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A FUNCTIONAL EQUATION WITH RESTRICTED ARGUMENT RELATED TO COSINE FUNCTION

PRASANNA SAHOO

Department of Mathematics, University of Louisville, Louisville, Kentucky 40292 USA

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E-mail address: sahoo@louisville.edu

ABSTRACT. Let G be an abelian group, $\mathbb{C}$ be the field of complex numbers, and $0 \neq \alpha \in G$ be a fixed element. In this paper, we determine the general solution

$f, g : G \to \mathbb{C}$ of the functional equation $f(x-y+\alpha)+g(x+y+\alpha) = 2\, f(x)\, f(y)$ for all $x, y \in G$.

Keywords and phrases. Abelian group, cosine functional equation, functional equation with restricted arguments, homomorphism.

1. Introduction

Let $\mathbb{R}$ be the field of real numbers and $\mathbb{C}$ be the field of complex numbers. Let G be an abelian group and α be a fixed nonzero element of G. Let $0 \in G$ be the identity element of G. A mapping h from the group G into the multiplicative group of nonzero complex numbers is said to be a multiplicative homomorphism if and only if $h(x + y) = h(x)h(y)$ for all $x, y \in G$. It is known that if $h(0) = 0$, then $h(x) = 0$ for all $x \in G$. It is easy to see that if $h \neq 0$, then $h(-x) = h(x)^{-1}$ for all $x \in G$. Similarly, a mapping a from the group G into the additive group of complex numbers is said to be an additive homomorphism if and only if $a(x+y) = a(x)+a(y)$ for all $x, y \in G$.

In 1910, Van Vleck [6] (see also [10] and [5]) proved the following result: The continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfies the functional equation

$$f(x - y + \alpha) - f(x + y + \alpha) = 2f(x)f(y) \tag{1}$$

for all $x, y \in \mathbb{R}$, if and only if f is given by either $f \equiv 0$ or

$$f(x) = \cos\left(\frac{\pi}{2\alpha}(x - \alpha)\right), \quad \forall x \in \mathbb{R}.$$

In [2], Kannappan considered the functional equation

$$f(x - y + \alpha) + f(x + y + \alpha) = 2f(x)f(y), \tag{2}$$

and proved the following result: The general solution $f : \mathbb{R} \to \mathbb{C}$ of the functional equation (2) is either $f \equiv 0$ or $f(x) = g(x - \alpha)$, where g is an arbitrary solution of the cosine functional equation $g(x + y) + g(x - y) = 2g(x)g(y)$ for all $x, y \in \mathbb{R}$ with period 2α.

Other similar functional equations solved in literature are

$$f(x + y + \alpha)\, f(x - y + \alpha) = f(x)^2 - f(y)^2 \tag{3}$$

and

$$f(x+y+\alpha)\, f(x-y+\alpha) = f(x)^2 + f(y)^2 - 1, \tag{4}$$

for all $x, y \in \mathbb{R}$. The functional equation (3) was considered by Kannappan in [3] while (4) was considered by Etigson in [1]. These functional equations are examples of functional equations with restricted argument where at least one of the variables is restricted to a certain discrete subset of the domain of the other variable(s). In particular, the subset may consist of a single element.

The goal of this paper is to determine the general solutions $f, g : G \to \mathbb{C}$ of the functional equation

$$f(x-y+\alpha) + g(x+y+\alpha) = 2\, f(x)\, f(y) \tag{5}$$

for all $x, y \in G$. The functional equation (1) is a special case of the above functional equation (5) where $g = -f$ and $G = \mathbb{R}$. If $G = \mathbb{R}$ and $g = f$, then the equation (5) reduces to the functional equation (2) studied by Kannappan in [2].

2. Preliminary results

The following results can be found in [5] and will be instrumental in proving the main result of this paper.

Lemma 1. *The function $f : G \to \mathbb{C}$ satisfies the functional equation*

$$f(x+y) + f(x-y) = f(x)\,[f(y) + f(-y)] \tag{6}$$

for all $x, y \in G$ if and only if

$$f(x) = c \tag{7}$$

or

$$f(x) = a\,\psi(x) + b\,\psi(x)^{-1}, \qquad \psi(x) \neq \psi(x)^{-1} \tag{8}$$

or

$$f(x) = [A(x)+1]\,\psi(x), \qquad \psi(x) = \psi(x)^{-1} \tag{9}$$

where $A : G \to \mathbb{C}$ is a additive homomorphism from G into the additive group of complex numbers, the function $\psi : G \to \mathbb{C}^\star$ is a multiplicative homomorphism from G into the multiplicative group of nonzero complex numbers, and a, b, c are complex constants satisfying $c(c-1) = 0$ and $a + b = 1$.

3. Main result

Now we are ready to prove our main result.

Theorem 1. *Let G be an abelian group and $0 \neq \alpha \in G$ be a fixed element. Suppose the functions $f, g : G \to \mathbb{C}$ satisfy the functional equation*

$$f(x-y+\alpha) + g(x+y+\alpha) = 2\, f(x)\, f(y) \tag{10}$$

for all $x, y \in G$. Then there exist multiplicative homomorphisms $h_1, h_2 : G \to \mathbb{C}^\star$ such that the solutions f and g are given by

$$f(x) = c, \quad g(x) = c\,(2c-1) \tag{11}$$

$$f(x) = \frac{h_1(x) - h_1(x)^{-1}}{2h_1(\alpha)}, \qquad g(x) = -\frac{h_1(x) - h_1(x)^{-1}}{2h_1(\alpha)}, \tag{12}$$

$$\begin{cases} f(x) = \begin{cases} f(0)\, h_2(x) & \text{if } h_2(x) = h_2(x)^{-1} \\ \frac{1}{2}\left[h_2(x-\alpha) + h_2(x-\alpha)^{-1}\right] & \text{if } h_2(x) \neq h_2(x)^{-1}, \end{cases} \\ g(x) = \begin{cases} f(0)\left[2f(0)h_2(\alpha)^{-1} - 1\right]h_2(x) & \text{if } h_2(x) = h_2(x)^{-1} \\ \frac{1}{2}\left[\frac{a}{b}h_2(x-\alpha) + \frac{b}{a}h_2(x-\alpha)^{-1}\right], & \text{if } h_2(x) \neq h_2(x)^{-1}, \end{cases} \end{cases} \tag{13}$$

where $h_1(\alpha) = -h_1(-\alpha)$, and $\gamma, a, b \in \mathbb{C}$ are arbitrary constants satisfying $a+b = 1$ together with $ah_2(\alpha) = bh_2(-\alpha) = 2abf(0)$.

Moreover, if $f(0) = 0$, then f and g are periodic functions of period 4α.

Proof. If f is a constant function, say $f(x) = c$ for all $x \in G$, then using the functional equation (10) we have $g(x) = c(2c-1)$ for any arbitrary constant $c \in \mathbb{C}$. This yields the asserted solution (11). Hence from now on we assume $f(x)$ is a non-constant function.

Letting $y = 0$ in (10), we have

$$f(x+\alpha) + g(x+\alpha) = 2f(x)f(0).$$

Hence

$$g(x) = 2f(x-\alpha)\, f(0) - f(x) \tag{14}$$

for all $x \in G$. Using (14) in (10), we obtain

$$f(x-y+\alpha) - f(x+y+\alpha) = 2f(x)\, f(y) - 2f(0)\, f(x+y) \tag{15}$$

for all $x, y \in G$.

Case 1. Suppose $f(0) = 0$. Then (15) reduces to

$$f(x-y+\alpha) - f(x+y+\alpha) = 2f(x)\, f(y) \tag{16}$$

for all $x, y \in G$.

Replacing y with $-y$ in (16), we get

$$f(x+y+\alpha) - f(x-y+\alpha) = 2f(x)\, f(-y) \tag{17}$$

for all $x, y \in G$. Adding (16) and (17), we have

$$f(x)\ [f(y) + f(-y)] = 0 \tag{18}$$

for all $x, y \in G$. Since f is non-constant, from (18), we get

$$f(-y) = -f(y), \quad \forall y \in G. \tag{19}$$

That is f is an odd function. Interchanging x with y in (16), we see that

$$f(y-x+\alpha) - f(y+x+\alpha) = 2f(y)\, f(x) \tag{20}$$

for all $x, y \in G$. Comparing (16) and (20), we obtain

$$f(x-y+\alpha) = f(y-x+\alpha) \tag{21}$$

Therefore, using (19) and (21), we see that

$$f(x-y+\alpha)=f(y-x+\alpha)=f(-(x-y-\alpha))=-f(x-y-\alpha).$$

Hence

$$f(x-y+\alpha)=-f(x-y-\alpha) \tag{22}$$

for all $x, y \in G$. Letting $y=0$ in (22), we have

$$f(x+\alpha)=-f(x-\alpha). \tag{23}$$

Hence, replacing x by $x+\alpha$ in (23), we obtain

$$f(x+2\alpha)=-f(x) \tag{24}$$

and

$$f(x+3\alpha)=-f(x+\alpha). \tag{25}$$

Using (23) and (25), we have

$$f(x+4\alpha)=f(x) \tag{26}$$

for all $x \in G$. From (26) and (14), we see that

$$g(x+4\alpha)=-f(x+4\alpha)=-f(x)=g(x) \tag{27}$$

for all $x \in G$. This proves that f and g are periodic functions of period 4α.

Replacing x with $x+\alpha$ and y with $y+\alpha$ in (16), we obtain

$$f(x-y+\alpha)-f(x+y+3\alpha)=2f(x+\alpha)f(y+\alpha) \tag{28}$$

for all $x, y \in G$. Using (25) in (28), we see that

$$f(x-y+\alpha)+f(x+y+\alpha)=2f(x+\alpha)f(y+\alpha) \tag{29}$$

for all $x, y \in G$. Defining $\ell : G \to \mathbb{C}$ by

$$\ell(x)=f(x+\alpha) \qquad \forall x \in G \tag{30}$$

and using it in (29), we obtain

$$\ell(x+y)+\ell(x-y)=2\,\ell(x)\,\ell(y) \tag{31}$$

for all $x, y \in G$. The general solution of (31) can be obtained from Lemma 1 as

$$\ell(x)=\frac{1}{2}\left[h_1(x)+h_1(x)^{-1}\right], \tag{32}$$

where $h_1 : G \to \mathbb{C}^\star$ is a homomorphism. Hence from (30) and (32), we get

$$f(x)=\frac{1}{2}\left[h_1(x-\alpha)+h_1(x-\alpha)^{-1}\right]. \tag{33}$$

Using (14), we have

$$g(x)=-\frac{1}{2}\left[h_1(x-\alpha)+h_1(x-\alpha)^{-1}\right]. \tag{34}$$

Since $h_1 \neq 0$, $h_1(-x)=h_1(x)^{-1}$ for all $x \in G$, and hence (33) simplifies to

$$f(x)=\frac{1}{2}\left[h_1(x)h_1(\alpha)^{-1}+h_1(x)^{-1}h_1(\alpha)\right]. \tag{35}$$

Hence

$$f(-x) = \frac{1}{2}\left[h_1(x)^{-1}h_1(\alpha)^{-1} + h_1(x)h_1(\alpha)\right]. \tag{36}$$

Since f is an odd function on G, using (35) and (36), we get

$$\left[h_1(x) + h_1(x)^{-1}\right]\left[h_1(\alpha) + h_1(\alpha)^{-1}\right] = 0 \tag{37}$$

for all $x \in G$. Since $h \neq 0$, we have

$$h_1(\alpha) + h_1(\alpha)^{-1} = 0 \tag{38}$$

which implies

$$h_1(\alpha)^2 = -1. \tag{39}$$

Using (39) in (33), we have

$$f(x) = \frac{1}{2\,h_1(\alpha)}\left[h_1(x) - h_1(x)^{-1}\right]. \tag{40}$$

From (39) and (34), we get

$$g(x) = -\frac{1}{2\,h_1(\alpha)}\left[h_1(x) - h_1(x)^{-1}\right]. \tag{41}$$

Next, we verify that (40) and (41) are the solution of the functional equation (10) in the case $f(0) = 0$. Inserting (40) and (41) into (10) and using the fact that $h_1(\alpha)^2 = -1$, we get

$$\begin{aligned}
&f(x-y+\alpha) + g(x+y+\alpha) - 2\,f(x)\,f(y)\\
&\quad = \frac{1}{2}\left[h_1(x-y) + h_1(x-y)^{-1}\right] - \frac{1}{2}\left[h_1(x+y) + h_1(x+y)^{-1}\right]\\
&\qquad - \frac{1}{2}\left[h_1(x-\alpha) + h_1(x-\alpha)^{-1}\right]\left[h_1(y-\alpha) + h_1(y-\alpha)^{-1}\right]\\
&\quad = \frac{1}{2}\left[h_1(x)h_1(y)^{-1} + h_1(x)^{-1}h_1(y)\right] - \frac{1}{2}\left[h_1(x)h_1(y) + h_1(x)^{-1}h_1(y)^{-1}\right]\\
&\qquad - \frac{1}{2}\left[h_1(x)h_1(\alpha)^{-1} + h_1(x)^{-1}h_1(\alpha)\right]\left[h_1(y)h_1(\alpha)^{-1} + h_1(y)^{-1}h_1(\alpha)\right]\\
&\quad = -\frac{1}{2}\left[h_1(x) - h_1(x)^{-1}\right]\left[h_1(y) - h_1(y)^{-1}\right]\\
&\qquad - \frac{1}{2h_1(\alpha)^2}\left[h_1(x) - h_1(x)^{-1}\right]\left[h_1(y) - h_1(y)^{-1}\right]\\
&\quad = -\frac{1}{2}\left[h_1(x) - h_1(x)^{-1}\right]\left[h_1(y) - h_1(y)^{-1}\right]\\
&\qquad + \frac{1}{2}\left[h_1(x) - h_1(x)^{-1}\right]\left[h_1(y) - h_1(y)^{-1}\right]\\
&\quad = 0.
\end{aligned}$$

Hence (40) and (41) are the solution of (10) in the case $f(0) = 0$. This is exactly what asserted in the solution (12).

Case 2. Next suppose $f(0) \neq 0$. Interchanging y with $-y$ in (15), we obtain

$$f(x+y+\alpha) - f(x-y+\alpha) = 2f(x)\,f(-y) - 2f(0)\,f(x-y) \tag{42}$$

for all $x, y \in G$. Adding (42) to (15), we see that

$$f(0)\,[f(x+y)+f(x-y)] = f(x)\,[f(y)+f(-y)] \tag{43}$$

for all $x, y \in G$. Define $\phi : G \to \mathbb{C}$ by

$$\phi(x) = \frac{f(x)}{f(0)}, \qquad \forall x \in G. \tag{44}$$

Then (43) and (44) yield

$$\phi(x+y)+\phi(x-y) = \phi(x)\,[\phi(y)+\phi(-y)] \tag{45}$$

for all $x, y \in G$. The general nontrivial solution of (45) is given by

$$\phi(x) = \begin{cases} a\,h_2(x) + b\,h_2(x)^{-1}, & \text{if } h_2(x) \not\equiv h_2(x)^{-1} \\ h_2(x)\;[A(x)+1]\,, & \text{if } h_2(x) \equiv h_2(x)^{-1}, \end{cases} \tag{46}$$

where $h_2 : G \to \mathbb{C}^\star$ is a homomorphism from the group G into the multiplicative group of nonzero complex numbers $\mathbb{C}^\star$, $A : G \to \mathbb{C}$ is an additive homomorphism, and $a, b \in \mathbb{C}$ are constants with $a + b = 1$.

Note that $\phi = 0$ is also a solution of (45). But in this case $f = 0$ and consequently $g = 0$. This is the trivial solution of (10).

Using (44) in (46), we see that

$$f(x) = \begin{cases} f(0)\;\left[a\,h_2(x) + b\,h_2(x)^{-1}\right], & \text{if } h_2(x) \not\equiv h_2(x)^{-1} \\ f(0)\,h_2(x)\;[A(x)+1]\,, & \text{if } h_2(x) \equiv h_2(x)^{-1}, \end{cases} \tag{47}$$

where a and b are complex numbers with $a + b = 1$.

Interchanging x with y in (15), we obtain

$$f(y-x+\alpha) - f(x+y+\alpha) = 2f(y)\,f(x) - 2f(0)\,f(x+y) \tag{48}$$

for all $x, y \in G$. Comparing (15) and (48), we conclude that

$$f(x-y+\alpha) = f(y-x+\alpha) \tag{49}$$

for all $x, y \in G$. Hence letting $x = 0$ in the above relation, we have

$$f(\alpha - y) = f(\alpha + y) \tag{50}$$

for all $y \in G$. Letting $y = \alpha$ in (50), we have

$$f(0) = f(2\,\alpha). \tag{51}$$

Now we consider two subcases.

Subcase 2.1. Suppose $h_2(x) \equiv h_2(x)^{-1}$. From (47), we get

$$f(x) = f(0)\,h_2(x)\,[A(x)+1]\,. \tag{52}$$

Substituting (52) in (50), we see that

$$f(0)\,h_2(\alpha)\,h_2(y)[A(\alpha)-A(y)+1] = f(0)\,h_2(\alpha)\,h_2(y)[A(\alpha)+A(y)+1].$$

Since $h_2 \neq 0$, from the above equality, we get

$$2\,A(y) = 0$$

for all $y \in G$. Hence $A = 0$. Thus (52) reduces to

$$f(x) = f(0)\, h_2(x). \tag{53}$$

Using (53) in (14), we get

$$\begin{aligned} g(x) &= 2\, f(0)\, f(x-\alpha) - f(x) \\ &= 2\, f(0)^2\, h_2(x)\, h_2(\alpha) - f(0)\, h_2(x) \\ &= f(0)\, h_2(x)\, \left[2 f(0)\, h_2(\alpha) - 1\right]. \end{aligned} \tag{54}$$

Next we check (53) and (54) are solution of (10) for this sub case. First, we compute

$$\begin{aligned} & f(x-y+\alpha) + g(x+y+\alpha) - 2\, f(x)\, f(y) \\ &\quad = f(0)\, h_2(x)\, h_2(y)^{-1}\, h_2(\alpha) \\ &\qquad\quad + f(0)\, h_2(x)\, h_2(y)\, h_2(\alpha)\, \left[2\, f(0)\, h_2(\alpha) - 1\right] - 2\, f(0)^2\, h_2(x)\, h_2(y)\big] \\ &\quad = f(0)\, h_2(x)\, h_2(y)\, h_2(\alpha) \\ &\qquad\quad + f(0)\, h_2(x)\, h_2(y)\, h_2(\alpha)\, [2\, f(0)\, h_2(\alpha) - 1] - 2\, f(0)^2\, h_2(x)\, h_2(y) \\ &\quad = 2\, f(0)^2\, h_2(x)\, h_2(y)\, h_2(\alpha)^2 - 2\, f(0)^2\, h_2(x)\, h_2(y) \\ &\quad = 2\, f(0)^2\, h_2(x)\, h_2(y)\, \left[h_2(\alpha)^2 - 1\right]. \end{aligned}$$

Hence

$$f(x-y+\alpha) + g(x+y+\alpha) - 2\, f(x)\, f(y) = 2\, f(0)^2\, h_2(x)\, h_2(y)\, \left[h_2(\alpha)^2 - 1\right] \tag{55}$$

for all $x, y \in G$. Since $f(0) = f(2\,\alpha)$, using this with (53), we have

$$f(0)\, h_2(0) = f(0)\, h_2(2\alpha). \tag{56}$$

The relation (56) yields

$$h_2(\alpha)^2 = 1. \tag{57}$$

Hence using (57) in (55), we have

$$f(x-y+\alpha) + g(x+y+\alpha) - 2\, f(x)\, f(y) = 0 \tag{58}$$

Thus for this sub case

$$\begin{cases} f(x) = f(0)\, h_2(x), \\ g(x) = f(0)\, h_2(x)\, [2 f(0)\, h_2(\alpha) - 1] \end{cases} \tag{59}$$

is the solution of (10) with $h(\alpha)$ satisfies $h_2(\alpha)^2 = 1$.

Subcase 2.2. Suppose $h(x) \neq h(x)^{-1}$ for all $x \in G$. From (47), the form of f is given by

$$f(x) = f(0)\, \left[a\, h_2(x) + b\, h_2(x)^{-1}\right], \tag{60}$$

where a, b are complex numbers with $a + b = 1$. From (50), we have

$$f(\alpha - x) = f(\alpha + x) \qquad \forall x \in G.$$

Computing $f(\alpha - x)$, we get

$$\begin{aligned} f(\alpha - x) &= f(0)\, \left[a\, h_2(\alpha - x) + b\, h_2(\alpha - x)^{-1}\right] \\ &= f(0)\, \left[a\, h_2(\alpha)\, h_2(x)^{-1} + b\, h_2(\alpha)^{-1} h_2(x)\right]. \end{aligned} \tag{61}$$

Next, we compute $f(\alpha+x)$ to get

(62) $$\begin{aligned} f(\alpha+x) &= f(0)\,\left[a\,h_2(\alpha+x)+b\,h_2(\alpha+x)^{-1}\right] \\ &= f(0)\,\left[a\,h_2(\alpha)\,h_2(x)+b\,h_2(\alpha)^{-1}h_2(x)^{-1}\right]. \end{aligned}$$

From (50), (61) and (62), we get

$$a\,h_2(\alpha)\,h_2(x)+b\,h_2(\alpha)^{-1}h_2(x)^{-1}=a\,h_2(\alpha)\,h_2(x)^{-1}+b\,h_2(\alpha)^{-1}h_2(x)$$

which yields

$$\left[a\,h_2(\alpha)-b\,h_2(\alpha)^{-1}\right]\,\left[h_2(x)-h_2(x)^{-1}\right]=0$$

for all $x\in G$. Since $h_2(x)\neq h_2(x)^{-1}$, we have

(63) $$a\,h_2(\alpha)=b\,h_2(\alpha)^{-1}.$$

Letting (60) into (15) and simplifying resulting expression using (63) and then using the fact that $a+b=1$, we get

$$\begin{aligned} &f(x-y+\alpha)-f(x+y+\alpha) \\ &\qquad= f(0)\,\left[a\,h_2(x)\,h_2(y)^{-1}h_2(\alpha)+b\,h_2(x)^{-1}\,h_2(y)\,h_2(\alpha)^{-1}\right] \\ &\qquad\quad - f(0)\,\left[a\,h_2(x)\,h_2(y)\,h_2(\alpha)+b\,h_2(x)^{-1}\,h_2(y)^{-1}\,h_2(\alpha)^{-1}\right] \\ &\qquad= -f(0)\,\left[h_2(y)-h_2(y)^{-1}\right]\,\left[a\,h_2(x)\,h_2(\alpha)-b\,h_2(x)^{-1}h_2(\alpha)^{-1}\right] \\ &\qquad= -f(0)\,\left[h_2(y)-h_2(y)^{-1}\right]\,\left[a\,h_2(x)\,h_2(\alpha)-a\,h_2(x)^{-1}h_2(\alpha)\right] \\ &\qquad= -f(0)\,a\,h_2(\alpha)\,\left[h_2(x)-h_2(x)^{-1}\right]\,\left[h_2(y)-h_2(y)^{-1}\right]. \end{aligned}$$

Hence, we have

(64)
$$f(x-y+\alpha)-f(x+y+\alpha)=-f(0)\,a\,h_2(\alpha)\,\left[h_2(x)-h_2(x)^{-1}\right]\,\left[h_2(y)-h_2(y)^{-1}\right].$$

Similarly, next we compute $2f(x)f(y)-2f(0)f(x+y)$ to get

$$\begin{aligned} &2f(x)f(y)-2f(0)f(x+y) \\ &\quad= 2f(0)^2\left[ah_2(x)+bh_2(x)^{-1}\right]\,\left[ah_2(y)+bh_2(y)^{-1}\right] \\ &\qquad - 2f(0)^2\left[ah_2(x)h_2(y)+bh_2(x)^{-1}h_2(y)^{-1}\right] \\ &\quad= 2f(0)^2[a^2h_2(x)h_2(y)+abh_2(x)^{-1}h_2(y)^{-1}+abh_2(x)^{-1}h_2(y) \\ &\qquad + b^2h_2(x)^{-1}h_2(y)^{-1}-ah_2(x)h_2(y)-bh_2(x)^{-1}h_2(y)^{-1}] \\ &\quad= 2f(0)^2[(a^2-a)h_2(x)h_2(y) \\ &\qquad + (b^2-b)h_2(x)^{-1}h_2(y)^{-1}+abh_2(x)h_2(y)^{-1}+abh_2(x)^{-1}h_2(y)] \\ &\quad= -2f(0)^2[abh_2(x)h_2(y)+abh_2(x)^{-1}h_2(y)^{-1} \\ &\qquad - abh_2(x)h_2(y)^{-1}-abh_2(x)^{-1}h_2(y)] \\ &\quad= -2f(0)^2ab\left[h_2(x)-h_2(x)^{-1}\right]\,\left[h_2(y)-h_2(y)^{-1}\right]. \end{aligned}$$

Thus, we have

(65) $$2f(x)f(y)-2f(0)f(x+y)=-2f(0)^2ab\big[h_2(x)-h_2(x)^{-1}\big]\big[h_2(y)-h_2(y)^{-1}\big].$$

Hence, from (64) and (65), we see that

$$f(x-y+\alpha)-f(x+y+\alpha)=2f(x)f(y)-2f(0)f(x+y)$$

implies

(66) $$f(0)\, a\, h_2(\alpha) = 2\, f(0)^2\, ab.$$

Hence

(67) $$h_2(\alpha) = 2\, f(0)\, b.$$

Since by (63), $a\, h_2(\alpha) = b\, h_2(\alpha)^{-1}$, (67) yields

$$a\, h_2(\alpha) = 2\, f(0)\, ab$$

which is

$$b\, h_2(\alpha)^{-1} = 2\, f(0)\, a\, b.$$

Hence

(68) $$h_2(\alpha)^{-1} = 2\, f(0)\, a.$$

From (67) and (68), we obtain

(69) $$f(0)^2 = \frac{1}{4ab}.$$

Using (69) in (66), we see that

(70) $$f(0)\, a\, h_2(\alpha) = \frac{1}{2}.$$

From (63), (70) and (60), we have

(71) $$f(x) = \frac{1}{2\,b}\, \left[a\, h_2(x) + b\, h_2(x)^{-1}\right]\, h_2(\alpha).$$

Using (14), we get

$$\begin{aligned}
g(x) &= 2\, f(0)\, f(x-\alpha) - f(x) \\
&= 2\, f(0)^2\, \left[a\, h_2(x)\, h_2(\alpha)^{-1} + b\, h_2(x)^{-1}\, h_2(\alpha)\right] - f(0)\, \left[a\, h_2(x) + b\, h_2(x)^{-1}\right] \\
&= \frac{2}{4ab}\left[a\, h_2(x)\, h_2(\alpha)^{-1} + b\, h_2(x)^{-1}\, h_2(\alpha)\right] - \frac{a\, h_2(\alpha)}{2ab}\left[a\, h_2(x) + b\, h_2(x)^{-1}\right] \\
&= \frac{1}{2ab}\left[a\, h(x)\, h(\alpha)^{-1} + b\, h_2(x)^{-1}\, h_2(\alpha) - a^2 h_2(x)\, h_2(\alpha) - ab h_2(x)^{-1} h_2(\alpha)\right] \\
&= \frac{1}{2ab}\left[a\, h_2(x)\left\{h_2(\alpha)^{-1} - a\, h_2(\alpha)\right\} + b\, h_2(x)^{-1}\left\{h_2(\alpha) - a\, h_2(\alpha)\right\}\right] \\
&= \frac{1}{2ab}\left[a\, h_2(x)\left\{h_2(\alpha)^{-1} - b\, h_2(\alpha)^{-1}\right\} + b\, h_2(x)^{-1}\left\{h_2(\alpha) - a\, h_2(\alpha)\right\}\right] \\
&= \frac{1}{2ab}\left[a^2\, h_2(x)\, h_2(\alpha)^{-1} + b^2\, h_2(x)^{-1} h_2(\alpha)\right].
\end{aligned}$$

The function $f(x)$ in (71) can be rewritten as

$$\begin{aligned}
f(x) &= \frac{1}{2b}\, \left[a\, h_2(x) + b\, h_2(x)^{-1}\right]\, h_2(\alpha) \\
&= \frac{1}{2ab}\, \left[a^2\, h_2(x)\, h_2(\alpha) + ab\, h_2(x)^{-1}\, h_2(\alpha)\right] \\
&= \frac{1}{2ab}\, \left[ab\, h_2(x)\, h_2(\alpha)^{-1} + ab\, h_2(x)^{-1}\, h_2(\alpha)\right] \\
&= \frac{1}{2}\, \left[h_2(x)\, h_2(\alpha)^{-1} + h_2(x)^{-1}\, h_2(\alpha)\right].
\end{aligned}$$

Thus the general solution for this case is given by

$$f(x) = \frac{1}{2}\left[h_2(x)h_2(\alpha)^{-1} + h_2(x)^{-1}h_2(\alpha)\right] \tag{72}$$

$$g(x) = \frac{1}{2}\left[\frac{a}{b}\, h_2(x)h_2(\alpha)^{-1} + \frac{b}{a}\, h_2(x)^{-1}h_2(\alpha)\right] \tag{73}$$

where a, b are complex numbers satisfying $a + b = 1$ and $ah_2(\alpha) = bh_2(\alpha)^{-1}$, and $f(0)^2 = \frac{1}{4ab}$.

Now we verify that the solution (f, g) given in (72) and (73) satisfy

$$2\, f(x)\, f(y) = f(x - y + \alpha) + g(x + y + \alpha).$$

For this let us compute

$$\begin{aligned}
&4\, f(x)\, f(y)\\
&\quad = [h_2(x-\alpha) + h_2(\alpha - x)]\; [h_2(y-\alpha) + h_2(\alpha - y)]\\
&\quad = \left[h_2(x)h_2(\alpha)^{-1} + h_2(\alpha)h_2(x)^{-1}\right]\; \left[h_2(y)h_2(\alpha)^{-1} + h_2(\alpha)h_2(y)^{-1}\right]\\
&\quad = \left[h_2(x)h_2(y)h_2(\alpha)^{-2} + h_2(x)h_2(y)^{-1} + h_2(x)^{-1}h_2(y) + h_2(x)^{-1}h_2(y)^{-1}h_2(\alpha)^2\right]\\
&\quad = \frac{a}{b}\, h_2(x)\, h_2(y) + h_2(x)\, h_2(y)^{-1} + h_2(x)^{-1}\, h_2(y) + \frac{b}{a}\, h_2(x)^{-1}\, h_2(y)^{-1}\\
&\quad = h_2(y)\, \left\{\frac{a}{b}\, h_2(x) + h_2(x)^{-1}\right\} + h_2(y)^{-1}\, \left\{h_2(x) + \frac{b}{a}\, h_2(x)^{-1}\right\}\\
&\quad = \frac{1}{b}\, h_2(y)\left\{a\, h_2(x) + b\, h_2(x)^{-1}\right\} + \frac{1}{a}\, h_2(y)^{-1}\left\{a\, h_2(x) + b\, h_2(x)^{-1}\right\}\\
&\quad = \frac{1}{ab}\, \left[a\, h_2(x) + b\, h_2(x)^{-1}\right]\, \left[a\, h_2(y) + b\, h_2(y)^{-1}\right].
\end{aligned}$$

Thus we have

$$2\, f(x)\, f(y) = \frac{1}{2ab}\, \left[a\, h_2(x) + b\, h_2(x)^{-1}\right]\, \left[a\, h_2(y) + b\, h_2(y)^{-1}\right] \tag{74}$$

for all $x, y \in G$. Next we compute $f(x - y + \alpha) + g(x + y + \alpha)$ to get

$$\begin{aligned}
&f(x - y + \alpha) + g(x + y + \alpha)\\
&\quad = \frac{1}{2}\, \left[h_2(x)\, h_2(y)^{-1}\, h_2(\alpha)\, h_2(\alpha)^{-1} + h_2(x)^{-1}\, h_2(y)\, h_2(\alpha)^{-1}\, h_2(\alpha)\right]\\
&\qquad\qquad + \frac{1}{2}\left[\frac{a}{b}\, h_2(x)\, h_2(y)\, h_2(\alpha)\, h_2(\alpha)^{-1} + \frac{b}{a}\, h_2(x)^{-1}\, h_2(y)^{-1}\, h_2(\alpha)^{-1}\, h_2(\alpha)\right]\\
&\quad = \frac{1}{2}\left[h_2(x)\, h_2(y)^{-1} + h_2(x)^{-1}\, h_2(y) + \frac{a}{b}\, h_2(x)\, h_2(y) + \frac{b}{a}\, h_2(x)^{-1}\, h_2(y)^{-1}\right]\\
&\quad = \frac{1}{2}\left[h_2(x)\, \left\{h_2(y)^{-1} + \frac{a}{b}\, h_2(y)\right\} + h_2(x)^{-1}\, \left\{h_2(y) + \frac{b}{a}\, h_2(y)^{-1}\right\}\right]\\
&\quad = \frac{1}{2}\left[\frac{1}{b}\, h_2(x)\, \left\{b\, h_2(y)^{-1} + a\, h_2(y)\right\} + \frac{1}{a}\, h_2(x)^{-1}\, \left\{a\, h_2(y) + b\, h_2(y)^{-1}\right\}\right]\\
&\quad = \frac{1}{2ab}\, \left[a\, h_2(x) + b\, h_2(x)^{-1}\right]\, \left[a\, h_2(y) + b\, h_2(y)^{-1}\right].
\end{aligned}$$

Therefore

(75) $f(x-y+\alpha)+g(x+y+\alpha)=\frac{1}{2ab}\left[ah_2(x)+bh_2(x)^{-1}\right]\left[ah_2(y)+bh_2(y)^{-1}\right]$

From (74) and (75), we see that (f,g) given in (72) and (73) is a solution of the equation (10) for this sub case.

Since there are no more cases left, this complete the proof of the theorem. □

4. An open problem

Let G be an abelian group and let 0 denote the identity element of G. Let $\sigma : G \to G$ be an involution, that is $\sigma(xy)=\sigma(y)+\sigma(x)$ and $\sigma(\sigma(x))=x$ for all $x,y\in G$. Find all functions $f,g : G \to \mathbb{C}$ satisfying

$$f(x+\sigma(y)+\alpha)+g(x+y+\alpha)=2f(x)\,f(y)$$

for all $x,y\in G$.

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