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RICCI ALMOST SOLITONS ON RIEMANNIAN MANIFOLDS

Shyamal Kumar Hui *Corresponding author
Department of Mathematics, The University of Burdwan
Golapbag, Burdwan-713104, West Bengal, India

Akshoy Patra
Department of Mathematics,
Govt. College of Engineering and Textile Technology
Berhampore, Murshidabad-742101, West Bengal, India

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Corresponding author. E-mail: shyamal_hui@yahoo.co.in

Abstract

The present paper deals with the study of Riemannian mani-folds whose metric is Ricci almost soliton with a conformal Killing vector field. We also study Ricci almost solitons on Riemannian manifolds with respect to semi-symmetric metric connection and obtain a necessary and sufficient condition of a Ricci almost soliton on Riemannian manifold with respect to semi-symmetric metric connection to be Ricci almost soliton on Riemannian manifold with respect to Levi-Civita connection.

1. Introduction

In 1982, Hamilton [10] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. Then Ricci flow has become a powerful tool for the study of Riemannian manifolds, especially for those manifolds with positive curvature. Perelman ([19], [20]) used Ricci flow and its surgery to prove Poincare conjecture. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined as follows:

$$\frac{\partial}{\partial t} g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton emerges as the limit of the solutions of the Ricci flow. A solution to the Ricci flow is called Ricci soliton if it moves only by a one parameter group of diffeomorphism and scaling. A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that [11]

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according as λ is negative, zero and positive respectively.

During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians. In particular, it has become more important after Perelman applied Ricci solitons to solve the long standing Poincare conjecture posed in 1904. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as Bagewadi et. al ([1], [2], [3], [16]), Bejan and Crasmareanu [4], Blaga [5], Chen and Deshmukh [7], Deshmukh et.

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al [8], He and Zhu [13], Hui et. al ([6], [14], [15]), Nagaraja and Premalatta [17], Tripathi [25] and many others.

Recently Pigola et. al [21] introduced and studied the notion of Ricci almost solitons on a Riemannian manifold (M, g), defined by the same equation (1.1) in which λ is a smooth function. For λ constant, the Ricci almost soliton will becomes the Ricci soliton.

The present paper deals with the study of Ricci almost soliton on Riemannian manifolds (M^n, g) with the potential vector field is conformal Killing. The paper is organized as follows. Section 2 is concerned with some preliminaries. Section 3 is devoted to the study of Riemannian manifolds (M^n, g) whose metric is Ricci almost soliton with a conformal Killing vector field. In section 4, we studied Ricci almost solitons on Riemannian manifolds with respect to semi-symmetric metric connection and obtain a necessary and sufficient condition of a Ricci almost soliton on Riemannian manifold with respect to semi-symmetric metric connection to be Ricci almost soliton on Riemannian manifold with respect to Levi-Civita connection.

2. Preliminaries

This section deals with some preliminaries, which will be required in the sequel.

Definition 2.1. [23] A vector field V in a Riemannian manifold (M^n, g) is said to be conformal Killing vector field if it satisfies

$$(\pounds_V g)(Y, Z) = 2\rho g(Y, Z) \tag{2.1}$$

for any vector fields Y, Z, where ρ is a smooth function on (M^n, g) and $\pounds$ is the operator of Lie differentiation.
In particular, if ρ is constant then V is called homothetic and if $\rho = 0$ then V is called isometric as well as killing vector field.

A Riemannian manifold is said to be Ricci symmetric if its Ricci tensor S of type (0,2) satisfies $\nabla S = 0$, where ∇ denotes the Riemannian connection. During the last five decades, the notion of Ricci symmetry has been weakened by many authors in several ways to a different extent such as Ricci recurrent manifold [18], Ricci semisymmetric manifold [24] etc. A Riemannian manifold (M^n, g) is said to be Ricci semisymmetric [24] if it satisfies $R(X, Y) \cdot S = 0$, where $R(X, Y)$ is the curvature operator and S is the Ricci tensor of type (0,2). Every Ricci symmetric manifold is Ricci semisymmetric but not conversely [24].

The Weyl tensor has the special property that it is invariant under conformal changes to the metric. For this reason the Weyl tensor is also called the conformal tensor. It follows that a necessary condition for a Riemannian manifold to be conformally flat is that the Weyl tensor vanish. In dimensions ≥ 4 this condition is sufficient as well. In dimension 3 the vanishing of the Cotton tensor is a necessary and sufficient condition for the Riemannian manifold being conformally flat. Any 2-dimensional (smooth) Riemannian manifold is conformally flat, a consequence of the existence of isothermal coordinates. Conformal transformations of a Riemannian structures are an important object of study in differential geometry.

The conformal transformation on a Riemannian manifold (M^n, g) is a transformation under which the angle between two curves remains invariant. The Weyl conformal curvature tensor C of type (1,3) of a Riemannian manifold (M^n, g) is defined by

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{2.2}$$

where R, S, Q and r are the Curvature tensor, Ricci tensor, Ricci-operator and scalar curvature on Riemannian manifold (M^n, g) respectively.

3. Ricci almost solitons on Riemannian manifolds with a conformal Killing vector field

This section deals with the study of of Riemannian manifolds whose metric is Ricci almost soliton with a conformal Killing vector field and we prove the following:

Theorem 3.1. *Let (g, V, λ) be a Ricci almost soliton on a Riemannian manifold (M^n, g). If V is conformal Killing vector field then the followings are eqivalent:*
(i) *(M^n, g) is Einstein.*
(ii) *$\lambda + \rho$ is constant though λ and ρ are smooth functions.*
(iii) *(M^n, g) is Ricci symmetric.*
(iv) *(M^n, g) is Ricci semisymmetric.*

Proof. Since (g, V, λ) is a Ricci almost soliton on a Riemannian manifold (M^n, g) with V is conformal Killing vector field then by virtue of (2.1) we obtain from (1.1) that

$$S(X,Y) = -(\lambda + \rho)g(X,Y), \tag{3.1}$$

which implies that the manifold under consideration is Einstein, i.e. (i) holds and hence $\lambda + \rho$ is always constant by Bianchi's identity, though λ and ρ are smooth functions, i.e. (ii) holds.

As (M^n, g) is Einstein, its Ricci tensor is parallel, i.e. (M^n, g) is Ricci symmetric, which implies (iii).

It is known that every Ricci symmetric manifold is Ricci semisymmetric but not conversely [24]. Now for any X, Y, Z, U on (M^n, g), we have

$$(R(X,Y) \cdot S)(Z,U) = -S(R(X,Y)Z, U) - S(Z, R(X,Y)U). \tag{3.2}$$

Using (3.1) in (3.2), we obtain

$$(R(X,Y)\cdot S)(Z,U) = (\lambda+\rho)[g(R(X,Y)Z,U)+g(Z,R(X,Y)U)] = 0, \quad (3.3)$$

which implies that the manifold under consideration is Ricci semisymmetric, i.e. (iv) holds.
We now consider a Riemannian manifold (M^n, g), which is conformally flat. Then $C(X,Y)Z = 0$ and hence from (2.2), we get

$$\begin{aligned}(3.4)\,R(X,Y)Z &= \frac{1}{n-2}\big[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ &- g(X,Z)QY\big] - \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\}.\end{aligned}$$

From (3.1) we get

$$QX = -(\lambda+\rho)X, \quad (3.5)$$

$$r = -n(\lambda+\rho), \quad (3.6)$$

where Q is the Ricci-operator such that $g(QX,Y) = S(X,Y)$ for all X, Y and r is the scalar curvature of the manifold.
In view of (3.1), (3.6) and (3.7), (3.4) yields

$$R(X,Y)Z = -\frac{1}{n-1}(\lambda+\rho)\{g(Y,Z)X - g(X,Z)Y\}, \quad (3.7)$$

which implies that the manifold under consideration is a manifold of constant curvature, as $\lambda+\rho$ is constant. Thus we can state the following:

Theorem 3.2. *Let (g, V, λ) be a Ricci almost soliton on a conformally flat Riemannian manifold (M^n, g). If V is a conformal Killing vector field then M is a manifold of constant curvature.*

Corollary 3.1. *Let (g, V, λ) be a Ricci almost soliton on a 3-dimensional Riemannian manifold (M^n, g). If V is a conformal Killing vector field then M is a manifold of constant curvature.*

4. Ricci almost soliton on Riemannian manifolds with respect to semi-symmetric metric connection

In [9] Friedmann and Schouten introduced the notion of semi-symmetric linear connection on a differentiable manifold. Then in 1932 Hayden [12] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold has been given by Yano in 1970 [26]. Thereafter semi-symmetric metric connection on Riemannian manifold is studied by various authors.

Let (M^n, g) be an n-dimensional Riemannian manifold of class C^∞ with the metric tensor g and ∇ be the Riemannian connection of the manifold (M^n, g).

A linear connection $\overline{\nabla}$ on (M^n, g) is said to be semi-symmetric [9] if the torsion tensor τ of the connection $\overline{\nabla}$ satisfies

$$\tau(X, Y) = \alpha(Y)X - \alpha(X)Y \tag{4.1}$$

for any vector field X, Y on M and α is an 1-form associated with the torsion tensor τ of the connection $\overline{\nabla}$ given by

$$\alpha(X) = g(X, \rho),$$

ρ being the vector field associated with the 1-form α. The 1-form α is called the associated 1-form of the semi-symmetric connection and the vector field ρ is called the associated vector field of the connection. A semi-symmetric connection $\overline{\nabla}$ is called a semi-symmetric metric connection [12] if in addition it satisfies

$$\overline{\nabla} g = 0. \tag{4.2}$$

The relation between the semi-symmetric connection $\overline{\nabla}$ and the Riemannian connection ∇ of (M^n, g) is given by [26]

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(Y)X - g(X, Y)\rho. \tag{4.3}$$

In particular, if the 1-form α vanishes identically then a semi-symmetric metric connection reduces to the Riemannian connection. The covariant differentiation of an 1-form ω with respect to $\overline{\nabla}$ is given by [26]

$$(\overline{\nabla}_X \omega)(Y) = (\nabla_X \omega)(Y) + \omega(X)\alpha(Y) - \omega(\rho)g(X, Y).$$

If R and $\overline{R}$ are respectively the curvature tensor of the Levi-Civita connection ∇ and the semi-symmetric metric connection $\tilde{\nabla}$, then we have ([22], [26])

$$\begin{aligned} \overline{R}(X, Y)Z &= R(X, Y)Z - P(Y, Z)X + P(X, Z)Y \\ &- g(Y, Z)LX + g(X, Z)LY, \end{aligned} \tag{4.4}$$

where P is a tensor field of type (0, 2) given by

$$P(X, Y) = g(LX, Y) = (\nabla_X \alpha)(Y) - \alpha(X)\alpha(Y) + \frac{1}{2}\alpha(\rho)g(X, Y) \tag{4.5}$$

for any vector fields X and Y. From (4.4), it follows that

$$\overline{S}(Y, Z) = S(Y, Z) - (n - 2)P(Y, Z) - ag(Y, Z), \tag{4.6}$$

where $\overline{S}$ and S denote respectively the Ricci tensor with respect to $\overline{\nabla}$ and ∇, $a = \text{trace } P$.

We now consider (g, V, λ) is a Ricci almost soliton on a Riemannian manifold M with respect to semi-symmetric metric connection. Then we have

$$(\overline{\pounds}_V g)(Y, Z) + 2\overline{S}(Y, Z) + 2\lambda g(Y, Z) = 0, \tag{4.7}$$

where $\overline{\pounds}_V$ is the Lie derivative along the vector field V on M with respect to semi-symmetric metric connection.
By virtue of (4.3), we have

$$
\begin{aligned}
(\overline{\pounds}_V g)(Y,Z) &= g(\overline{\nabla}_Y V, Z) + g(Y, \overline{\nabla}_Z V) \\
&= g(\nabla_Y V + \alpha(V)Y - g(Y,V)\rho, Z) \\
&+ g(Y, \nabla_Z V + \alpha(V)Z - g(Z,V)\rho) \\
&= (\pounds_V g)(Y,Z) + 2\alpha(V)g(Y,Z) \\
&- [\alpha(Z)g(Y,V) + \alpha(Y)g(Z,V)].
\end{aligned}
\tag{4.8}
$$

Using (4.6) and (4.8) in (4.7), we get

$$
\begin{aligned}
&(\pounds_V g)(Y,Z) + 2S(Y,Z) + 2\lambda g(Y,Z) \\
+ \ &2\{\alpha(V) - a\}g(Y,Z) - 2(n-2)P(Y,Z) \\
- \ &[\alpha(Z)g(Y,V) + \alpha(Y)g(Z,V)] = 0.
\end{aligned}
\tag{4.9}
$$

If (g, V, λ) is a Ricci almost soliton on a Riemannian manifold with respect to Levi-Civita connection then (1.1) holds. Thus from (1.1) and (4.9) we can state the following:

Theorem 4.1. *A Ricci almost soliton (g, V, λ) on a Riemannian manifold is invariant under semi-symmetric metric connection if and only if the relation*

$$
\begin{aligned}
&2\{\alpha(V) - a\}g(Y,Z) - 2(n-2)P(Y,Z) \\
- \ &[\alpha(Z)g(Y,V) + \alpha(Y)g(Z,V)] = 0
\end{aligned}
$$

holds for arbitrary vector fields Y, Z and V.

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