{2-\alpha}(t_1-s)^{\alpha-1}}{\Gamma(\alpha)} s^k(1-s)^l ds \\
& + M_r \int_{t_1}^{t_2} \left(\frac{ac}{\delta} + \frac{ad(1-s)^{\alpha-1}}{\delta\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha)}\right) s^k(1-s)^l ds \\
& + \int_0^1 \left(\frac{ac[t_1-t_2]}{\delta} + M_r\frac{ad[t_1-t_2](1-s)^{\alpha-1}}{\delta\Gamma(\alpha)}\right) s^k(1-s)^l ds.
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero. Therefore, $t^{2-\alpha}TM$ is equicontinuous.

Note that $e^{\lambda} - f^{\lambda} \leq (e-f)^{\lambda}$ for all $e > f > 0$ and $\lambda \in (0,1]$. For each subinterval $[a,b] \subset (0,1]$

and $t_1, t_2 \in [a, b]$ withhh $t_1 < t_2$, similarly, we have

$$|t_1^{2+\beta-\alpha} D_{0^+}^{\beta}(Ty)(t_1) - t_2^{2+\beta-\alpha} D_{0^+}^{\beta}(Ty)(t_2)|$$

$$\leq \left| t_1^{2+\beta-\alpha} \int_0^{t_1} \frac{(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s, y(s), D_{0^+}^{\beta} y(s)) ds - t_2^{2+\beta-\alpha} \int_0^{t_2} \frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} f(s, y(s), D_{0^+}^{\beta} y(s) ds \right|$$

$$+ \left[\frac{ac}{\delta} \int_0^1 |f(s, y(s), D_{0^+}^{\beta} y(s))| ds + \frac{ad}{\delta} \int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} |f(s, y(s), D_{0^+}^{\beta} y(s))| ds \right] \frac{\Gamma(\alpha)|t_1-t_2|}{\gamma(\alpha-\beta)}$$

$$\leq \left| t_1^{2+\beta-\alpha} - t_2^{2+\beta-\alpha} \right| \int_0^{t_2} \frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} M_r s^k (1-s)^l ds + t_1^{2+\beta-\alpha} \int_{t_1}^{t_2} \frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} M_r s^k (1-s)^l ds$$

$$+ t_1^{2+\beta-\alpha} \int_0^{t_1} \frac{|(t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}|}{\Gamma(\alpha-\beta)} M_r s^k (1-s)^l ds$$

$$+ \left[\frac{ac}{\delta} M_r \mathbf{B}(l+1, k+1) + \frac{ad}{\delta} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} M_r \right] \frac{\Gamma(\alpha)|t_1-t_2|}{\gamma(\alpha-\beta)}$$

$$\leq \left| t_1^{2+\beta-\alpha} - t_2^{2+\beta-\alpha} \right| \max\{a^{\alpha-\beta+k+l}, b^{\alpha-\beta+k+l}\} \frac{\mathbf{B}(\alpha-\beta+l, k+1)}{\Gamma(\alpha-\beta)} M_r$$

$$+ M_r b^{2+\beta-\alpha} \max\{a^{\alpha-\beta+k+l}, b^{\alpha-\beta+k+l}\} \int_{\frac{t_1}{t_2}}^{1} \frac{(1-w)^{\alpha-\beta+l-1}}{\Gamma(\alpha-\beta)} w^k dw$$

$$+ M_r b^{2+\beta-\alpha} \begin{cases} \int_0^{t_1} \frac{(t_2-s)^{\alpha-\beta-1} - (t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} s^k (1-s)^l ds, \beta < \alpha - 1 \\ 0, \qquad \beta = \alpha - 1, \\ \int_0^{t_1} \frac{(t_1-s)^{\alpha-\beta-1} - (t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} s^k (t_1-s)^l ds, \beta > \alpha - 1 \end{cases}$$

$$+ \left[\frac{ac}{\delta} M_r \mathbf{B}(l+1, k+1) + \frac{ad}{\delta} \frac{\mathbf{B}(\alpha+l, k+1)}{\Gamma(\alpha)} M_r \right] \frac{\Gamma(\alpha)|t_1-t_2|}{\gamma(\alpha-\beta)}$$

$$\leq \left| t_1^{2+\beta-\alpha} - t_2^{2+\beta-\alpha} \right| \max\{a^{\alpha-\beta+k+l}, b^{\alpha-\beta+k+l}\} \frac{\mathbf{B}(\alpha-\beta+l, k+1)}{\Gamma(\alpha-\beta)} M_r$$

$$+ M_r b^{2+\beta-\alpha} \max\{a^{\alpha-\beta+k+l}, b^{\alpha-\beta+k+l}\} \int_{\frac{t_1}{t_2}}^{1} \frac{(1-w)^{\alpha-\beta+l-1}}{\Gamma(\alpha-\beta)} w^k dw$$

$$+ M_r b^{2+\beta-\alpha} \begin{cases} (t_2-t_1)^{\alpha-\beta-1} \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 s^k (1-s)^l ds, \beta < \alpha - 1 \\ 0, \qquad \beta = \alpha - 1, \\ \int_0^{t_1} \frac{(t_1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} s^k (t_1-s)^l ds - \int_0^{t_1} \frac{(t_2-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} s^k (t_2-s)^l ds, \beta > \alpha - 1 \end{cases}$$

$$+\left[\frac{ac}{\delta}M_r\mathbf{B}(l+1,k+1)+\frac{ad}{\delta}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}M_r\right]\frac{\Gamma(\alpha)|t_1-t_2|}{\gamma(\alpha-\beta)}$$

$$\leq\left|t_1^{2+\beta-\alpha}-t_2^{2+\beta-\alpha}\right|\max\{a^{\alpha-\beta+k+l},b^{\alpha-\beta+k+l}\}\frac{\mathbf{B}(\alpha-\beta+l,k+1)}{\Gamma(\alpha-\beta)}M_r$$

$$+M_rb^{2+\beta-\alpha}\max\{a^{\alpha-\beta+k+l},b^{\alpha-\beta+k+l}\}\int_{\frac{t_1}{t_2}}^1\frac{(1-w)^{\alpha-\beta+l-1}}{\Gamma(\alpha-\beta)}w^kdw$$

$$+M_rb^{2+\beta-\alpha}\begin{cases}(t_2-t_1)^{\alpha-\beta-1}\frac{1}{\Gamma(\alpha-\beta)}\mathbf{B}(l+1,k+1),\beta<\alpha-1\\ 0,\qquad \beta=\alpha-1,\\ t_1^{\alpha-\beta+k+l}\int_0^1\frac{(1-w)^{\alpha+l-\beta-1}}{\Gamma(\alpha-\beta)}w^kdw-t_2^{\alpha-\beta+k+l}\int_0^{\frac{t_1}{t_2}}\frac{(1-w)^{\alpha+l-\beta-1}}{\Gamma(\alpha-\beta)}w^kdw,\beta>\alpha-1\end{cases}$$

$$+\left[\frac{ac}{\delta}M_r\mathbf{B}(l+1,k+1)+\frac{ad}{\delta}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}M_r\right]\frac{\Gamma(\alpha)|t_1-t_2|}{\gamma(\alpha-\beta)}.$$

As $t_1\to t_2$, the right-hand side of the above inequality tends to zero. Therefore, $t^{2+\beta-\alpha}D_{0+}^{\beta}TM$ is equicontinuous on $[a,b]$.

$$\left|t^{2+\beta-\alpha}D_{0+}^{\beta}(Ty)(t)-\left[\frac{bc\Gamma(\alpha)}{\delta}\int_0^1h(s)ds+\frac{bd}{\delta}\int_0^1(1-s)^{\alpha-1}h(s)ds\right]\frac{\Gamma(\alpha-1)}{\Gamma(\alpha-\beta-1)}\right|$$

$$\leq t^{2+\beta-\alpha}\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,y(s),D_{0+}^{\beta}y(s))|ds$$

$$\left[\frac{ac}{\delta}\int_0^1|f(s,y(s),D_{0+}^{\beta}y(s))|ds+\frac{ad}{\delta}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,y(s),D_{0+}^{\beta}y(s))|)ds\right]\frac{\Gamma(\alpha)t}{\gamma(\alpha-\beta)}$$

$$\leq M_rt^{2+k+l}\frac{\mathbf{B}(\alpha-\beta+l,k+1)}{\Gamma(\alpha-\beta)}+M_r\left[\frac{ac}{\delta}\mathbf{B}(l+1,k+1)+\frac{ad}{\delta}\frac{\mathbf{B}(\alpha+l,k+1)}{\Gamma(\alpha)}\right]\frac{\Gamma(\alpha)t}{\gamma(\alpha-\beta)}$$

$$\to 0 \text{ uniformly as } t\to 0.$$

Therefore, $t^{2+\beta-\alpha}D_{0+}^{\beta}TM$ is equiconvergent at $t=0$.

The Arzela-Ascoli theorem implies that $\overline{T(M)}$ is relatively compact. Thus, the operator $T:P\to P$ is completely continuous.

3 Main Results

In this section, we prove the main results.

Theorem 3.1. Suppose that $\delta=bd\Gamma(\alpha)+ad+ac\Gamma(\alpha)>0$, and $f:(0,1)\times[0,\infty)\times R\to[0,\infty)$ satisfies

(i) $t\to f(t,t^{\alpha-2}u,t^{2+\beta-\alpha}v)$ is continuous,

(ii) $(u,v)\to f(t,t^{\alpha-2}u,t^{2+\beta-\alpha}v)$ is continuous,

(iii) for each $r>0$ there exist $M_r>0$, $k>-1$ and $l\in(\max\{-1,\beta-\alpha\},0]$ such that $|f(t,t^{\alpha-2}u,t^{2+\beta-\alpha}v)|\leq Mt^k(1-t)^l$ for all $t\in(0,1)$ and $|u|,|v|\leq r$.

(iv) there exist numbers $k_1, k_2 > -1$ and $l_1, l_2 > \max\{-1, \beta-\alpha]$, $M_1, M_2 \geq 0$ such that f satisfies $f(t,0,0) \not\equiv 0$ for $t \in (0,1)$ and

$$|f(t, t^{\alpha-2}u_1, t^{\alpha-\beta-2}v_1) - f(t, t^{\alpha-2}u_1, t^{\alpha-\beta-2}v_2)| \leq M_1 t^{k_1}(1-t)^{l_1}|u_1-u_2| + M_2 t^{k_2}(1-t)^{l_2}|v_1-v_2|,$$

holds for all $t \in (0,1), u_1, u_2 \in [0,\infty)$ and $v_1, v_2 \in R$.

Then BVP(2) has a unique positive solution if

$$M < 1, \tag{6}$$

where

$$M = \max\Big\{ \tfrac{ad+bc+bd+ac\Gamma(\alpha)+bc\Gamma(\alpha)}{\delta\Gamma(\alpha)}[M_1\mathbf{B}(l_1+1,k_1+1) + M_2\mathbf{B}(l_2+1,k_2+1)],$$

$$[M_1\mathbf{B}(\alpha-\beta+l_1,k_1+1) + M_2\mathbf{B}(\alpha-\beta+l_2,k_2+1) + \tfrac{adM_1+bd\Gamma(\alpha-1)M_1}{\delta}\mathbf{B}(\alpha+l_1,k_1+1)$$

$$+\tfrac{ac\Gamma(\alpha)M_1+|\alpha-\beta-1|\Gamma(\alpha-1)\Gamma(\alpha)bcM_1}{\delta}\mathbf{B}(l_1+1,k_1+1) + \tfrac{ac\Gamma(\alpha)M_2+ad\Gamma(\alpha)M_2}{\delta}\mathbf{B}(l_2+1,k_2+1)$$

$$+\tfrac{|\alpha-\beta-1|\Gamma(\alpha)\Gamma(\alpha-1)bcM_2+bd\Gamma(\alpha-1)M_2}{\delta}\mathbf{B}(l_2+1,k_2+1)\Big]\,\tfrac{1}{\Gamma(\alpha-\beta)}\Big\}.$$

Proof. We shall prove that under the assumptions supposed, T is a contraction operator. Indeed, by the definition of $G(t,s)$ for $u, v \in P$, from Lemma 2.3, we have the estimate

$$\begin{aligned}
& t^{2-\alpha}|(Tu)(t) - (Tv)(t)| \\
\leq\ & \int_0^1 t^{2-\alpha}G(t,s)|f(s,u(s),D_{0+}^{\beta}u(s)) - f(s,v(s),D_{0+}^{\beta}v(s))|ds \\
\leq\ & \left(\frac{ad+ac\Gamma(\alpha)}{\delta\Gamma(\alpha)}t + \frac{bc+bd+bc\Gamma(\alpha)}{\delta}\right)\times \\
& \int_0^1 \left[M_1 s^{k_1}(1-s)^{l_1}s^{2-\alpha}|u(s)-v(s)| + M_2 s^{k_2}(1-s)^{l_2}s^{2+\beta-\alpha}|D_{0+}^{\beta}u(s) - D_{0+}^{\beta}v(s)|\right]ds \\
\leq\ & \frac{ad+bc+bd+ac\Gamma(\alpha)+bc\Gamma(\alpha)}{\delta\Gamma(\alpha)}[M_1\mathbf{B}(l_1+1,k_1+1) + M_2\mathbf{B}(l_2+1,k_2+1)]||u-v|| \\
\leq\ & M||u-v||.
\end{aligned}$$

Similarly we get

$$\begin{aligned}
& t^{2+\beta-\alpha}|D_{0+}^{\beta}(Tu)(t) - D_{0+}^{\beta}(Tv)(t)| \\
\leq\ & t^{2+\beta-\alpha}\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}|f(s,u(s),D_{0+}^{\beta}u(s)) - f(s,v(s),D_{0+}^{\beta}v(s))|ds \\
& + \left[\frac{ac}{\delta}\int_0^1 |f(s,u(s),D_{0+}^{\beta}u(s)) - f(s,v(s),D_{0+}^{\beta}v(s))|ds \right. \\
& \left. + \frac{ad}{\delta}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s),D_{0+}^{\beta}u(s)) - f(s,v(s),D_{0+}^{\beta}v(s))|ds\right]\frac{\Gamma(\alpha)t}{\Gamma(\alpha-\beta)}
\end{aligned}$$

$$
\begin{aligned}
& +\left[\frac{bc\Gamma(\alpha)}{\delta}\int_0^1 |f(s,u(s),D_{0+}^{\beta}u(s))-f(s,v(s),D_{0+}^{\beta}v(s))|ds\right.\\
& \left.+\frac{bd}{\delta}\int_0^1 (1-s)^{\alpha-1}|f(s,u(s),D_{0+}^{\beta}u(s))-f(s,v(s),D_{0+}^{\beta}v(s))|ds\right]\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-\beta-1)|}\\
\leq\ & t^{2+\beta-\alpha}\int_0^t \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}[M_1s^{k_1}(1-s)^{l_1}+M_2s^{k_2}(1-s)^{l_2}]ds||u-v||\\
& +\left[\frac{ac}{\delta}\int_0^1[M_1s^{k_1}(1-s)^{l_1}+M_2s^{k_2}(1-s)^{l_2}]ds||u-v||\right.\\
& \left.+\frac{ad}{\delta}\int_0^1\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}[M_1s^{k_1}(1-s)^{l_1}+M_2s^{k_2}(1-s)^{l_2}]ds||u-v||\right]\frac{\Gamma(\alpha)t}{\Gamma(\alpha-\beta)}\\
& +\left[\frac{bc\Gamma(\alpha)}{\delta}\int_0^1[M_1s^{k_1}(1-s)^{l_1}+M_2s^{k_2}(1-s)^{l_2}]ds||u-v||\right.\\
& \left.+\frac{bd}{\delta}\int_0^1(1-s)^{\alpha-1}[M_1s^{k_1}(1-s)^{l_1}+M_2s^{k_2}(1-s)^{l_2}]ds||u-v||\right]\frac{\Gamma(\alpha-1)}{|\Gamma(\alpha-\beta-1)|}\\
\leq\ & [M_1\mathbf{B}(\alpha-\beta+l_1,k_1+1)+M_2\mathbf{B}(\alpha-\beta+l_2,k_2+1)\\
& +\frac{adM_1+bd\Gamma(\alpha-1)M_1}{\delta}\mathbf{B}(\alpha+l_1,k_1+1)\\
& +\frac{ac\Gamma(\alpha)M_1+|\alpha-\beta-1|\Gamma(\alpha-1)\Gamma(\alpha)bcM_1}{\delta}\mathbf{B}(l_1+1,k_1+1)\\
& +\frac{ac\Gamma(\alpha)M_2+ad\Gamma(\alpha)M_2}{\delta}\mathbf{B}(l_2+1,k_2+1)\\
& \left.+\frac{|\alpha-\beta-1|\Gamma(\alpha)\Gamma(\alpha-1)bcM_2+bd\Gamma(\alpha-1)M_2}{\delta}\mathbf{B}(l_2+1,k_2+1)\right]\frac{||u-v||}{\Gamma(\alpha-\beta)}\\
\leq\ & M||u-v||.
\end{aligned}
$$

Therefor, we get that

$$||Tu-T||\leq M||u-v||.$$

Hence the contraction map principle implies that Tb has a fixed point $x\in P$. Since $T:P\to P$ and $f(t,0,0)\not\equiv 0$ for $t\in(0,1)$, then x is positive. So BVP(2) has an unique positive solution. The proof is completed.

4 An example

In this section, we give an example to illustrate the main theorem.

Example 4.1. Let$\lambda>0$. Consider the following BVP

$$
\begin{cases}
D_{0+}^{\frac{3}{2}}u(t)+t^{-\frac{1}{8}}(1-t)^{-\frac{1}{8}}\left(A+Bt^{\frac{1}{2}}u(t)+Dt^{\frac{9}{4}}|D_{0+}^{\frac{5}{4}}u(t)|\right)=0, \quad t\in(0,1],1<\alpha<2,\\
\lim_{t\to 0}t^{\frac{1}{2}}u(t)-\lim_{t\to 0}\ D_{0+}^{\frac{1}{2}}u(t)=0,\\
D_{0+}^{\frac{1}{2}}u(1)+u(1)=0.
\end{cases}
\tag{7}
$$

where $A, B, D \geq 0$.

Corresponding to BVP(2), we find that $\alpha = \frac{3}{2}$, $\beta = \frac{5}{4}$, $a = b = c = d = 1$ and

$$f(t,x,y) = t^{-\frac{1}{8}}(1-t)^{-\frac{1}{8}}\left(A + Bt^{\frac{1}{2}}x + Dt^{\frac{9}{4}}|y|\right).$$

One sees that $\delta = 1 + 2\Gamma(3/2) > 0$, $f(t,0,0) \not\equiv 0$ for $t \in (0,1]$, $f(t,u)$ satisfies

$$|f(t, t^{\frac{3}{2}-2}u_1, t^{\frac{3}{2}-\frac{5}{4}-2}v_1) - f(t, t^{\frac{3}{2}-2}u_2, , t^{\frac{3}{2}-\frac{5}{4}-2}v_2)|$$

$$\leq t^{-\frac{1}{8}}(1-t)^{-\frac{1}{8}}[B|u_1-u_2| + D|v_1-v_2|], t \in (0,1), u_1, u_2 \in [0,\infty), v_1, v_2 \in R.$$

Let $M_1 = B, M_2 = D$ and $k_1 = k_2 = -\frac{1}{8}$ and $l_1 = l_2 = -\frac{1}{8}$. Hence Theorem 3.1 implies that BVP(7) has a unique positive solution if

$$\begin{aligned} M = \max\Big\{ & \tfrac{3+2\Gamma(3/2)}{(1+2\Gamma(3/2))\Gamma(3/2)}[B\mathbf{B}(7/8,7/8) + D\mathbf{B}(7/8,7/8)], \\ & [B\mathbf{B}(1/8,7/8) + D\mathbf{B}(1/8,7/8) \ + B\tfrac{1+\Gamma(1/2)}{1+2\Gamma(3/2)}\mathbf{B}(11/8,7/8) \\ & +B\tfrac{\Gamma(3/2)+\frac{3}{4}\Gamma(1/2)\Gamma(3/2)}{1+2\Gamma(3/2)}\mathbf{B}(7/8,7/8) \ + D\tfrac{\Gamma(3/2)+\Gamma(3/2)}{1+2\Gamma(3/2)}\mathbf{B}(7/8,7/8) \\ & +D\tfrac{\frac{3}{4}\Gamma(3/2)\Gamma(1/2)+\Gamma(1/2)}{1+2\Gamma(3/2)}\mathbf{B}(7/8,7/8)\Big] \tfrac{1}{\Gamma(1/4)}\Big\} < 1. \end{aligned} \tag{8}$$

Since (8) holds for sufficiently small B and D. It is easy to see that BVP(7) has a unique positive solution for sufficiently small B and D.

References

[1] R.P. Agarwal, M. Benchohra, S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl Math, 109(2010)973-1033.

[2] A. Arara, M. Benchohra, N. Hamidi, and J.J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Anal., 72(2010)580-586,

[3] D. Guo, and J. Sun, it Nonlinear Integral Equations, Shandong Science and Technology Press, Jinan, 1987 (in Chinese).

[4] A. A. Kilbas, O.I. Marichev, and S. G. Samko, *Fractional Integral and Derivatives (Theory and Applications)*, Gordon and Breach, Switzerland, 1993.

[5] A. A. Kilbas, and J.J. Trujillo, *Differential equations of fractional order: methods, results and problems-I*, Applicable Analysis, 78(2001)153–192.

[6] A. Arara, M. Benchohra, N. Hamidi, and J. Nieto, *Fractional order differential equations on an unbounded domain*, Nonlinear Analysis, 72(2010)580-586.

[7] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equation*, Wiley, New York, 1993.

[8] S. Z. Rida, H.M. El-Sherbiny, and A. Arafa, *On the solution of the fractional nonlinear Schrodinger equation*, Physics Letters A, 372(2008)553-558.

[9] S. Samko, A. Kilbas, and O. Marichev, *Fractional Integral and Derivative, Theory and Applications*, Gordon and Breach, 1993.

[10] S. Zhang, *The existence of a positive solution for a nonlinear fractional differential equation*, J. Math. Anal. Appl. 252(2000)804–812.

[11] E. Kaufmann, E. Mboumi, *Positive solutions of a boundary value problem for a nonlinear fractional differential equation*, Electronic Journal of Qualitative Theory of Differential Equations, 3(2008)1-11.

[12] R. Dehghant and K. Ghanbari, *Triple positive solutions for boundary value problem of a nonlinear fractional differential equation*, Bulletin of the Iranian Mathematical Society, 33(2007)1-14.

[13] Y. Liu, *Positive solutions for singular FDES*, U.P.B. Sci. Series A, 73(2011)89-100.

[14] L. Erbe and M. Tang, *Existence and multiplicity of positive solutions to nonlinear boundary value problems*, Diff. Equs. Dynam. Systems, 4(1996)313-320.

[15] L. Erbe and M. Tang, *On the existence of positive solutions of ordinary differential equations*, Proc. Amer. math. Soc. 120(1994)743-748.

[16] P. J. Torvik, R. L. Bagley, *On the appearance of the fractional derivative in the behavior of real materials*, J. Appl. Mech. 51 (1984) 294-298.

[17] Y. Liu, T. He, H. Shi, Existence of positive solutions for Sturm-Liouville BVPs of singular fractional differential equations, U.P.B. Sci. Bull., Series A, 74(2012)93-108.

[18] S. Liang, J. Zhang, Existence and uniqueness of positive solutions to m-point boundary value problem for nonlinear fractional differential equation, Journal of Applied Mathematics and Computing, 38(2012) 225-241.

[19] Y. Liu, W. Zhang, Multiplicity Result of Positive Solutions for Nonlinear Differential Equation of Fractional Order, Abstr. Appl. Anal. doi:10.1155/2012/540846, 2012 (2012), 15 pages