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On the expansion of Ramanujan's continued fraction of order sixteen *

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Abstract

We give 2-, 4-, 8- and 16-dissections of a continued fraction of order sixteen. We show that the sign of the coefficients in the power series expansion of the continued fraction of order sixteen is periodic with period 16. We also give combinatorial interpretations for the coefficients in the power series expansion of the continued fraction of order sixteen.

Keywords and Phrases: *Continued fractions, dissections, periodicity of sign of coefficients.*

1. Introduction

Throughout this paper, we assume that $|q| < 1$ and use the standard notation

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),$$

$$(a_1, a_2, a_3, \ldots, a_n; q)_\infty = (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty \ldots, (a_n;q)_\infty,$$
$$[z;q]_\infty := (z;q)_\infty(z^{-1}q;q)_\infty$$

and

$$[z_1, z_2, z_3, \ldots, z_n; q]_\infty = [z_1;q]_\infty[z_2;q]_\infty[z_3;q]_\infty \ldots, [z_n;q]_\infty.$$

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The celebrated Rogers–Ramanujan continued fraction and its product representation is

$$R(q) := \frac{1}{1}\,{}_{+}\,\frac{q}{1}\,{}_{+}\,\frac{q^2}{1}\,{}_{+}\,\frac{q^3}{1}\,{}_{+}\cdots \quad = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}, \tag{1.1}$$

where

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}\, b^{n(n-1)/2}, \qquad |ab| < 1,$$

is Ramanujan's general theta function. The function $f(a,b)$ satisfy the well known Jacobi triple product identity

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

On page 36 of his "lost" notebook, Ramanujan recorded four q-series representations of the famous Rogers-Ramanujan continued fraction. Recently, C. Adiga, N. A. S. Bulkhali, Y. Simsek and H. M. Srivastava [3] have established two q-series representations of Ramanujan's continued fraction found in his "lost" notebook. They have also established three equivalent integral representations and modular equations for a special case of this continued fraction. On employing q-identities of Ramanujan found in his lost notebook, Srivastava et al. [19] established a number of results involving continued fractions of the form involved in (1.1).
G. E. Andrews et al. [7] investigated combinatorial partition identities associated with the following general family:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{8\binom{n}{2}+tn} \sum_{j=0}^{[n/u]} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}(q^{uv};q^{uv})_j}. \tag{1.2}$$

In [18], Srivastava and M. P. Chaudhary, proved several results associated with the family $R(s,t,l,u,v,w)$ which depict the inter relationship between q-product identities, continued fraction identities and combinatorial partition identities.

S. Ramanujan [16, p. 50] gave the 2-dissections and 5-dissections of $R(q)$ and its reciprocal, and these were first proved by Andrews [5] and M. D. Hirschhorn [9] respectively. The periodic behavior of the sign of the coefficients in the series expansion of $R(q)$ and its reciprocal were observed by

B. Richmond and G. Szekers [17]. Hirschhorn [9], conjectured formulas for the 4-dissections of $R(q)$ and its reciprocal, and these were first proved by R. Lewis and Z. -G. Liu [13]. In [21], O. X. M. Yao and E. X. W. Xia gave generalizations of Hirschhorn's formulas on the 4-dissections of $R(q)$ and its reciprocal. Recently, Hirschhorn [11] gave a simple proof of the 2-dissections and 4-dissections of $R(q)$ and its reciprocal by using Jacobi's triple product identity.

The beautiful Ramanujan–Göllnitz–Gordon continued fraction and its product representation is

$$G(q) := \frac{1}{1+q} \, {}_{+} \, \frac{q^2}{1+q^3} \, {}_{+} \, \frac{q^4}{1+q^5} \, {}_{+} \cdots \quad = \frac{f(-q,-q^7)}{f(-q^3,-q^5)}.$$

In [10], Hirschhorn found the 8-dissections of $G(q)$ and its reciprocal and also proved that the sign of the coefficients in the power series expansion of $G(q)$ and its reciprocal are periodic with period 8 and in particular that certain coefficients are zero, a phenomenon first observed and proved by Richmond and Szekers [17]. Recently, S. H. Chan and H. Yesilyurt [8] show the periodicity of signs of a large number of quotients of certain infinite products. Their results include as special cases, results of Ramanujan, Andrews, Bressoud, Richmond, Szekers and Hirschhorn.

The famous Ramanujan's cubic continued fraction and its product representation is

$$H(q) := \frac{1}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^2+q^4}{1} \, {}_{+} \cdots \quad = \frac{f(-q,-q^5)}{f(-q^3,-q^3)}.$$

In [20], B. Srivastava has studied the 2- and 4-dissections of the continued fraction of $H(q)^{-1}$. In [12], Hirschhorn and Roselin have studied the 2-, 3-, 4- and 6-dissections of $H(q)$ and its reciprocal and also shows the sign of the coefficients in the power series expansion of $H(q)$ and its reciprocal, are periodic with period 3 and 6 respectively.

The fascinating continued fraction of order twelve and its product representation is

$$U(q) := \frac{(1-q)}{(1-q^3)} \, {}_{+} \, \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)} \, {}_{+} \cdots \quad = \quad \frac{f(-q,-q^{11})}{f(-q^5,-q^7)}.$$

In [14], B. L. S. Lin has studied the 2-, 3-, 4-, 6- and 12-dissections of $U(q)$ and its reciprocal and also shows the sign of the coefficients in the power series

expansion of $U(q)$ and its reciprocal, are periodic with period 12.
The Ramanujan's continued fraction of order six and its product representation is

$$X^*(q) := \frac{(q^{1/2}-q^{-3/2})}{(1-q^{-3/2})} + \frac{(q^{1/4}-q^{-1/4})(q^{-7/4}-q^{7/4})}{(1-q^{-3/2})(1+q^3)} + \cdots = \frac{f(-q,-q^5)}{f(-q^2,-q^4)}.$$

In [4], Adiga et al. have studied the 2- and 4-dissections of $X^*(q)$ and its reciprocal and also shows that the sign of the coefficients in the power series expansion of $X^*(q)$ and its reciprocal, are periodic with period 2 and 6 respectively. In [2], Adiga, Bulkhali, D. Ranganatha and Srivastava have established several modular relations for the Rogers-Ramanujan type functions of order eleven which are analogous to Ramanujan's forty identities for Rogers-Ramanujan functions. Furthermore, they gave interesting partition theoretic interpretation of some of the modular relations. Motivated by the above works on continued fractions, we shall consider the power series expansion of the following continued fraction of order sixteen:

$$A(q) := \frac{f(-q,-q^{15})}{f(-q^7,-q^9)} = \frac{(1-q)}{(1-q^4)} + \frac{q^4(1-q^3)(1-q^5)}{(1-q^4)(1+q^8)} + \frac{q^4(1-q^{11})(1-q^{13})}{(1-q^4)(1+q^{16})} + \cdots. \tag{1.3}$$

Ramanujan has recorded several continued fraction in his notebooks. One of the fascinating continued fraction identities recorded by Ramanujan as Entry 12 in his second notebook [15] is

$$\frac{(a^2q^3;q^4)_\infty(b^2q^3;q^4)_\infty}{(a^2q;q^4)_\infty(b^2q;q^4)_\infty} = \frac{1}{(1-ab)} + \frac{(a-bq)(b-aq)}{(1-ab)(1+q^2)} + \frac{(a-bq^3)(b-aq^3)}{(1-ab)(1+q^4)} + \cdots, \quad |q|<1, \quad |ab|<1. \tag{1.4}$$

For a proof of (1.4), see Adiga et al. [1].
In (1.4), replacing q by q^4 and then setting $a = q^{3/2}$ and $b = q^{5/2}$, we obtain (1.3).
For $n \geq 0$, we define $a(n)$ and $b(n)$ by

$$A(q) = \frac{f(-q,-q^{15})}{f(-q^7,-q^9)} = \sum_{n=0}^{\infty} a(n)q^n, \tag{1.5}$$

$$B(q) = \frac{f(-q^7,-q^9)}{f(-q,-q^{15})} = \sum_{n=0}^{\infty} b(n)q^n. \tag{1.6}$$

The main object of this paper is to study different dissections of $A(q)$ and $B(q)$. In Section 2, we prove the 2- and 4-dissections of $A(q)$ and $B(q)$. In Section 3, we prove the 8-dissections of $A(q)$ and $B(q)$. Furthermore, we prove that the signs of the coefficients in the power series expansion of $A(q)$ and $B(q)$ are periodic with period 16. In Section 4, we prove the 16-dissections of $A(q)$ and $B(q)$. In the last section, we give combinatorial interpretations of the coefficients in the power series expansion of $A(q)$ and $B(q)$.

2. 2- and 4-Dissections of $A(q)$ and $B(q)$

In this section, we present 2- and 4-dissections of $A(q)$ and $B(q)$. To prove our results, we need the following Lemmas.

Lemma 2.1. *If $ab = cd$, then*

$$f(a,b)f(c,d) = f(ac,bd)f(ad,bc) + af(b/c, ac^2d)f(b/d, acd^2). \tag{2.1}$$

Proof. Adding Entries 29(i) and 29(ii) in [1], we obtain the result. □

Lemma 2.2. *[1, p. 46, Entry 30 (iv)] One has*

$$f(a,b)f(-a,-b) = f(-a^2,-b^2)f(-ab,-ab). \tag{2.2}$$

Theorem 2.3. *Let $a(n)$ be as defined in (1.5). Then*

$$\sum_{n=0}^{\infty} a(2n)q^n = \frac{[q^4,q^5;q^{16}]_\infty}{[q^7,q^8;q^{16}]_\infty}, \tag{2.3}$$

$$\sum_{n=0}^{\infty} a(2n+1)q^n = -\frac{[q^3,q^4;q^{16}]_\infty}{[q^7,q^8;q^{16}]_\infty}. \tag{2.4}$$

Proof. We have

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{f(-q,-q^{15})}{f(-q^7,-q^9)} = \frac{f(-q,-q^{15})f(q^7,q^9)}{f(-q^7,-q^9)f(q^7,q^9)}. \tag{2.5}$$

By employing (2.1) with $a=-q$, $b=-q^{15}$, $c=q^7$ and $d=q^9$, we obtain

$$f(-q,-q^{15})f(q^7,q^9) = f(-q^8,-q^{24})f(-q^{10},-q^{22}) - qf(-q^8,-q^{24})f(-q^6,-q^{26}). \tag{2.6}$$

Setting $a=q^7$ and $b=q^9$ in (2.2), we find that

$$f(q^7,q^9)f(-q^7,-q^9)=f(-q^{14},-q^{18})f(-q^{16},-q^{16}). \tag{2.7}$$

Combining (2.5), (2.6) and (2.7), we obtain

$$\sum_{n=0}^{\infty} a(n)q^n=\frac{f(-q^8,-q^{24})f(-q^{10},-q^{22})-qf(-q^8,-q^{24})f(-q^6,-q^{26})}{f(-q^{14},-q^{18})f(-q^{16},-q^{16})}.$$

It follows immediately that

$$\sum_{n=0}^{\infty} a(2n)q^{2n}=\frac{f(-q^8,-q^{24})f(-q^{10},-q^{22})}{f(-q^{14},-q^{18})f(-q^{16},-q^{16})},$$

$$\sum_{n=0}^{\infty} a(2n+1)q^{2n+1}=-q\frac{f(-q^8,-q^{24})f(-q^6,-q^{26})}{f(-q^{14},-q^{18})f(-q^{16},-q^{16})}.$$

Changing q to $q^{1/2}$ in the above equations, we obtain (2.3) and (2.4). □

Theorem 2.4. *Let $b(n)$ be as defined in* (1.6). *Then*

$$\sum_{n=0}^{\infty} b(2n)q^n=\frac{[q^4,q^5;q^{16}]_\infty}{[q,q^8;q^{16}]_\infty}, \tag{2.8}$$

$$\sum_{n=0}^{\infty} b(2n+1)q^n=\frac{[q^3,q^4;q^{16}]_\infty}{[q,q^8;q^{16}]_\infty}. \tag{2.9}$$

Proof. The proof of Theorem 2.4 is similar to that of Theorem 2.3. □

Theorem 2.5. *We have*

$$\sum_{n=0}^{\infty} a(4n)q^n=\frac{[q^2,q^6,q^6;q^{16}]_\infty}{[q^4,q^4,q^8;q^{16}]_\infty}, \tag{2.10}$$

$$\sum_{n=0}^{\infty} a(4n+1)q^n=-\frac{[q^2,q^5,q^6,q^6;q^{16}]_\infty}{[q^4,q^4,q^7,q^8;q^{16}]_\infty}, \tag{2.11}$$

$$\sum_{n=0}^{\infty} a(4n+2)q^n=-q^2\frac{[q,q^2,q^2,q^6;q^{16}]_\infty}{[q^4,q^4,q^7,q^8;q^{16}]_\infty}, \tag{2.12}$$

$$\sum_{n=0}^{\infty} a(4n+3)q^n=q\frac{[q^2,q^2,q^3,q^6;q^{16}]_\infty}{[q^4,q^4,q^7,q^8;q^{16}]_\infty}. \tag{2.13}$$

Proof. From (2.3), we see that

$$\sum_{n=0}^{\infty} a(2n)q^n = \frac{[q^4;q^{16}]_\infty}{[q^8;q^{16}]_\infty} \frac{f(-q^5,-q^{11})f(q^7,q^9)}{f(-q^7,-q^9)f(q^7,q^9)}. \tag{2.14}$$

By employing (2.1) with $a=-q^5$, $b=-q^{11}$, $c=q^7$ and $d=q^9$, we obtain

$$f(-q^5,-q^{11})f(q^7,q^9) = f(-q^{12},-q^{20})f(-q^{14},-q^{18}) - q^5 f(-q^4,-q^{28})f(-q^2,-q^{30}).$$

Combining above identity, (2.7) and (2.14), we obtain

$$\sum_{n=0}^{\infty} a(2n)q^n = \frac{[q^4;q^{16}]_\infty}{[q^8;q^{16}]_\infty} \left\{ \frac{f(-q^{12},-q^{20})f(-q^{14},-q^{18}) - q^5 f(-q^4,-q^{28})f(-q^2,-q^{30})}{f(-q^{14},-q^{18})f(-q^{16},-q^{16})} \right\}.$$

It follows immediately that

$$\sum_{n=0}^{\infty} a(4n)q^n = \frac{[q^2,q^6,q^6;q^{16}]_\infty}{[q^4,q^4,q^8;q^{16}]_\infty},$$

$$\sum_{n=0}^{\infty} a(4n+2)q^n = -q^2 \frac{[q,q^2,q^2,q^6;q^{16}]_\infty}{[q^4,q^4,q^7,q^8;q^{16}]_\infty}.$$

Proofs of (2.11) and (2.13) are similar. □

Theorem 2.6. *We have*

$$\sum_{n=0}^{\infty} b(4n)q^n = \frac{[q^2,q^3,q^6,q^6;q^{16}]_\infty}{[q,q^4,q^4,q^8;q^{16}]_\infty}, \tag{2.15}$$

$$\sum_{n=0}^{\infty} b(4n+1)q^n = \frac{[q^2,q^2,q^6,q^7;q^{16}]_\infty}{[q,q^4,q^4,q^8;q^{16}]_\infty}, \tag{2.16}$$

$$\sum_{n=0}^{\infty} b(4n+2)q^n = \frac{[q^2,q^2,q^5,q^6;q^{16}]_\infty}{[q,q^4,q^4,q^8;q^{16}]_\infty}, \tag{2.17}$$

$$\sum_{n=0}^{\infty} b(4n+3)q^n = \frac{[q^2,q^6,q^6;q^{16}]_\infty}{[q^4,q^4,q^8;q^{16}]_\infty}. \tag{2.18}$$

Proof. The proof of Theorem 2.6 is similar to that of Theorem 2.5. □

By Theorems 2.3, 2.4, 2.5 and 2.6, we have the following corollary:

Corollary 2.7. *For $n \geq 0$,*

$$a(4n) = b(4n+3)$$

and

$$\frac{\sum_{n=0}^{\infty} a(2n)q^n}{\sum_{n=0}^{\infty} a(2n+1)q^n} = -\frac{\sum_{n=0}^{\infty} b(2n)q^n}{\sum_{n=0}^{\infty} b(2n+1)q^n}, \qquad \frac{\sum_{n=0}^{\infty} a(4n)q^n}{\sum_{n=0}^{\infty} a(4n+1)q^n} = -\frac{\sum_{n=0}^{\infty} b(4n+1)q^n}{\sum_{n=0}^{\infty} b(4n+2)q^n}$$

and

$$\frac{\sum_{n=0}^{\infty} a(4n+2)q^n}{\sum_{n=0}^{\infty} a(4n+3)q^n} = -q\frac{\sum_{n=0}^{\infty} b(4n+3)q^n}{\sum_{n=0}^{\infty} b(4n)q^n}.$$

Remark 2.8. *Theorem 2.5 and Theorem 2.6 can also be proved by applying an identity proved by Andrews and D. Bressoud [6, Theorem 1].*

3. 8-Dissections of $A(q)$ and $B(q)$

In this section, we present 8-dissections of $A(q)$ and $B(q)$ and also present the signs of the coefficients in the power series expansion of $A(q)$ and $B(q)$ are periodic with period 16.

Theorem 3.1. *We have*

$$\sum_{n=0}^{\infty} a(8n)q^n = \frac{[q, q^3, q^3; q^8]_\infty}{[q^2, q^2, q^4; q^8]_\infty}, \tag{3.1}$$

$$\sum_{n=0}^{\infty} a(8n+1)q^n = -\frac{[q, q^3, q^3, q^5, q^5, q^7; q^{16}]_\infty}{[q^2, q^2, q^4, q^4, q^6, q^8; q^{16}]_\infty}, \tag{3.2}$$

$$\sum_{n=0}^{\infty} a(8n+2)q^n = -\, q\frac{[q,q,q^3,q^5,q^5,q^7;q^{16}]_\infty}{[q^2,q^2,q^4,q^6,q^6,q^8;q^{16}]_\infty}, \tag{3.3}$$

$$\sum_{n=0}^{\infty} a(8n+3)q^n = -\, q^2\frac{[q,q,q^3,q^3,q^5,q^7;q^{16}]_\infty}{[q^2,q^4,q^4,q^6,q^6,q^8;q^{16}]_\infty}, \tag{3.4}$$

$$\sum_{n=0}^{\infty} a(8n+4)q^n = 0, \tag{3.5}$$

$$\sum_{n=0}^{\infty} a(8n+5)q^n = q^2\frac{[q,q,q^3,q^3,q^5,q^5;q^{16}]_\infty}{[q^2,q^4,q^4,q^6,q^6,q^8;q^{16}]_\infty}, \tag{3.6}$$

$$\sum_{n=0}^{\infty} a(8n+6)q^n = q\frac{[q,q,q^3,q^3,q^5,q^7;q^{16}]_\infty}{[q^2,q^2,q^4,q^6,q^6,q^8;q^{16}]_\infty}, \tag{3.7}$$

$$\sum_{n=0}^{\infty} a(8n+7)q^n = \frac{[q,q,q^3,q^5,q^5,q^7;q^{16}]_\infty}{[q^2,q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty}. \tag{3.8}$$

Proof. Using Theorem 2.5, it is easy to prove the above eight identities as we do in obtaining the 2-dissections of $A(q)$, so we omit the details. □

Using Theorem 2.6, we establish the following eight identities.

Theorem 3.2. *We have*

$$\sum_{n=0}^{\infty} b(8n)q^n = \frac{[q^3,q^3,q^5,q^5,q^7,q^7;q^{16}]_\infty}{[q^2,q^4,q^4,q^6,q^6,q^8;q^{16}]_\infty}, \tag{3.9}$$

$$\sum_{n=0}^{\infty} b(8n+1)q^n = \frac{[q,q^3,q^5,q^5,q^7,q^7;q^{16}]_\infty}{[q^2,q^2,q^4,q^6,q^6,q^8;q^{16}]_\infty}, \tag{3.10}$$

$$\sum_{n=0}^{\infty} b(8n+2)q^n = \frac{[q,q^3,q^3,q^5,q^7,q^7;q^{16}]_\infty}{[q^2,q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty}, \tag{3.11}$$

$$\sum_{n=0}^{\infty} b(8n+3)q^n = \frac{[q,q^3,q^3,q^5,q^5,q^7;q^{16}]_\infty}{[q^2,q^2,q^4,q^4,q^6,q^6;q^{16}]_\infty}, \tag{3.12}$$

$$\sum_{n=0}^{\infty} b(8n+4)q^n = \frac{[q,q^3,q^3,q^5,q^5,q^7;q^{16}]_\infty}{[q^2,q^2,q^4,q^4,q^6,q^8;q^{16}]_\infty}, \tag{3.13}$$

$$\sum_{n=0}^{\infty} b(8n+5)q^n = \frac{[q,q^3,q^3,q^5,q^7,q^7;q^{16}]_\infty}{[q^2,q^2,q^4,q^6,q^6,q^8;q^{16}]_\infty}, \tag{3.14}$$
$$\sum_{n=0}^{\infty} b(8n+6)q^n = \frac{[q,q^3,q^5,q^5,q^7,q^7;q^{16}]_\infty}{[q^2,q^4,q^4,q^6,q^6,q^8;q^{16}]_\infty}, \tag{3.15}$$
$$\sum_{n=0}^{\infty} b(8n+7)q^n = 0. \tag{3.16}$$

Corollary 3.3. *We have*

$$\frac{\sum_{n=0}^{\infty} a(8n)q^n}{\sum_{n=0}^{\infty} a(8n+1)q^n} = -\frac{\sum_{n=0}^{\infty} b(8n+3)q^n}{\sum_{n=0}^{\infty} b(8n+4)q^n}, \qquad \frac{\sum_{n=0}^{\infty} a(8n+1)q^n}{\sum_{n=0}^{\infty} a(8n+7)q^n} = -\frac{\sum_{n=0}^{\infty} b(8n)q^n}{\sum_{n=0}^{\infty} b(8n+6)q^n},$$

$$\frac{\sum_{n=0}^{\infty} a(8n+2)q^n}{\sum_{n=0}^{\infty} a(8n+6)q^n} = -\frac{\sum_{n=0}^{\infty} b(8n+1)q^n}{\sum_{n=0}^{\infty} b(8n+5)q^n}, \qquad \frac{\sum_{n=0}^{\infty} a(8n+3)q^n}{\sum_{n=0}^{\infty} a(8n+5)q^n} = -\frac{\sum_{n=0}^{\infty} b(8n+2)q^n}{\sum_{n=0}^{\infty} b(8n+4)q^n}$$

and

$$\frac{\sum_{n=0}^{\infty} a(8n+7)q^n}{\sum_{n=0}^{\infty} a(8n+2)q^n} = -q\frac{\sum_{n=0}^{\infty} b(8n+4)q^n}{\sum_{n=0}^{\infty} b(8n+5)q^n}.$$

Proof. Proof follows from Theorem 3.1 and Theorem 3.2. □

Theorem 3.4. *We have $a(2) = a(3) = a(5) = a(6) = a(8n+4) = 0$. The remaining coefficients $a(n)$ satisfy the inequalities*

$$a(16n), a(16n+2), a(16n+5), a(16n+7), a(16n+9), a(16n+11), a(16n+14) > 0,$$
$$a(16n+1), a(16n+3), a(16n+6), a(16n+8), a(16n+10), a(16n+13), a(16n+15) < 0.$$

Proof. From (3.1), we have

$$\begin{aligned}\sum_{n=0}^{\infty} a(8n)(-q)^n &= \frac{[-q,-q^3,-q^3;q^8]_\infty}{[q^2,q^2,q^4;q^8]_\infty} \\ &= \frac{(-q;q^2)_\infty(-q^3,-q^5;q^8)_\infty}{(q^2,q^4,q^6;q^8)_\infty^2}.\end{aligned}$$

From the above equality, it follows that $a(16n) > 0$ and $a(16n+8) < 0$. Similarly, we can determine the signs of the remaining subsequences for $a(n)$. □

Theorem 3.5. *We have* $b(8) = b(8n+7) = 0$. *The remaining coefficients* $b(n)$ *satisfy the inequalities*

$$\begin{aligned}&b(16n), b(16n+1), b(16n+2), b(16n+3), b(16n+4), b(16n+5), b(16n+6) > 0,\\ &b(16n+8), b(16n+9), b(16n+10), b(16n+11), b(16n+12), b(16n+13), b(16n+14) < 0.\end{aligned}$$

Proof. From (3.9), we have

$$\sum_{n=0}^{\infty} b(8n)(-q)^n = \frac{[-q^3,-q^3,-q^5,-q^5,-q^7,-q^7;q^{16}]_\infty}{[q^2,q^4,q^4,q^6,q^6,q^8;q^{16}]_\infty}$$

From the above equality, it follows that $b(16n) > 0$ and $b(16n+8) < 0$. Similarly, we can determine the signs of the remaining subsequences for $b(n)$. □

4. 16-Dissections of $A(q)$ and $B(q)$

In this section, we present 16-dissections of $A(q)$ and $B(q)$. In these dissections components are not single products. One can establish the following two theorems on using 8-dissections of $A(q)$ and $B(q)$. The proof is similar to that of 8-dissections of $A(q)$ and $B(q)$, so we omit the details.

Theorem 4.1. *We have*

$$
\begin{aligned}
\sum_{n=0}^{\infty} a(16n)q^n =& \frac{X}{f(-q,-q)f(-q^2,-q^2)},\\
\sum_{n=0}^{\infty} a(16n+1)q^n =& \frac{-RX}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)},\\
\sum_{n=0}^{\infty} a(16n+2)q^n =& \frac{qQ\{f(-q^6,-q^{10})f(-q^7,-q^9)M+q^2f(-q^2,-q^{14})f(-q,-q^{15})N\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},\\
\sum_{n=0}^{\infty} a(16n+3)q^n =& \frac{-qP\{f(-q^5,-q^{11})f(-q^6,-q^{10})N+q^2f(-q^3,-q^{13})f(-q^2,-q^{14})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},\\
\sum_{n=0}^{\infty} a(16n+5)q^n =& \frac{qPf(-q^4,-q^{12})\{f(-q^5,-q^{11})X+qf(-q^3,-q^{13})Y\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},\\
\sum_{n=0}^{\infty} a(16n+6)q^n =& \frac{-qQ\{f(-q^6,-q^{10})f(-q^5,-q^{11})M+qf(-q^2,-q^{14})f(-q^3,-q^{13})N\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},\\
\sum_{n=0}^{\infty} a(16n+7)q^n =& \frac{R\{f(-q^6,-q^{10})f(-q^7,-q^9)N+q^3f(-q^2,-q^{14})f(-q,-q^{15})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},\\
\sum_{n=0}^{\infty} a(16n+8)q^n =& \frac{-Y}{f(-q,-q)f(-q^2,-q^2)},\\
\sum_{n=0}^{\infty} a(16n+9)q^n =& \frac{RY}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)},\\
\sum_{n=0}^{\infty} a(16n+10)q^n =& \frac{-Q\{f(-q^6,-q^{10})f(-q^7,-q^9)N+q^3f(-q^2,-q^{14})f(-q,-q^{15})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},\\
\sum_{n=0}^{\infty} a(16n+11)q^n =& \frac{qP\{f(-q^5,-q^{11})f(-q^6,-q^{10})M+qf(-q^3,-q^{13})f(-q^2,-q^{14})N\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},\\
\sum_{n=0}^{\infty} a(16n+13)q^n =& \frac{-qPf(-q^4,-q^{12})\{f(-q^3,-q^{13})X+f(-q^5,-q^{11})Y\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},\\
\sum_{n=0}^{\infty} a(16n+14)q^n =& \frac{Q\{f(-q^6,-q^{10})f(-q^5,-q^{11})N+q^2f(-q^2,-q^{14})f(-q^3,-q^{13})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},\\
\sum_{n=0}^{\infty} a(16n+15)q^n =& \frac{-R\{f(-q^6,-q^{10})f(-q^7,-q^9)M+q^2f(-q^2,-q^{14})f(-q,-q^{15})N\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q^7,-q^9)},
\end{aligned}
$$

where,

$$
\begin{aligned}
P :=& f(-q,-q^7), \quad Q := f(-q^2,-q^6), \quad R := f(-q^3,-q^5),\\
X :=& f(q^3,q^5)f(q^7,q^9) + q^2 f(q,q^7)f(q,q^{15}),\\
Y :=& f(q,q^7)f(q^7,q^9) + q f(q^3,q^5)f(q,q^{15}),\\
M :=& f(q^3,q^5)f(q^3,q^{13}) + f(q,q^7)f(q^5,q^{11}),\\
N :=& f(q^3,q^5)f(q^5,q^{11}) + q f(q,q^7)f(q^3,q^{13}).
\end{aligned}
$$

Theorem 4.2. *We have*

$$
\begin{aligned}
\sum_{n=0}^{\infty} b(16n)q^n =& \frac{Pf(-q^4,-q^{12})\{f(-q^{11},-q^5)X - qf(-q^3,-q^{13})Y\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},\\
\sum_{n=0}^{\infty} b(16n+1)q^n =& \frac{Q\{f(-q^6,-q^{10})f(-q^3,-q^{13})N - qf(-q^2,-q^{14})f(-q^5,-q^{11})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},\\
\sum_{n=0}^{\infty} b(16n+2)q^n =& \frac{R\{f(-q^2,-q^{14})f(-q^7,-q^9)N - qf(-q^6,-q^{10})f(-q,-q^{15})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},\\
\sum_{n=0}^{\infty} b(16n+3)q^n =& \frac{X}{f(-q,-q)f(-q^2,-q^2)},\\
\sum_{n=0}^{\infty} b(16n+4)q^n =& \frac{RX}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)},\\
\sum_{n=0}^{\infty} b(16n+5)q^n =& \frac{Q\{f(-q^2,-q^{14})f(-q^7,-q^9)N - qf(-q^6,-q^{10})f(-q,-q^{15})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},\\
\sum_{n=0}^{\infty} b(16n+6)q^n =& \frac{P\{f(-q^6,-q^{10})f(-q^3,-q^{13})N - qf(-q^2,-q^{14})f(-q^5,-q^{11})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},\\
\sum_{n=0}^{\infty} b(16n+8)q^n =& \frac{Pf(-q^4,-q^{12})\{f(-q^3,-q^{13})X - f(-q^5,-q^{11})Y\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},
\end{aligned}
$$

$$\sum_{n=0}^{\infty} b(16n+9)q^n = \frac{Q\{f(-q^2,-q^{14})f(-q^5,-q^{11})N - f(-q^6,-q^{10})f(-q^3,-q^{13})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},$$

$$\sum_{n=0}^{\infty} b(16n+10)q^n = \frac{R\{f(-q^6,-q^{10})f(-q,-q^{15})N - f(-q^2,-q^{14})f(-q^7,-q^9)M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},$$

$$\sum_{n=0}^{\infty} b(16n+11)q^n = \frac{-Y}{f(-q,-q)f(-q^2,-q^2)},$$

$$\sum_{n=0}^{\infty} b(16n+12)q^n = \frac{-RY}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)},$$

$$\sum_{n=0}^{\infty} b(16n+13)q^n = \frac{Q\{f(-q^6,-q^{10})f(-q,-q^{15})N - f(-q^2,-q^{14})f(-q^7,-q^9)M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},$$

$$\sum_{n=0}^{\infty} b(16n+14)q^n = \frac{P\{f(-q^2,-q^{14})f(-q^5,-q^{11})N - f(-q^6,-q^{10})f(-q^3,-q^{13})M\}}{f(-q,-q)f(-q^2,-q^2)f(-q^4,-q^4)f(-q^8,-q^8)f(-q,-q^{15})},$$

where P, Q, R, X, Y, M, N are as defined in Theorem 4.1.

5. Combinatorial Interpretations of $a(n)$ and $b(n)$

In this section, we present combinatorial interpretations of $a(n)$ and $b(n)$. For simplicity, we define

$$(q^{r\pm}; q^s)_\infty := (q^r, q^{s-r}; q^s)_\infty,$$

where r and s are positive integers and $r < s$.

Definition 5.1. *A positive integer n has k colors if there are k copies of n available and all of them are viewed as distinct objects. Partitions of positive integer into parts with colors are called "colored partitions".*

For example, if 2 is allowed to have two colors, say r (red) and g (green) and odd parts being distinct, then all colored partitions of 3 are $3, 2_r + 1, 2_g + 1$. An important fact is that

$$\frac{1}{(q^u; q^v)_\infty^k},$$

is the generating function for the number of partitions of n, where all the parts are congruent to $u \pmod{v}$ and have k colors.

Theorem 5.2. *Let $P_0(n)$ denote the number of partitions of n with parts not congruent to $8 \pmod{16}$ and all having two colors except for parts congruent to $\pm 1, \pm 7 \pmod{16}$ and odd parts being distinct.*
Let $P_1(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 1, \pm 6, \pm 7 \pmod{16}$ and odd parts being distinct.
Let $P_2(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 3, \pm 4, \pm 7 \pmod{16}$ and odd parts being distinct.
Let $P_3(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 2, \pm 5, \pm 7 \pmod{16}$ and odd parts being distinct.
Let $P_4(n)$ denote the number of partitions of n with parts not congruent to $\pm 7 \pmod{16}$ and all having two colors except for parts congruent to $\pm 2 \pmod{16}$ and odd parts being distinct.
Let $P_5(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 4, \pm 5, \pm 7 \pmod{16}$ and odd parts being distinct.
Let $P_6(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 3, \pm 6, \pm 7 \pmod{16}$ and odd parts being distinct.
Then, we have

$$(-1)^n a(8n) = P_0(n), \quad (-1)^n a(8n+1) = -P_1(n), \quad (-1)^n a(8n+2) = P_2(n-1),$$
$$(-1)^n a(8n+3) = -P_3(n-2), \quad (-1)^n a(8n+5) = P_4(n-2),$$
$$(-1)^n a(8n+6) = -P_5(n-1), \quad (-1)^n a(8n+7) = P_6(n).$$

Proof. Replacing q to $-q$ in (3.1), we obtain

$$\sum_{n=0}^{\infty} (-1)^n a(8n) q^n = \frac{(-q^{\pm 1}, -q^{\pm 3}, -q^{\pm 3}, -q^{\pm 5}, -q^{\pm 5}, -q^{\pm 7}; q^{16})_\infty}{(q^{\pm 2}, q^{\pm 2}, q^{\pm 4}, q^{\pm 4}, q^{\pm 6}, q^{\pm 6}; q^{16})_\infty}.$$

Observe that the product on the right is the generating function for $P_0(n)$ and so
$(-1)^n a(8n) = P_0(n)$. Similar arguments can be used to derive the remaining equations. □

Example 5.3. *By Maple, we have been able to find the following series expansion for $A(q)$:*

$$A(q) = 1 - q + q^7 - q^8 + q^9 - q^{10} + q^{14} - 2q^{15} + 2q^{16} - 2q^{17} + 2q^{18} - q^{19} + q^{21} - 2q^{22} + 2q^{23} - 4q^{24} + 4q^{25} - 2q^{26} + 2q^{27} - 2q^{29} + 2q^{30} - 5q^{31} + 7q^{32} - 7q^{33} + 5q^{34} - \cdots .$$

The following table verifies the case $n = 3$ in Theorem 5.2.

$P_0(3) = 4 = -a(24)$	$3_r,\ 3_g,\ 2_r + 1,\ 2_g + 1$
$P_1(3) = -4 = -a(25)$	$3_r,\ 3_g,\ 2_r + 1,\ 2_g + 1$
$P_2(2) = 2 = -a(26)$	$2_r,\ 2_g$
$P_3(1) = -2 = -a(27)$	$1_r,\ 1_g$
$P_4(1) = 2 = -a(29)$	$1_r,\ 1_g$
$P_5(2) = -2 = -a(30)$	$2_r,\ 2_g$
$P_6(3) = 5 = -a(31)$	$3,\ 2_r + 1_r,\ 2_r + 1_g,$ $2_g + 1_r,\ 2_g + 1_g$

Theorem 5.4. *Let $P_0(n)$ denote the number of partitions of n with parts not congruent to $\pm 1 \pmod{16}$ and all having two colors except for parts congruent to $\pm 2 \pmod{16}$ and odd parts being distinct.*
Let $P_1(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 1, \pm 3, \pm 4 \pmod{16}$ and odd parts being distinct.
Let $P_2(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 1, \pm 5, \pm 6 \pmod{16}$ and odd parts being distinct.
Let $P_3(n)$ denote the number of partitions of n with parts not congruent to $8 \pmod{16}$ and all having two colors except for parts congruent to $\pm 1, \pm 7 \pmod{16}$ and odd parts being distinct.
Let $P_4(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 1, \pm 6, \pm 7 \pmod{16}$ and odd parts being distinct.
Let $P_5(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 1, \pm 4, \pm 5 \pmod{16}$ and odd parts being distinct.
Let $P_6(n)$ denote the number of partitions of n with parts congruent to all parts $\pmod{16}$ and all having two colors except for parts congruent to $\pm 1, \pm 2, \pm 3$

(mod 16) *and odd parts being distinct.*
Then, we have

$$(-1)^n b(8n+k) = P_k(n) \quad for \quad 0 \leq k \leq 6.$$

Proof. Replacing q to $-q$ in (3.9), we obtain

$$\sum_{n=0}^{\infty}(-1)^n b(8n)q^n = \frac{(-q^{\pm 3}, -q^{\pm 3}, -q^{\pm 5}, -q^{\pm 5}, -q^{\pm 7}, -q^{\pm 7}; q^{16})_\infty}{(q^{\pm 2}, q^{\pm 4}, q^{\pm 4}, q^{\pm 6}, q^{\pm 6}, q^8, q^8; q^{16})_\infty}.$$

Observe that the product on the right is the generating function for $P_0(n)$ and so
$(-1)^n b(8n) = P_0(n)$. Similar arguments can be used to derive the remaining equations. □

Example 5.5. *By Maple, we have been able to find the following series expansion for $B(q)$:*

$$B(q) = 1 + q + q^2 + q^3 + q^4 + q^5 + q^6 - q^9 - q^{10} - q^{11} - q^{12} - q^{13} - q^{14} + q^{16} + 2q^{17} + 2q^{18} + 2q^{19} + 2q^{20} + 2q^{21} + q^{22} - 2q^{24} - 3q^{25} - 4q^{26} - 4q^{27} - 4q^{28} - 4q^{29} - 2q^{30} + \cdots.$$

The following table verifies the case $n = 3$ in Theorem 5.4.

$P_0(3) = 2 = -b(24)$	$3_r,\ 3_g$
$P_1(3) = 3 = -b(25)$	$3,\ 2_r+1,\ 2_g+1$
$P_2(3) = 4 = -b(26)$	$3_r,\ 3_g,\ 2_r+1,\ 2_g+1$
$P_3(3) = 4 = -b(27)$	$3_r,\ 3_g,\ 2_r+1,\ 2_g+1$
$P_4(3) = 4 = -b(28)$	$3_r,\ 3_g,\ 2_r+1,\ 2_g+1$
$P_5(3) = 4 = -b(29)$	$3_r,\ 3_g,\ 2_r+1,\ 2_g+1$
$P_6(3) = 2 = -b(30)$	$3,\ 2+1$

Remark 5.6. *Theorem 3.4 and Theorem 3.5 also follows from Theorem 5.2 and Theorem 5.4 respectively.*

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