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ON STRONGLY CEZ`ARO SUMMABLE SEQUENCES

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Abstract. The object of this paper is to study the sequence spaces : $\Delta W(A, M, \overline{q}, p, u, s)$, $\Delta W_0(A, M, \overline{q}, p, u, s)$ and $\Delta W_\infty(A, M, \overline{q}, p, u, s)$ associated with strongly Cezàro summable sequences and discuss some topological properties of these spaces and other related results.

1. Definitions and notations

Let w denote the set of all sequences $x = (x_n)$, real or complex, let $p = (p_n), q = (q_n)$ and $\overline{q} = (\overline{q}_n)$ denote the sequences of positive real numbers and the sequence $\overline{Q} = (\overline{Q}_n)$ is such that :

$$\overline{Q}_n = \overline{q}_1 + \overline{q}_2 + \overline{q}_3 +, \cdots + \overline{q}_n \neq 0.$$

For a sequence $x = (x_n)$, we write

$$\bar{t}_n(x) = \frac{1}{\overline{Q}_n} \sum_{k=1}^{n} \overline{q}_k \mid x_k \mid^{p_k} .$$

An Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, nondecreasing, and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$, as $x \to \infty$, (see Krasnoselskii and Rutickii [6]).

If convexity of M is replaced by $M(x + y) \leq M(x) + M(y)$, then it is called a modulus function, defined and studied by Nakano [10], Ruckle [12], Maddox [9] and others.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of l, if there exist a constant $K > 0$ such that $M(2l) \leq KM(l)(l \geq 0)$ (see Krasnoselskii and Rutickii [6]).

Lindenstrauss and Tzafriri [7] defined the Orlicz sequence space :

$$l_M = \{x \in w : \sum_{k=1}^{\infty} M(\frac{\mid x_k \mid}{\rho}) < \infty, \text{ for some } \rho > 0\},$$

which is a Banach space with the norm :

$$\parallel x \parallel_M = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{\mid x_k \mid}{\rho}) \leq 1\}.$$

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Different Orlicz sequence spaces were studied by several mathematicians as Bilgen [1], Güngör et al [4], Tripathy and Mahanta [14], Esi and Et [2], Parashar and Choudhary [11] and many others.

A paranorm on a linear topological space X is a function $g : X \to \mathbb{R}$ which satisfies the following axioms :

for any $x, y, x_0 \in X$ and $\lambda, \lambda_0 \in \mathbb{C}$,

(i) $g(\theta) = 0$, where $\theta = (0, 0, 0, \cdots)$, the zero sequence,

(ii) $g(x) = g(-x)$,

(iii) $g(x + y) \leq g(x) + g(y)$ (subadditivity),

and

(iv) the scalar multiplication is continuous, that is,

$$\lambda \to \lambda_0, x \to x_0 \text{ imply } \lambda x \to \lambda_0 x_0 \ ;$$

in other words,

$$\mid \lambda - \lambda_0 \mid \to 0, g(x - x_0) \to 0 \text{ imply } g(\lambda x - \lambda_0 x_0) \to 0.$$

A paranormed space is a linear space X with a paranorm g and is written (X, g).

Any function g which satisfies all the conditions (i)-(iv) together with the condition :

(v) $g(x) = 0$ if and only if $x = \theta$,

is called a total paranorm on X, and the pair (X, g) is called a total paranormed space, (see Maddox [8]).

Let $A = (a_{ik})$ be an infinite matrix of complex numbers and let $(E, \parallel . \parallel)$ be a Banach space over the complex field. We write $A(\Delta^u_x) = (A_i(\Delta^u_x))$ where $A_i(\Delta^u_x) = A_i(u_k \Delta x_k) = \sum_{k=1}^{\infty} a_{ik}(u_k x_k - u_{k+1}x_{k+1})$ which converges for each i.

Now, Let $u = (u_i)$ be any sequence such that $u_i \neq 0$ for each i and s is any real number such that $s \geq 0$, then we define the following sequence spaces :

$$\begin{aligned} \Delta W(A, M, \overline{q}, p, u, s) &= \{x \in w : \frac{1}{\overline{Q}_n} \sum_{k=1}^{n} k^{-s} \overline{q}_k [M(\frac{\parallel A(\Delta^u_x - L_i e_i \parallel)}{\rho})]^{p_k} \to 0, \\ \text{as } n &\to \infty, \text{ for some } \rho > 0, \ L = (L_1, L_2, L_3, \cdots) \in E, L_i \in \mathbb{C}\}, \end{aligned}$$

$$\begin{aligned} \Delta W_0(A, M, \overline{q}, p, u, s) &= \{x \in w : \frac{1}{\overline{Q}_n} \sum_{k=1}^{n} k^{-s} \overline{q}_k [M(\frac{\parallel A(\Delta^u_x \parallel)}{\rho})]^{p_k} \to 0 \\ \text{as } n &\to \infty, \text{ for some } \rho > 0\}, \end{aligned}$$

and

$$\Delta W_\infty(A, M, \overline{q}, p, u, s) = \{x \in w : \sup_n \frac{1}{\overline{Q}_n} \sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u \|)}{\rho})] < \infty$$
$$, \text{for some } \rho > 0\}$$

where

$$e_i = \begin{cases} 1, & \text{at the i-th place} \\ 0, & \text{otherwise} \end{cases}$$

For more details on Cezàro sequences, one may see (Khan and Rahman[5], Etgin[3] and Shiue[13]).

2. Main results

In this section, we prove the following theorems :

Theorem 2.1. *Let* $p = (p_n)$ *be bounded. Then* $\Delta W(A, M, \overline{q}, p, u, s)$, $\Delta W_0(A, M, \overline{q}, p, u, s)$ *and* $\Delta W_\infty(A, M, \overline{q}, p, u, s)$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. Let $x, y \in \Delta W(A, M, \overline{q}, p, u, s)$ and $\alpha, \beta \in \mathbb{C}$. Then there exists some positive numbers ρ_1 and ρ_2 such that :

$$\frac{1}{\overline{Q}_n} \sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u - L_i e_i \|)}{\rho_1})]^{p_k} \to 0, \text{as } n \to \infty,$$
$$\text{where } L = (L_1, L_2, L_3, \cdots) \in E, L_i \in \mathbb{C}\},$$

and

$$\frac{1}{\overline{Q}_n} \sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_y^u - l_i e_i \|)}{\rho_2})]^{p_k} \to 0, \text{as } n \to \infty,$$
$$\text{where } l = (l_1, l_2, l_3, \cdots) \in E, l_i \in \mathbb{C}\}.$$

Define $\rho_3 = \max(2 \mid \alpha \mid \rho_1, 2 \mid \beta \mid \rho_2)$. Then since M is nondecreasing and convex, we see that :

$$\begin{aligned}
& \frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\alpha\Delta_x^u+\beta\Delta_y^u-(L_ie_i+l_ie_i)\ \|)}{\rho_3})]^{p_k} \\
\leq\ & \frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\alpha\Delta_x^u-L_ie_i)\ \|)}{\rho_3})]^{p_k} \\
& +\frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\beta\Delta_y^u-l_ie_i)\ \|)}{\rho_3})]^{p_k} \\
\leq\ & \frac{1}{\overline{Q}_n}\sum_{k=1}^{n} \frac{1}{2}k^{-s}\overline{q}_k[M(\frac{\| A(\alpha\Delta_x^u-L_ie_i)\ \|)}{\rho_1})]^{p_k} \\
& +\frac{1}{\overline{Q}_n}\sum_{k=1}^{n} \frac{1}{2}k^{-s}\overline{q}_k[M(\frac{\| A(\beta\Delta_y^u-l_ie_i)\ \|)}{\rho_2})]^{p_k} \\
\to\ & 0,\ \text{as } n\to\infty.
\end{aligned}$$

This shows that $\alpha x+\beta y\in \Delta W(A,M,\overline{q},p,u,s)$.

Similarly, it can be proved that $\Delta W_0(A,M,\overline{q},p,u,s)$ and $\Delta W_\infty(A,M,\overline{q},p,u,s)$ are also linear spaces. □

Theorem 2.2. *Let M be an Orlicz function which satisfies the Δ_2-condition. Then*
$\Delta W(A,\overline{q},u,s)\subseteq \Delta W(A,M,\overline{q},u,s)$,
$\Delta_0 W(A,\overline{q},u,s)\subseteq \Delta_0 W(A,M,\overline{q},u,s)$
and $\Delta_\infty W(A,\overline{q},u,s)\subseteq \Delta_\infty W(A,M,\overline{q},u,s)$.

Proof. Let $x\in \Delta W(A,\overline{q},u,s)$. Then

$$\frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[\quad \| \quad A(\Delta_x^u-L_ie_i\ \|] \to 0, \text{as } n\to\infty,$$
$$\text{where } L\ =\ (L_1,L_2,L_3,\cdots)\ \in E, L_i\in\mathbb{C}\}.$$

Now, if $\epsilon>0$ is given, one can choose δ such that $0<\delta<1$ and $M(t)<\epsilon$, for $0\leq t\leq\delta$. Let $y_k=\| A(\Delta_x^u-L_ie_i\ \|$ and $\sum_{k=1}^{n}\overline{q}_k[M(\frac{y_k}{\rho})]=\sum_1+\sum_2$,

where $\sum_1$ over $y_k\leq\delta$ and $\sum_2$ is over $y_k>\delta$. Then using the continuity of M we get that $\sum_1<\overline{Q}_n\epsilon$ and for $y_k>\delta$ we use the inequality $y_k<\frac{y_k}{\delta}<1+\frac{y_k}{\delta}$.

But M is nondecreasing and convex which implies that :

$$M(\frac{y_k}{\rho})<M(1+\frac{y_k}{\delta})<\frac{1}{2}M(2)+\frac{1}{2}M(\frac{2y_k}{\delta}).$$

Since M satisfies the Δ_2-condition, we see that :

$$M(\frac{y_k}{\rho}) < \frac{1}{2}L(\frac{y_k}{\delta})M(2) + \frac{1}{2}L(\frac{y_k}{\delta})M(2) = L(\frac{y_k}{\delta})M(2).$$

This yields that :

$$\sum_{2} \overline{q}_k M(\frac{y_k}{\delta}) \leq \frac{L}{\delta}M(2)\overline{Q}_n[\frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k(\frac{\| A(\Delta_x^u - L_ie_i) \|)}{\rho})$$

Hence $\Delta W(A, \overline{q}, u, s) \subseteq \Delta W(A, M, \overline{q}, u, s)$. A similar proof can be done for the other two inclusions. □

Theorem 2.3. *(i) Let* $0 < \inf_k p_k \leq p_k < 1$. *Then*
$\Delta W(A, M, \overline{q}, p, u, s) \subseteq \Delta W(A, M, \overline{q}, u, s)$.
(ii) Let $1 \leq p_k \leq \sup_k p_k < \infty$. *Then*
$\Delta W(A, M, \overline{q}, u, s) \subseteq \Delta W(A, M, \overline{q}, p, u, s)$.

Proof. (i) Let $x \in \Delta W(A, M, \overline{q}, p, u, s)$.Then since $0 < \inf_k p_k \leq p_k < 1$ we conclude that :

$$\begin{aligned} & \frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u - L_ie_i) \|)}{\rho})] \\ \leq \; & \frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u - L_ie_i) \|)}{\rho})]^{p_k}. \end{aligned}$$

Therefore $x \in \Delta W(A, M, \overline{q}, u, s)$.

(ii) Let $x \in \Delta W(A, M, \overline{q}, u, s)$.Then for all ϵ such that $0 < \epsilon < 1$, there exists a positive integer N such that :

$$\frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u - L_ie_i) \|)}{\rho})] \leq \epsilon < 1, \text{for all } n \geq N.$$

Now since $1 \leq p_k \leq \sup_k p_k < \infty$, we have :

$$\begin{aligned} & \frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u - L_ie_i) \|)}{\rho})]^{p_k} \\ \leq \; & \frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u - L_ie_i) \|)}{\rho})]. \end{aligned}$$

Therefore $x \in \Delta W(A, M, \overline{q}, p, u, s)$ and this completes the proof. □

Theorem 2.4. *Let* $H = \sup_k p_k \leq p_k$. *Then* $\Delta W_0(A, M, \overline{q}, p, u, s)$ *is a linear topological space paranormed by* h *definrd as :*

$$h(x)=\inf_n\{\rho^{\frac{p_n}{H}}:(\frac{1}{\overline{Q}_n}\sum_{k=1}^{n}k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u))\|}{\rho})]^{p_k})^{\frac{1}{H}}\leq 1,n=1,2,3,\cdots\}$$

Proof. Clearly $h(-x)=h(x)$. For $\alpha=\beta=1$ and using the linearity of $\Delta W_0(A,M,\overline{q},p,u,s)$, we get that :

Then since $0<\inf_k p_k\leq p_k<1$ we conclude that :

$$h(\Delta_x^u+\Delta_y^u)\leq h(\Delta_x^u)+h(\Delta_y^u).$$

Since $M(0)=0$, we see that $\inf_n \rho^{\frac{p_n}{H}}=0$, for $x=0$.

Conversely, suppose that $h(x)=0$, then

$$(\inf_n\{\rho^{\frac{p_n}{H}}:(\frac{1}{\overline{Q}_n}\sum_{k=1}^{n}k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u))\|}{\rho})]^{p_k})^{\frac{1}{H}}\leq 1)=0.$$

This yields that for a given $\epsilon>0$, there exists some $\rho_\epsilon(0<\rho_\epsilon<\epsilon)$ such that :

$$(\frac{1}{\overline{Q}_n}\sum_{k=1}^{n}k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u))\|}{\rho_\epsilon})]^{p_k})^{\frac{1}{H}}\leq 1.$$

Therefore

$$\begin{aligned}&(\frac{1}{\overline{Q}_n}\sum_{k=1}^{n}k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u))\|}{\epsilon})]^{p_k})^{\frac{1}{H}}\\ \leq\ &(\frac{1}{\overline{Q}_n}\sum_{k=1}^{n}k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u))\|}{\rho_\epsilon})]^{p_k})^{\frac{1}{H}}\leq 1.\end{aligned}$$

Suppose $x_{n_m}\neq 0$, for some m. Let $\epsilon\to 0$,then $\| \frac{A_i(u_{n_m}\Delta x_{n_m})}{\rho}\|\to\infty$, which is a contradiction.

Hence $x_{n_m}\neq 0$, for each m. Finally, we prove that scalar multiplication is continuous. Let λ be any number. Then

$$h(\lambda x)=\inf_n\{\rho^{\frac{p_n}{H}}:(\frac{1}{\overline{Q}_n}\sum_{k=1}^{n}k^{-s}\overline{q}_k[M(\frac{\| A(\lambda\Delta_x^u))\|}{\rho})]^{p_k})^{\frac{1}{H}}\leq 1,n=1,2,3,\cdots\}$$

and therefore

$$h(\lambda x)=\inf_n\{(\lambda r)^{\frac{p_n}{H}}:(\frac{1}{\overline{Q}_n}\sum_{k=1}^{n}k^{-s}\overline{q}_k[M(\frac{\| A(\lambda\Delta_x^u))\|}{\rho})]^{p_k})^{\frac{1}{H}}\leq 1,n=1,2,3,\cdots\},$$

where $r=\frac{\rho}{\lambda}$.

Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$, we get that $|\lambda|^{\frac{p_k}{H}} \leq (\max(1, |\lambda|^H))^{\frac{1}{H}}$. This implies that :

$$\begin{aligned} h(\lambda x) &\leq (\max(1, |\lambda|^H))\frac{1}{H}\inf_n\{(r)^{\frac{p_n}{H}} : (\frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\lambda\Delta_x^u)) \|}{\rho})]^{p_k})^{\frac{1}{H}} \\ &\leq 1, n = 1, 2, 3, \cdots\} \to 0 \text{ as } h(x) \to 0 \text{ in } \Delta W_0(A, M, \overline{q}, p, u, s). \end{aligned}$$

Now, let $\lambda \to 0$ and $x \in \Delta W_0(A, M, \overline{q}, p, u, s)$, then for $\epsilon > 0$, let N be a positive integer such that :

$$\frac{1}{\overline{Q}_N}\sum_{k=1}^{N} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u)) \|}{\rho})]^{p_k})^{\frac{1}{H}} < \frac{\epsilon}{2}, \text{ for some } \rho > 0.$$

Therefore

$$\frac{1}{\overline{Q}_N}\sum_{k=1}^{N} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u)) \|}{\rho})]^{p_k})^{\frac{1}{H}} < \frac{\epsilon}{2}.$$

Now, if $0 < |\lambda| < 1$ and using the convexity of M, we see that :

$$\frac{1}{\overline{Q}_N}\sum_{k=1}^{N} k^{-s}\overline{q}_k[M(\frac{\| A(\Delta_x^u)) \|}{\rho})]^{p_k} < \frac{1}{\overline{Q}_N}\sum_{k=1}^{N} k^{-s}\overline{q}_k[|\lambda| M(\frac{\| A(\Delta_x^u)) \|}{\rho})]^{p_k} < (\frac{\epsilon}{2})^H.$$

Since M is continuous everywhere in $[0, \infty]$, we have $f(t) = \frac{1}{\overline{Q}_N}\sum_{k=1}^{N} k^{-s}\overline{q}_k[M(\frac{\|A(t\Delta_x^u))\|}{\rho})]$ is continuous at 0 and so there exists $\delta(0 < \delta < 1)$ such that $|f(t)| < \frac{\epsilon}{2}$, for $0 < t < \delta$.

Let K be such that $|\lambda_n| < \delta$,for $n > K$. Then for $n > K$,

$$(\frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\lambda_n\Delta_x^u)) \|}{\rho})]^{p_k})^{\frac{1}{H}} < \frac{\epsilon}{2}.$$

Thus

$$(\frac{1}{\overline{Q}_n}\sum_{k=1}^{n} k^{-s}\overline{q}_k[M(\frac{\| A(\lambda_n\Delta_x^u)) \|}{\rho})]^{p_k})^{\frac{1}{H}} < \epsilon, \text{ for } n > K.$$

This completes the proof of the theorem. □

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