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Inclusion theorems on general convergence and statistical convergence of $(L, 1, 1)$ - summability using generalized Tauberian conditions *

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Abstract

Statistical convergence has attracted the attention of researchers due to the fact that it is more general than classical convergence. Recently Móricz [Analysis Mathematica, 40(3) (2014), 231-242] has established a result on statistical extension of classical Tauberian theorem by $(L, 1)$ - summability for a function of a single variable. In this paper some

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new inclusion theorems are established under Tauberian conditions by $(L, 1, 1)$ - summability of double integrable functions of two variables using oscillating behavior and De la vallée Poussin mean.

Keywords and Phrases: *Harmonic Summability, Slow oscillation, Improper Integral, Tauberian theorem, $(L, 1, 1)$-summability, Statistical summability*

1. Introduction and definition

From calculus of two variables, we may remember that a complex valued function $f(x, y)$ is locally integrable over R_+^2 [i.e., over $[0, \infty) \times [0, \infty)$], if $\forall\ 0 < p < \infty$, $0 < q < \infty$, the integral

$$s(p, q) = \int_0^p \int_0^q f(x, y) dx dy$$

exists in the Lebesgue's sense. Also, we may recall (Hardy [7], p.11 or Titchmarsh [1], p.26) that a continuous function $a(t)$, integrable over $(0, x)$ with $s(x) = \int_0^x a(t)dt$ is integrable in Pickwickian sense if $s(x)$ has p limit as $x \to \infty$.

Let $f(t)$ be a real - valued continuous function which is measurable (in Lebesgue's sense) on $[1, \infty)$, then its logarithmic $(L, 1)$ (also called harmonic) mean $\tau(t)$ of order 1 is given by [3]

$$\tau(t) = \frac{1}{\log t} \int_1^t \frac{s(x)}{x} dx. \tag{1.1}$$

Let us consider a function of two variables $f(x, y)$ which is real - valued, continuous and is measurable (in Lebesgue's sense) on $[1, \infty) \times [1, \infty)$. We set the partial sum

$$s(x, y) = \int_1^x \int_1^y f(\zeta, \eta) d\zeta d\eta, \forall\ 1 < x, y < \infty.$$

We may define the logarithmic $(L, 1, 1)$ (also called harmonic) mean of $s(x, y)$ as

$$\sigma_{11}(s(x, y)) = \sigma(s(x, y)) = \frac{1}{\log x \log y} \int_1^x \int_1^y \frac{s(\zeta, \eta)}{\zeta\eta} d\zeta d\eta. \tag{1.2}$$

Analogous to (1.2), the $(L, 1, 0)$ and $(L, 0, 1)$ means of $s(x, y)$ can be viewed as

$\sigma_{1,0}(s(x,y)) = \frac{1}{\log x}\int_1^x \frac{s(\zeta,y)}{\zeta}d\zeta$ and $\sigma_{0,1}(s(x,y)) = \frac{1}{\log y}\int_1^y \frac{s(x,\eta)}{\eta}d\eta$ respectively.

Remark. Our choice of $s(x,y)$ in (1.2) generalizes (1.1).

For a function of a single variable, we may recall (see [4]) that a function $s(x)$ has a statistical limit at ∞, if there exists a number l such that for every $\epsilon > 0$,

$$\lim_{b\to\infty}\frac{1}{b-a}|\{x \in (a,b) : |s(x) - l| > \epsilon\}| = 0. \tag{1.3}$$

Here we may say that, the function $s(x,y)$ is said to be statistically converging to a finite number l at ∞, if for every $\epsilon > 0$,

$$\lim_{u,v\to\infty}\frac{1}{uv}|\,\{1 < x,y < u,v : |s(x,y) - l| \geq \epsilon\}\,| = 0. \tag{1.4}$$

Symbolically, we write $s(x,y) \xrightarrow{st} l$, as $x,y \to \infty$. Since the function $s(x,y)$ converges to l, as $x,y \to (a,b)$ (Pringsheim's sense), for every $\epsilon > 0$, $\exists$ a $\delta > 0$, such that $|s(x,y) - l| < \epsilon$, whenever $|x-a| < \delta,\ |y-b| < \delta$.
It can be written as $s(x,y) \to l$, as $x,y \to \infty$.
The function $s(t)$ is said to be $(L,1)$ - summable to a finite number l [3], if

$$\tau(s(t)) \to l, \quad as \quad t \to \infty. \tag{1.5}$$

Also, we may say that, the function $s(x,y)$ is said to be $(L,1,1)$ - summable to a finite number l, if

$$\sigma(s(x,y)) \to l, \quad as \quad x,y \to \infty. \tag{1.6}$$

Analogous to (1.6), the function $s(x,y)$ is said to be $(L,1,1)$ - statistically summable to a finite number l, if

$$\sigma(s(x,y)) \xrightarrow{st} l, \quad as \quad x,y \to \infty. \tag{1.7}$$

Clearly, if the function

$$s(x,y) \xrightarrow{st} l, \quad as\ x,y \to \infty \tag{1.8}$$

exists, then (1.7) also exists. But (1.7) does not imply (1.8). Again, if the function $s(x,y)$ converges to l, i.e.,

$$s(x,y) \to l, \quad as \quad x,y \to \infty \tag{1.9}$$

exists; then (1.8), (1.7) and (1.6) also exist. But the converse is not true in general. We prove the converse part by using some conditions like oscillatory behavior and De la Vallée Poussin - mean of double integral. Such conditions are called Tauberian conditions and theorems with Tauberian conditions are called Tauberian theorems.
A function s(x) is said to be oscillating slowly with respect to $(L,1)$ - summability if

$$\lim_{\lambda\to 1^+}\limsup_{x\to\infty}\sup_{\log x<\log t\leq\lambda\log x}(s(t)-s(x))=0 \tag{1.10}$$

exists [3]. It is easy to check that the condition (1.10) is satisfied if and only if for every $\epsilon>0$, $\exists\, x_0=x_0(\epsilon)>1$ and $\lambda=\lambda(\epsilon)>1$, the latter one is as close to 1 as we want, such that

$$|s(t)-s(x)|\leq\epsilon,\ \textit{whenever}\ x<t<x^\lambda. \tag{1.11}$$

Analogous to (1.10), we say that $s(x,y)$ is oscillating slowly with respect to $(L,1,1)$ - summability, if

$$\lim_{\lambda\to 1^+}\limsup_{x,y\to\infty}\sup_{x,y<\zeta,\eta\leq x^\lambda,y^\lambda}(s(\zeta,\eta)-s(x,t))=0 \tag{1.12}$$

exists, i.e., for every $\epsilon>0$, $\exists\, x_0=x_0(\epsilon)>1,\ y_0=y_0(\epsilon)>1$ and $\lambda=\lambda(\epsilon)>1$, the latter one is as close to 1 as we want, such that

$$|s(\zeta,\eta)-s(x,y)|\leq\epsilon,\ \textit{whenever}\ x,y<\zeta,\eta<x^\lambda,y^\lambda. \tag{1.13}$$

If a function $f(x,y)$ is such that

$$\log u\log v|f(u,v)|\leq C. \tag{1.14}$$

at almost every $u,v>\zeta_0,\eta_0$, where C,ζ_0,η_0 are constants, then the function $s(x,y)$ is oscillating slowly w.r.t. $(L,1,1)$ - summability. The equation (1.14) is known as two - sided Tauberian condition. Authors like, Landau [2] and Hardy [8] have used the two - sided Tauberian conditions for $(C,1)$ - summability methods of numerical sequences.

The De la Vallée Poussin - mean of the double integral $\int_1^x\int_1^y f(\zeta,\eta)d\zeta d\eta$ is defined by

$$\tau(s(x,y))=\frac{1}{(\lambda\log x-\log x)(\lambda\log y-\log y)}\int_x^{x^\lambda}\int_y^{y^\lambda}s(\zeta,\eta)d\zeta d\eta,\ \lambda\in(1,\infty);$$

$$\tau(s(x,y)) = \frac{1}{(\log x - \lambda \log x)(\log y - \lambda \log y)} \int_{x^\lambda}^{x} \int_{y^\lambda}^{y} s(\zeta,\eta) d\zeta d\eta, \ \lambda \in (0,1).$$

For each nonnegative integers k and r, we define

$$\sigma_{k,r}(s(x,y)) = \begin{cases} \frac{1}{\log x \log y} \int_1^x \int_1^y \sigma_{(k-1,r-1)} s(\zeta,\eta) d\zeta d\eta, & for \ k,r \geq 1 \\ \\ \int_1^x \int_1^y s(\zeta,\eta) d\zeta d\eta, & for \ k,r = 0 \end{cases}$$

and we notice that $\sigma_{11}(s(x,y)) = \sigma(s(x,y))$.

The double integral $\int_1^\infty \int_1^\infty f(x,y) dx dy$ is known to be (L,k,r) - statistically summable to l if $\sigma_{k,r}(s(x,y))$ converges statistically to l. If $k=1$ and $r=1$, then (L,k,r) - statistical summability reduces to $(L,1,1)$ - statistical summability. Again, if $k \neq 0$ and $r=0$, then (L,k,r) - statistical summability reduces to $(L,k,0)$ - statistical summability. Further, if $k=0$ and $r \neq 0$ then (L,k,r) - statistical summability reduces to $(L,0,r)$ - statistical summability. Let

$$s(x,y) - \sigma(s(x,y)) = v(f(x,y)) \tag{1.15}$$

$$where, \ v(f(x,y)) = v_{11}(f(x,y)) = \frac{1}{\log x \log y} \int_1^x \int_1^y \zeta\eta f(\zeta,\eta) d\zeta d\eta.$$

Note that, $\sigma'(s(x,y)) = \frac{v(f(x,y))}{(x \log x) \ (y \log y)}$.

For each positive integers k, r, we define

$$v_{k,r}(f(x,y)) = \begin{cases} \frac{1}{\log x \log y} \int_1^x \int_1^y v_{(k-1,r-1)} \zeta\eta f(\zeta,\eta) d\zeta d\eta, & for \ k,r \geq 1 \\ \\ \int_1^x \int_1^y \zeta\eta f(\zeta,\eta) d\zeta d\eta, & for \ k,r = 0 \end{cases}$$

The double integral $\int_1^\infty \int_1^\infty xy f(x,y) dx dy$ is (L,k,r) - statistical summable to l, if $v_{k,r}(f(x,y))$ converges statistically to l. If $k = 1$ and $r = 1$, then (L,k,r) - statistical summability reduces to $(L,1,1)$ - statistical summability. Again, if $k \neq 0$ and $r = 0$, then (L,k,r) - statistical summability reduces to $(L,k,0)$ - statistical summability. Further, if $k = 0$ and $r \neq 0$, then (L,k,r) - statistical summability reduces to $(L,0,r)$ - statistical summability.

2. Known results

The concept of statistical convergence was introduced by Fast [9]. Fridly and Khan [10] implemented Hardy's [8] and Landau's [2] Tauberian theorems to the case of statistical convergence. In 2003, Móricz [5], [6] has established some Tauberian theorems for statistical Cesàro summability of single and double sequences. Subsequently, in 2014, Móricz [3] has established the following theorems on statistical convergence of Harmonic summability of a single - valued function.

Theorem 1. [3] If $s(x) \in \mathbb{R}$ is $(L,1)$ - summable and $s(x)$ is decreasing slowly, then original converges follows from statistical converges of $s(x)$.

Theorem 2. [3] If $s(x) \in \mathbb{C}$ is $(L,1)$ - summable and $s(x)$ is oscillating slowly, then original converges follows from statistical converges of $s(x)$.

Theorem 3. [3] If $s(x) \in \mathbb{R}$ is $(L,1)$ - summable and $s(x)$ is oscillating slowly, then original converges follows from $(L,1)$ mean of statistical converges.

Theorem 4. [3] If $s(x) \in \mathbb{C}$ is $(L,1)$ - summable and $s(x)$ is oscillating slowly, then original converges follows from $(L,1)$ mean of statistical converges.

3. Main result

In an attempt to enrich further studies in this direction, two new inclusion theorems on $(L,1,1)$ - summability of double integrable functions of two variables are established under Tauberian conditions as follows.

Theorem 1. If $s(x,y)$ is $(L,1,1)$ - statistical summable to l and $s(x,y)$ is oscillating slowly, then $s(x,y) \to l$, as $x,y \to \infty$.

For proving theorem 1, we need the following lemmas:

Lemma 1. (i) For $\lambda > 1$,

$$\begin{aligned}
& s(x,y) - \sigma(s(x^\lambda, y^\lambda)) \\
&\quad = \frac{1}{(\lambda-1)^2}\left(\sigma(s(x^\lambda, y^\lambda)) - \sigma(s(x,y))\right) + \frac{2}{(\lambda-1)}\sigma(s(x^\lambda, y^\lambda)) \\
&\quad - \frac{1}{(\lambda \log x - \log x)(\lambda \log y - \log y)} \int_x^{x^\lambda} \int_y^{y^\lambda} (s(\zeta,\eta) - s(x,y)) d\zeta d\eta.
\end{aligned}$$

(ii) For $0 < \lambda < 1$,

$$\begin{aligned}
& s(x,y) - \sigma(s(x^\lambda, y^\lambda)) \\
&\quad = \frac{1}{(1-\lambda)^2}(\sigma(s(x,y)) - \sigma(s(x^\lambda, y^\lambda))) + \frac{2}{(1-\lambda)}\sigma(s(x^\lambda, y^\lambda)) \\
&\quad - \frac{1}{(\log x - \lambda \log x)(\log y - \lambda \log y)} \int_{x^\lambda}^{x} \int_{y^\lambda}^{y} (s(x,y) - s(\zeta,\eta)) d\zeta d\eta.
\end{aligned}$$

Proof. (i) We have by De la Vallée Poussin mean of $s(x,y)$,

$$\begin{aligned}
\tau(s(x,y)) &= \frac{1}{(\lambda \log x - \log x)(\lambda \log y - \log y)} \int_x^{x^\lambda} \int_y^{y^\lambda} s(\zeta,\eta) d\zeta d\eta \\
&= \left(\frac{1}{\log x(\lambda-1)\log y(\lambda-1)}\right) \\
&\quad \times \left(\int_1^{x^\lambda}\int_1^{y^\lambda} s(\zeta,\eta)d\zeta d\eta - \int_1^x \int_1^y s(\zeta,\eta) d\zeta d\eta\right), \ for\ \lambda > 1.
\end{aligned}$$

Since,

$$\sigma(s(x^\lambda, y^\lambda)) = \frac{1}{\lambda \log x \ \lambda \log y} \int_1^{x^\lambda} \int_1^{y^\lambda} s(\zeta,\eta) d\zeta d\eta \ and$$

$$\sigma(s(x,y)) = \frac{1}{\log x \ \log y} \int_1^{x} \int_1^{y} s(\zeta,\eta) d\zeta d\eta,$$

we get,

$$\begin{aligned}\tau(s(x,y)) &= \frac{\lambda^2}{(\lambda-1)^2}\sigma(s(x^\lambda,y^\lambda)) - \frac{1}{(\lambda-1)^2}\sigma(s(x,y))\\ &= \left(1+\frac{1}{(\lambda-1)}\right)^2\sigma(s(x^\lambda,y^\lambda)) - \frac{1}{(\lambda-1)^2}\sigma(s(x,y)).\\ \Rightarrow \tau(s(x,y)) - \sigma(s(x^\lambda,y^\lambda)) &= \frac{1}{(\lambda-1)^2}\sigma(s(x^\lambda,y^\lambda)) + \frac{2}{(\lambda-1)}\sigma(s(x^\lambda,y^\lambda))\\ &\quad - \frac{1}{(\lambda-1)^2}\sigma(s(x,y)). \qquad (3.1)\end{aligned}$$

$$\textit{we have, } s(x,y) = \tau(s(x,y)) - \frac{1}{(\lambda\log x - \log x)(\lambda\log y - \log y)} \times \left(\int_x^{x^\lambda}\int_y^{y^\lambda}(s(\zeta,\eta)-s(x,y))d\zeta d\eta\right),$$

Subtracting $\sigma(s(x^\lambda,y^\lambda))$ from the identity, we have

$$\begin{aligned}s(x,y)-\sigma(s(x^\lambda,y^\lambda)) &= \tau(s(x,y)-\sigma(s(x^\lambda,y^\lambda))\\ &\quad - \frac{1}{(\lambda\log x - \log x)(\lambda\log y - \log y)}\\ &\qquad \times \left(\int_x^{x^\lambda}\int_y^{y^\lambda}(s(\zeta,\eta)-s(x,y))d\zeta d\eta.\right) \qquad (3.2)\end{aligned}$$

From equation (3.1) and (3.2), we have

$$\begin{aligned}&s(x,y)-\sigma(s(x^\lambda,y^\lambda))\\ &\quad = \frac{1}{(\lambda-1)^2}\big(\sigma(s(x^\lambda,y^\lambda)) - \sigma(s(x,y))\big) + \frac{2}{(\lambda-1)}\sigma(s(x^\lambda,y^\lambda))\\ &\quad - \frac{1}{(\lambda\log x - \log x)(\lambda\log y - \log y)}\int_x^{x^\lambda}\int_y^{y^\lambda}(s(\zeta,\eta)-s(x,y))d\zeta d\eta. \qquad (3.3)\end{aligned}$$

This completes the proof of lemma 1(i) and similarly we can prove 1(ii).

Lemma 2. $s(x,y)$ is oscillating slowly if and only if $v(f(x,y))$ is bounded and oscillating slowly.

Proof. Let $s(x,y)$ is oscillating slowly. First of all let us show that $v(f(x,y)) = O(1)$ as $x, y \to \infty$.
We have,

$$\int_1^x \int_1^y wzf(w,z)dwdz = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \int_{x/2^{i+1}}^{x/2^i} \int_{y/2^{j+1}}^{y/2^j} wzf(w,z)dwdz. \tag{3.4}$$

It follows from the identity,

$$\begin{aligned}
&\int_\alpha^\beta \int_\gamma^\delta wzf(w,z)dwdz = \int_\alpha^\beta \int_\gamma^\delta wzs'(w,z)dwdz \\
&= \int_\alpha^\beta z\left(\int_\gamma^\delta ws'(w,z)dw\right)dz \\
&= \int_\alpha^\beta z\left[w[s(w,z)]_\gamma^\delta - \int_\gamma^\delta s(w,z)dw\right]dz \\
&= -\int_\alpha^\beta \int_\gamma^\delta zs(w,z)dwdz + \delta\int_\alpha^\beta zs(\delta,z)dz - \gamma\int_\alpha^\beta zs(\gamma,z)dz \\
&-\gamma\int_\alpha^\beta zs(\delta,z)dz + \gamma\int_\alpha^\beta zs(\delta,z)dz \\
&= -\int_\alpha^\beta \int_\gamma^\delta zs(w,z)dwdz + (\delta-\gamma)\int_\alpha^\beta zs(\delta,z)dz \\
&\quad +\gamma\left(\int_\alpha^\beta zs(\delta,z)dz - \int_\alpha^\beta zs(\gamma,z)dz)\right) \\
&= -\int_\alpha^\beta \int_\gamma^\delta \left[z(s(w,z)-s(\delta,z))\right]dzdw + \gamma\left(\int_\alpha^\beta zs(\delta,z)dz - \int_\alpha^\beta zs(\gamma,z)dz\right) \\
&= (\beta-\alpha)(\delta-\gamma)\max_{\alpha,\gamma\le x,y\le\beta,\delta}|s(x,y)-s(\beta,\delta)| + \gamma\left|\int_\alpha^\beta z(s(\delta,z)-s(\gamma,z))dz\right|.
\end{aligned}$$

Choosing $\beta = x/2^i$, $\beta/\alpha \le 2$ and $\delta = y/2^j$, $\delta/\gamma \le 2$, we obtain,

$$\left|\int_1^x \int_1^y wzf(w,z)dwdz\right| \le A\sum_{i=1,j=1}^{\infty,\infty} \frac{xy}{2^{i+j}} = O(xy),\ \ as\ x,y\to\infty.$$

Now we have to show that $\sigma(s(x,y))$ is oscillating slowly.

Since, $\sigma'(s(x,y)) = \frac{v(f(x,y))}{(x\log x)\ (y\log y)}$, we get

$$\begin{aligned}|\sigma(s(\zeta,\eta)) - \sigma(s(x,y))| &= \left|\int_x^\zeta\int_y^\eta \sigma'(s(w,z))dwdz\right|\\ &= \left|\int_x^\zeta\int_y^\eta f(w,z)dwdz\right|\\ &\leq C\int_x^\zeta\left(\int_y^\eta \frac{dw}{w}\right)\frac{dz}{z}\\ &= C\log(\eta/y)\log(\zeta/x),\end{aligned}$$

for any $x,y \leq \zeta,\eta \leq x^\lambda, y^\lambda$.

Hence, we conclude that,

$$\max_{x,y\leq\zeta,\eta\leq x^\lambda,y^\lambda} |\sigma(s(\zeta,\eta)) - \sigma(s(x,y))| \leq C(\lambda-1)^2 \log x \log y.$$

Taking the limit sup to both sides as $\lambda \to 1^+$, we get

$$\lim_{\lambda\to 1^+} \lim \sup_{x,y\to\infty} \max_{x,y\leq\zeta,\eta\leq x^\lambda,y^\lambda} |\sigma(s(\zeta,\eta) - \sigma(s(x,y))| = 0.$$

Hence, $v(f(x,y))$ is oscillating slowly by Kronecker identity (1.15).
Conversely, let us suppose $v(f(x,y))$ is bounded and oscillating slowly. Thus the boundedness of $v(f(x,y))$ implies that $\sigma(s(x,y))$ is oscillating slowly. Again, since $v(f(x,y))$ is oscillating slowly, so $s(x,y)$ is oscillating slowly by Kronecker identity (1.15). Hence the proof of the lemma 2 follows from lemma 1.

Proof of Theorem 1.
Let $s(x,y)$ be $(L,1,1)$- statistical summable to l, then $\sigma(s(x,y)) \xrightarrow{st} l$ with respect to $(L,1,1)$ - summability. Now from equation (1.15), $v(f(x,y))$ is $(L,1,1)$ - statistical summable to zero. Thus, $v(f(x,y))$ is oscillating slowly

by lemma 2. Again by lemma 1(i), we get

$$\begin{aligned} v(f(x,y)) - \sigma(v(f(x^\lambda, y^\lambda))) &= \frac{1}{(\lambda-1)^2}(\sigma(v(f(x^\lambda, y^\lambda))) - \sigma(v(f(x,y)))) \\ &+ \frac{2}{(\lambda-1)}\sigma(v(f(x^\lambda, y^\lambda))) - \frac{1}{(\lambda\log x - \log x)(\lambda \log y - \log y)} \\ &\times \left(\int_x^{x^\lambda}\int_y^{y^\lambda}(v(f(\zeta,\eta)) - v(f(x,y)))d\zeta d\eta\right). \end{aligned} \tag{3.5}$$

We have by (3.5),

$$\begin{aligned} |v(f(x,y)) - \sigma(v(f(x,y)))| \le \frac{1}{(\lambda-1)^2}&|(\sigma(v(f(x^\lambda, y^\lambda))) - \sigma(v(f(x,y))))| \\ &+ \frac{2}{(\lambda-1)}|\sigma(v(f(x^\lambda, y^\lambda)))| \\ &+ \max_{x,y\le\zeta,\eta\le x^\lambda,y^\lambda}|v(f(\zeta,\eta)) - v(f(x,y))|. \end{aligned} \tag{3.6}$$

Now taking limit sup to both sides of equation (3.6) as $x, y \to \infty$, we have

$$\begin{aligned} &\lim\sup_{x,y\to\infty}|v(f(x,y)) - \sigma(v(f(x,y)))| \\ &\qquad \le \lim\sup_{x,y\to\infty}\frac{1}{(\lambda-1)^2}|(\sigma(v(f(x^\lambda, y^\lambda))) - \sigma(v(f(x,y))))| \\ &\qquad + \lim\sup_{x,y\to\infty}\frac{2}{(\lambda-1)}|\sigma(v(f(x^\lambda, y^\lambda)))| \\ &\qquad + \lim\sup_{x,y\to\infty}\max_{x,y\le\zeta,\eta\le x^\lambda,y^\lambda}|v(f(\zeta,\eta)) - v(f(x,y))|. \end{aligned} \tag{3.7}$$

Since $\sigma(v(f(x^\lambda, y^\lambda)))$ converges, first and second term in the right hand side of equation (3.7), must be zero.
This implies,

$$\begin{aligned} &\lim\sup_{x,y\to\infty}|v(f(x,y)) - \sigma(v(f(x,y)))| \\ &\qquad \le \lim\sup_{x,y\to\infty}\max_{x,y\le\zeta,\eta\le x^\lambda,y^\lambda}|v(f(\zeta,\eta)) - v(f(x,y))|. \end{aligned} \tag{3.8}$$

As $\lambda \to 1^+$ in (3.8), we get

$$\lim\sup_{x,y\to\infty}|v(f(x,y)) - \sigma(v(f(x,y)))| \le 0. \tag{3.9}$$

Clearly, $v(f(x,y)) = o(1)$, as $x,y \to \infty$. Again, since $s(x,y)$ is statistical summable to l by $(L,1,1)$ means and $v(f(x,y)) = o(1)$, as $x,y \to \infty$, so $s(x,y) \to l, \;\; as \;\; x,y \to \infty$.

Corollary 1. If $s(x,y)$ is (L,k,r) - statistical summable to l and $s(x,y)$ is oscillating slowly, then $s(x,y) \to l$ as $x,y \to \infty$.

Proof. By lemma 2, $s(x,y)$ being oscillating slowly, so $\sigma_{kr}(s(x,y))$ is oscillating slowly. Further in analogous to theorem-1, $s(x,y)$ is (L,k,r) - statistically summable to l, this implies

$$\sigma_{kr}(s(x,y)) \xrightarrow{st} l \; as \; x,y \to \infty. \tag{3.10}$$

Next from the definition,

$$\sigma_{kr}(s(x,y)) = \sigma_{11}(s(x,y))(\sigma_{k-1,r-1}(s(x,y))). \tag{3.11}$$

Clearly, equation (3.10) and (3.11) implies $s(x,y)$ is $(L,k-1,r-1)$ - statistical summable to l.
Again by lemma 2, $\sigma_{k-1,r-1}(s(x,y))$ is also oscillating slowly. Hence by theorem -1, we have

$$\sigma_{k-1,r-1}(s(x,y)) \to l \; as \; x,y \to \infty.$$

Continuing in this way, we get $s(x,y) \to l$ as $x,y \to \infty$.

Theorem 2. If $s(x,y)$ is $(L,1,1)$ - statistical summable to l and $v(f(x,y))$ is oscillating slowly, then $s(x,y) \to l$, as $x,y \to \infty$.

Proof. As $s(x,y)$ is $(L,1,1)$ - statistical summable to l, then $\sigma_{11}(s(x,y)) \xrightarrow{st} l$. So, $v(f(x,y)) \xrightarrow{st} 0$ with respect to $(L,1,1)$ summability, by equation (1.15). Again imposing identity (1.15) to $v(f(x,y))$, we get $v(v(f(x,y))) \xrightarrow{st} 0$, by $(L,1,1)$ summability. Hence, $v(v(f(x,y)))$ is oscillating slowly by lemma 2.

Now by lemma 1(i),

$$\begin{aligned}
&v(v(f(x,y))) - \sigma(v(v(f(x^\lambda,y^\lambda))))\\
&\quad = \frac{1}{(\lambda-1)^2}[\sigma(v(v(f(x^\lambda,y^\lambda)))) - \sigma(v(v(f(x,y))))]\\
&\qquad + \frac{2}{(\lambda-1)}(\sigma(v(v(f(x^\lambda,y^\lambda))))) - \frac{1}{(\lambda\log x - \log x)(\lambda \log y - \log y)}\\
&\qquad\qquad \times \left(\int_x^{x^\lambda}\int_y^{y^\lambda}(v(v(f(\zeta,\eta))) - v(v(f(x,y))))d\zeta d\eta.\right) \qquad (3.12)
\end{aligned}$$

We have,

$$\begin{aligned}
&|v(v(f(x,y))) - \sigma(v(v(f(x,y))))|\\
&\qquad \le \frac{1}{(\lambda-1)^2}|\sigma(v(v(f(x^\lambda,y^\lambda)))) - \sigma(v(v(f(x,y))))|\\
&\qquad\quad + \frac{2}{(\lambda-1)}|\sigma(v(v(f(x^\lambda,y^\lambda))))|\\
&\qquad\quad + \max_{x,y\le\zeta,\eta\le x^\lambda,y^\lambda}|(v(v(f(\zeta,\eta))) - v(v(f(x,y))))|. \qquad (3.13)
\end{aligned}$$

Now taking limit sup to both sides of equation (3.13) as $x, y \to \infty$, we have

$$\begin{aligned}
&\lim\sup_{x,y\to\infty}|v(v(f(x,y))) - \sigma(v(v(f(x,y))))|\\
&\quad \le \lim\sup_{x,y\to\infty}\frac{1}{(\lambda-1)^2}|\sigma(v(v(f(x^\lambda,y^\lambda)))) - \sigma(v(v(f(x,y))))|\\
&\qquad + \lim\sup_{x,y\to\infty}\frac{2}{(\lambda-1)}|\sigma(v(v(f(x^\lambda,y^\lambda))))|\\
&\qquad + \lim\sup_{x,y\to\infty}\max_{x,y\le\zeta,\eta\le x,y^\lambda}|v(v(f(\zeta,\eta))) - v(v(f(x,y)))|. \qquad (3.14)
\end{aligned}$$

Since $\sigma(v(v(f(x^\lambda,y^\lambda))))$ converges, first and second term in right hand side of equation (3.14) must be zero. This implies

$$\begin{aligned}
&\lim\sup_{x,y\to\infty}|v(v(f(x,y))) - \sigma(v(v(f(x,y))))|\\
&\qquad \le \lim\sup_{x,y\to\infty}\max_{x,y\le\zeta,\eta\le x^\lambda,y^\lambda}|v(v(f(\zeta,\eta))) - v(v(f(x,y)))|. \qquad (3.15)
\end{aligned}$$

As $\lambda \to 1^+$ in (3.15), we get

$$\lim_{x,y\to\infty} \sup |v(v(f(x,y))) - \sigma(v(v(f(x,y))))| \leq 0. \tag{3.16}$$

Clearly, $v(v(f(x,y))) = o(1)$, as $x, y \to \infty$. Further, since $s(x,y)$ is statistically summable to l by $(L,1,1)$ mean and $v(v(f(x,y))) = o(1)$, as $x, y \to \infty$, $s(x,y) \to l$ as $x, y \to \infty$.

Corollary 2. If $s(x,y)$ is (L,k,r) - statistical summable to l and $v(f(x,y))$ is oscillating slowly, then $s(x,y) \to l$, as $x, y \to \infty$.

Proof. As $v(f(x,y))$ is oscillating slowly, setting $v(f(x,y))$ in place of $s(x,y)$; $\sigma_{k,r}(v(f(x,y)))$ is oscillating slowly by lemma 2. Again as $v(f(x,y))$ is (L,k,r) - statistical summable to l; by theorem 2, we have

$$\sigma_{k,r}(v(f(x,y))) \xrightarrow{st} l \ as \ x, y \to \infty. \tag{3.17}$$

By definition,

$$\lim_{x,y\to\infty} \sigma_{k,r}(v(f(x,y))) = \sigma_{1,1}(v(f(x,y)))(\sigma_{(k-1,r-1)}(v(f(x,y)))). \tag{3.18}$$

From (3.17) and (3.18), we have $v(f(x,y))$ is $(L, k-1, r-1)$ - statistical summable to l. Again (by lemma 2), since $\sigma_{(k-1,r-1)}(v(f(x,y)))$ is oscillating slowly, we have $\lim_{x,y\to\infty} \sigma_{(k-1,r-1)}(v(f(x,y))) = l$ (by theorem 2). Continuing in this way, we get $\lim_{x,y\to\infty} v(f(x,y)) = l$, implies $s(x,y) \to l$,
as $x, y \to \infty$.

4. Conclusion

Statistical convergence has become an active area of research due to the fact that it is more general than classical convergence, i.e., a sequence may be convergent in statistical mean even if it is not convergent in classical mean. The result established in this paper for a function of two variables generalizes the earlier existing results for the function of a single variable. Further, it will be encouraging if one can extend the result for function of several variables.

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