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Infinite Families of Congruences for 2-Color overpartitions *

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Abstract

Let $\overline{p}_3(n)$ denote the number of overpartitions of n with 2-color in which one of the colors appears only in parts that are multiples of 3. In this work, we establish several infinite families of congruences modulo powers of 2 and 3 for $\overline{p}_3(n)$. We also show that for each $n \geq 0$ and $\alpha \geq 0$,

$$\overline{p}_3(6 \cdot 5^{2\alpha+4}n + (30i+25)5^{2\alpha+2}) \equiv 0 \pmod{18},$$

where $i = 1, 2, 3, 4$.

Keywords: *Color partition, Overpartition.*

1. Introduction

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. The number of partitions of n is denoted by $p(n)$ and we set $p(0) = 1$.

Corteel and Lovejoy [6] have introduced the combinatorial object known as overpartition of a nonnegative integer n, which is a non-increasing sequence of a natural number, whose sum is n and the first occurrence of parts of each size may be over lined. For example, the eight overpartitions of 3 are

$$3, \quad \overline{3}, \quad 2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1}, \quad 1+1+1, \quad \overline{1}+1+1.$$

We denote the number of overpartitions of n by $\overline{p}(n)$ and set $\overline{p}(0) = 1$. As noted in [6], the generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \prod_{n\geq 1} \frac{1+q^n}{1-q^n} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2}, \tag{1.1}$$

where $(q;q)_\infty = (1-q)(1-q^2)(1-q^3)\cdots$.

We define

$$\sum_{n=0}^{\infty} p_3(n)q^n = \frac{1}{(q;q)_\infty (q^3;q^3)_\infty}, \tag{1.2}$$

where $p_3(n)$ is the number of 2-color partitions of n, in which one of the colors appears only in parts that are multiples of 3. We also set $p_3(0) = 1$.

For example, there are four partitions of 2-color partitions of 3:

$$3_a, \quad 3_b, \quad 2_a + 1_a, \quad 1_a + 1_a + 1_a.$$

In this paper, we define

$$\sum_{n=0}^{\infty} \overline{p}_3(n) q^n = \frac{(-q;q)_\infty (-q^3;q^3)_\infty}{(q;q)_\infty (q^3;q^3)_\infty}. \tag{1.3}$$

Let $\overline{p}_3(n)$ denote the number of overpartitions of n with 2-color in which one of the colors appears only in parts that are multiples of 3. For example, there are ten partitions of 2-color overpartitions of 3:

3_a, $\overline{3}_a$, 3_b, $\overline{3}_b$, $2_a + 1_a$, $\overline{2}_a + 1_a$, $2_a + \overline{1}_a$, $\overline{2}_a + \overline{1}_a$, $1_a + 1_a + 1_a$, $\overline{1}_a + 1_a + 1_a$.

For any positive integer k, f_k is defined by

$$f_k := \prod_{i=1}^{\infty} (1 - q^{ki}). \tag{1.4}$$

The important special cases of Ramanujan's general theta function $f(a,b)$ are

$$\varphi(q) := f(q,q) = 1 + 2\sum_{n=1}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty (-q^2;q^2)_\infty} = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.5}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1}, \tag{1.6}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1, \tag{1.7}$$

where

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2} \tag{1.8}$$

$$= (-a;ab)_\infty (-b;ab)_\infty (ab;ab)_\infty, \quad |ab| < 1. \tag{1.9}$$

2. Preliminary results

In this section, we collect some results which are useful in proving our main results.

Lemma 2.1. *[7, p. 212] We have the following 5-dissection*

$$f_1 = f_{25}\left(a - q - q^2/a\right), \tag{2.1}$$

where

$$a := a(q) := \frac{(q^{10}, q^{15}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty}. \tag{2.2}$$

Lemma 2.2. *The following 2-dissections hold:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{2.3}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{2.4}$$

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \tag{2.5}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}. \tag{2.6}$$

Hirschhorn, Garvan and Borwein [2] proved equation (2.3). For proof of (2.4), see [4]. Replacing q by $-q$ in (2.3) and (2.4) , we obtain (2.5) and (2.6) respectively.

Lemma 2.3. *The following 2-dissections hold:*

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{2.7}$$

$$\frac{1}{f_1^2 f_3^2} = \frac{f_8^5 f_{24}^5}{f_2^5 f_6^5 f_{16}^2 f_{48}^2} + 2q\frac{f_4^4 f_{12}^4}{f_2^6 f_6^6} + 4q^4\frac{f_4^2 f_{12}^2 f_{16}^4 f_{48}^2}{f_2^5 f_6^5 f_8 f_{24}}. \tag{2.8}$$

Equations (2.7) and (2.8) were proved by Baruah and Ojah [5].

Lemma 2.4. *[1, p. 40, Entry 25]. We have the following 2-dissection holds:*

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}. \tag{2.9}$$

Lemma 2.5. *[1, p. 345, Entry 1 (iv)]. We have the following 3-dissection*

$$f_1^3 = \frac{f_6 f_9^6}{f_3 f_{18}^3} + 4q^3\frac{f_3^2 f_{18}^6}{f_6^2 f_9^3} - 3q f_9^3. \tag{2.10}$$

Lemma 2.6. *[1, p. 49] We have*

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}), \tag{2.11}$$

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{2.12}$$

Lemma 2.7. *The following 3-dissection holds:*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q\frac{f_6^3 f_9^3}{f_3^7} + 4q^2\frac{f_6^2 f_{18}^3}{f_3^6}. \tag{2.13}$$

Equation (2.13) was proved by Hirschhorn and Sellers [3].

3. Main Results

In this section, we establish several infinite families of congruences modulo powers of 2 and 3.

By binomial theorem, it is easy to see that

$$f_{2m} \equiv f_m^2 \pmod 2, \tag{3.1}$$
$$f_{2m}^2 \equiv f_m^4 \pmod{2^2}, \tag{3.2}$$
$$f_{2m}^4 \equiv f_m^8 \pmod{2^3}, \tag{3.3}$$
$$f_{3m} \equiv f_m^3 \pmod 3, \tag{3.4}$$
$$f_{3m}^3 \equiv f_m^9 \pmod{3^2}, \tag{3.5}$$
$$f_{3m}^9 \equiv f_m^{27} \pmod{3^3}. \tag{3.6}$$

Theorem 3.1. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_3(12\cdot 2^{\alpha}n) \equiv \overline{p}_3(6n) \pmod 9, \tag{3.7}$$

$$\overline{p}_3(4\cdot 3^{\alpha+3}n + 2\cdot 3^{\alpha+3}) \equiv \overline{p}_3(36n+18) \pmod 9, \tag{3.8}$$

$$\overline{p}_3(12n+6) \equiv 6\cdot \overline{p}_3(6n+3) \pmod 9, \tag{3.9}$$

$$\overline{p}_3(108n+18) \equiv \overline{p}_3(36n+6) \pmod 9, \tag{3.10}$$

$$\overline{p}_3(36n+30) \equiv 0 \pmod 9, \tag{3.11}$$

$$\overline{p}_3(6n+4) \equiv 0 \pmod{18}. \tag{3.12}$$

Proof. We have

$$\sum_{n=0}^{\infty} \overline{p}_3(n)q^n = \frac{f_2 f_6}{f_1^2 f_3^2}. \tag{3.13}$$

Substituting (2.13) in (3.13), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(n)q^n = \frac{f_6^5 f_9^6}{f_3^{10} f_{18}^3} + 2q\frac{f_6^4 f_9^3}{f_3^9} + 4q^2\frac{f_6^3 f_{18}^3}{f_3^8}. \tag{3.14}$$

Extracting the terms involving q^{3n+1}, dividing by q and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+1)q^n = 2\frac{f_2^4 f_3^3}{f_1^9}. \tag{3.15}$$

Invoking (3.5) into (3.15), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+1)q^n \equiv 2f_2^4 \pmod{18}. \tag{3.16}$$

Extracting the terms involving q^{2n+1} from (3.16), we obtain (3.12).

From (3.14), we have

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n = \frac{f_2^5 f_3^6}{f_1^{10} f_6^3}. \tag{3.17}$$

Invoking (3.5) into (3.17), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n \equiv \frac{f_3^3}{f_1 f_2^4} \pmod 9. \tag{3.18}$$

Employing (2.3) into (3.18), we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_3(3n)q^n \equiv \frac{f_4^3 f_6^2}{f_2^6 f_{12}} + q\frac{f_{12}^3}{f_2^4 f_4} \pmod 9. \tag{3.19}$$

Extracting the terms involving q^{2n} from (3.19) and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty}\overline{p}_3(6n)q^n \equiv \frac{f_2^3 f_3^2}{f_1^6 f_6} \pmod 9. \tag{3.20}$$

Invoking (3.5) into (3.20), we find that

$$\sum_{n=0}^{\infty}\overline{p}_3(6n)q^n \equiv \frac{f_1^3 f_2^3}{f_3 f_6} \pmod 9. \tag{3.21}$$

Employing (2.6) into (3.21), we have

$$\sum_{n=0}^{\infty}\overline{p}_3(6n)q^n \equiv \frac{f_2^3 f_4^3}{f_6 f_{12}} + 6q\frac{f_2^5 f_{12}^3}{f_4 f_6^3} \pmod 9. \tag{3.22}$$

Extracting the terms involving q^{2n+1} from (3.22), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(12n+6)q^n \equiv 6\frac{f_1^5 f_6^3}{f_2 f_3^3} \pmod 9. \tag{3.23}$$

Invoking (3.4) into (3.23), we get

$$\sum_{n=0}^{\infty}\overline{p}_3(12n+6)q^n \equiv 6\frac{f_1^2 f_6^3}{f_2 f_3^2} \pmod 9. \tag{3.24}$$

Replacing q by $-q$ in (2.11) and using the fact that

$$\phi(-q) = \frac{f_1^2}{f_2}, \tag{3.25}$$

we find that

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}. \tag{3.26}$$

Again employing (3.26) into (3.24), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(12n+6)q^n \equiv 6\frac{f_6^3f_9^2}{f_3^2f_{18}} - 12q\frac{f_6^2f_{18}^2}{f_3f_9} \pmod 9. \tag{3.27}$$

Congruence (3.11) follows by extracting the terms involving q^{3n+2} on both sides of (3.27).

Extracting the terms involving q^{3n+1} from (3.27) and dividing by q, then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty}\overline{p}_3(36n+18)q^n \equiv 6\frac{f_2^2f_6^2}{f_1f_3} \pmod 9. \tag{3.28}$$

It follows from (2.12) that

$$\frac{f_2^2}{f_1} = \frac{f_6f_9^2}{f_3f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{3.29}$$

Employing (3.29) into (3.28), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(36n+18)q^n \equiv 6\frac{f_6^3f_9^2}{f_3^2f_{18}} + 6q\frac{f_6^2f_{18}^2}{f_3f_9} \pmod 9. \tag{3.30}$$

Extracting the terms involving q^{3n+1} from (3.30) and dividing by q, then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty}\overline{p}_3(108n+54)q^n \equiv 6\frac{f_2^2f_6^2}{f_1f_3} \pmod 9. \tag{3.31}$$

In view of congruences (3.28) and (3.31), we have

$$\overline{p}_3(108n+54) \equiv \overline{p}_3(36n+18) \pmod 9. \tag{3.32}$$

Utilizing (3.32) and by mathematical induction on α, we get (3.8).

Extracting the terms involving q^{3n} from (3.27) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty}\overline{p}_3(36n+6)q^n \equiv 6\frac{f_2^3f_3^2}{f_1^2f_6} \pmod 9. \tag{3.33}$$

Extracting the terms involving q^{3n} from (3.30) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{p}_3(108n+18)q^n \equiv 6\frac{f_2^3 f_3^2}{f_1^2 f_6} \pmod{9}. \tag{3.34}$$

In view of congruences (3.33) and (3.34), we arrive at (3.10).

Extracting the terms involving q^{2n} from both sides of (3.22) and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(12n)q^n \equiv \frac{f_1^3 f_2^3}{f_3 f_6} \pmod{9}. \tag{3.35}$$

In view of congruences (3.21) and (3.35), we have

$$\overline{p}_3(12n) \equiv \overline{p}_3(6n) \pmod{9}. \tag{3.36}$$

Utilizing (3.36) and by mathematical induction on α, we arrive at (3.7).

Extracting the terms involving q^{2n+1} from (3.19) and dividing by q, then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+3)q^n \equiv \frac{f_6^3}{f_1^4 f_2} \pmod{9}. \tag{3.37}$$

Extracting the terms involving q^{2n+1} from (3.22) and dividing by q, then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+6)q^n \equiv 6\frac{f_1^5 f_6^3}{f_2 f_3^3} \pmod{9}. \tag{3.38}$$

Invoking (3.5) into (3.38), we have

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+6)q^n \equiv 6\frac{f_6^3}{f_1^4 f_2} \pmod{9}. \tag{3.39}$$

In view of congruences (3.37) and (3.39), we arrive at (3.9). □

Theorem 3.2. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_3(6 \cdot 5^{2\alpha+4}n + (30i + 25)5^{2\alpha+2}) \equiv 0 \pmod{18}, \tag{3.40}$$

where $i = 1, 2, 3, 4$.

Proof. Extracting the terms involving q^{2n} from (3.16) and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{p}_3(6n + 1)q^n \equiv 2f_1^4 \pmod{18}. \tag{3.41}$$

Employing (2.1) into (3.41) and extracting the terms involving q^{5n+4}, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(30n + 25)q^n \equiv 8f_5^4 \pmod{18}. \tag{3.42}$$

Extracting the terms involving q^{5n} from (3.42) and replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} \overline{p}_3(150n + 25)q^n \equiv 8f_1^4 \pmod{18}. \tag{3.43}$$

From (3.41) and (3.43), we find that

$$\overline{p}_3(150n + 25) \equiv 4\overline{p}_3(6n + 1) \pmod{18}. \tag{3.44}$$

Utilizing (3.44) and by mathematical induction on α, we obtain

$$\overline{p}_3(6 \cdot 5^{2\alpha+2}n + 5^{2\alpha+2}) \equiv 4^{\alpha+1}\overline{p}_3(6n + 1) \pmod{18}. \tag{3.45}$$

From (3.42), we get

$$\overline{p}_3(150n + 30i + 25) \equiv 0 \pmod{18}, \quad i = 1, 2, 3, 4. \tag{3.46}$$

Using (3.45) and (3.46), we obtain (3.40). □

Theorem 3.3. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_3(12n + 10) \equiv 0 \pmod{27}, \tag{3.47}$$

$$\overline{p}_3(3 \cdot 4^{\alpha+2}n + 10 \cdot 4^{\alpha+1}) \equiv 0 \pmod{27}. \tag{3.48}$$

Proof. From (3.15), we have

$$\sum_{n=0}^{\infty}\overline{p}_3(3n+1)q^n = 2f_2^4\left(\frac{f_3}{f_1^3}\right)^3. \tag{3.49}$$

Employing (2.4) into (3.49) and invoking (3.6), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(3n+1)q^n \equiv 2\frac{f_4^{18}f_6^9}{f_2^{23}f_{12}^6} + 18q\frac{f_4^{14}f_6^7}{f_2^{21}f_{12}^2} \pmod{27}. \tag{3.50}$$

Extracting the terms involving q^{2n+1} from (3.50), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(6n+4)q^n \equiv 18\frac{f_2^{14}f_3^7}{f_1^{21}f_6^2} \pmod{27}. \tag{3.51}$$

Invoking (3.4) into (3.51), we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_3(6n+4)q^n \equiv 18f_2^5f_6 \pmod{27}. \tag{3.52}$$

Congruence (3.47) follows by extracting the terms involving q^{2n+1} from both sides of (3.52).

Extracting the terms involving q^{2n} from (3.52) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(12n+4)q^n \equiv 18f_1^5f_3 \pmod{27}. \tag{3.53}$$

Using (3.4) into (3.53), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(12n+4)q^n \equiv 18f_1^8 \pmod{27}. \tag{3.54}$$

Invoking (2.9) into (3.54), we find that

$$\sum_{n=0}^{\infty}\overline{p}_3(12n+4)q^n \equiv 18\frac{f_4^{20}}{f_2^4f_8^8} + 18q^2\frac{f_2^4f_8^8}{f_4^4} + 18qf_4^8 \pmod{27}. \tag{3.55}$$

Extracting the terms involving q^{2n+1} from (3.55), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(24n+16)q^n \equiv 18f_2^8 \pmod{27}. \tag{3.56}$$

We can rewrite the above equation as

$$\sum_{n=0}^{\infty} \overline{p}_3(24n+16)q^n \equiv 18f_2^5 f_6 \pmod{27}. \tag{3.57}$$

In view of congruences (3.52) and (3.57), we have

$$\overline{p}_3(6n+4) \equiv \overline{p}_3(24n+16) \pmod{27}. \tag{3.58}$$

Utilizing (3.58) and by mathematical induction on α, we arrive at

$$\overline{p}_3(6n+4) \equiv \overline{p}_3(6\cdot 4^{\alpha+1}n+4^{\alpha+2}) \pmod{27}. \tag{3.59}$$

Using (3.59) and (3.47), we get (3.48). □

Theorem 3.4. *For each $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_3(3^\alpha n) \equiv \overline{p}_3(n) \pmod{8}, \tag{3.60}$$

$$\overline{p}_3(18n+6) \equiv 2\overline{p}_3(9n+3) \pmod{8}, \tag{3.61}$$

$$\overline{p}_3(3\cdot 4^{\alpha+1}n+10\cdot 4^\alpha) \equiv 0 \pmod{16}, \tag{3.62}$$

$$\overline{p}_3(6\cdot 4^{\alpha+1}n+5\cdot 4^{\alpha+1}) \equiv 0 \pmod{32}, \tag{3.63}$$

$$\overline{p}_3(6n+5) \equiv 0 \pmod{32}, \tag{3.64}$$

$$\overline{p}_3(18n+15) \equiv 0 \pmod{8}. \tag{3.65}$$

Proof. Invoking (3.3) into (3.17), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n \equiv \frac{f_2 f_6}{f_1^2 f_3^2} \pmod{8}. \tag{3.66}$$

In view of congruences (3.13) and (3.66), we have

$$\overline{p}_3(3n) \equiv \overline{p}_3(n) \pmod{8}. \tag{3.67}$$

Utilizing (3.67) and by mathematical induction on α, we arrive at (3.60).

Invoking (3.3) into (3.15), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+1)q^n \equiv 2\frac{f_3^3}{f_1} \pmod{16}. \tag{3.68}$$

Using (2.3) in (3.68) and extracting the terms involving q^{2n+1}, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+4)q^n \equiv 2\frac{f_6^3}{f_2} \pmod{16}, \tag{3.69}$$

which implies

$$\overline{p}_3(12n+10) \equiv 0 \pmod{16} \tag{3.70}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+4)q^n \equiv 2\frac{f_3^3}{f_1} \pmod{16}. \tag{3.71}$$

Using (3.68) and (3.71), we find that

$$\overline{p}_3(12n+4) \equiv \overline{p}_3(3n+1) \pmod{16}. \tag{3.72}$$

By mathematical induction on α, we get

$$\overline{p}_3(3 \cdot 4^{\alpha+1}n + 4^{\alpha+1}) \equiv \overline{p}_3(3n+1) \pmod{16}. \tag{3.73}$$

Congruence (3.62) follows from (3.70) and (3.73).

Equating the terms containing q^{3n+2} from both sides of (3.14), dividing by q^2 and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+2)q^n = 4\frac{f_2^3 f_6^3}{f_1^8}. \tag{3.74}$$

Invoking (3.3) into (3.74), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+2)q^n \equiv 4\frac{f_6^3}{f_2} \pmod{32}, \tag{3.75}$$

which implies

$$\overline{p}_3(6n+5) \equiv 0 \pmod{32} \tag{3.76}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+2)q^n \equiv 4\frac{f_3^3}{f_1} \pmod{32}. \tag{3.77}$$

Employing (2.3) into (3.77), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+2)q^n \equiv 4\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + 4q\frac{f_{12}^3}{f_4} \pmod{32}. \tag{3.78}$$

Extracting the terms involving q^{2n+1} from (3.78), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+8)q^n \equiv 4\frac{f_6^3}{f_2} \pmod{32}, \tag{3.79}$$

which implies

$$\overline{p}_3(24n+20) \equiv 0 \pmod{32} \tag{3.80}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_3(24n+8)q^n \equiv 4\frac{f_3^3}{f_1} \pmod{32}. \tag{3.81}$$

In view of congruences (3.77) and (3.81), and by mathematical induction on α, we find that

$$\overline{p}_3(6\cdot 4^{\alpha+1}n + 2\cdot 4^{\alpha+1}) \equiv \overline{p}_3(6n+2) \pmod{32}. \tag{3.82}$$

Congruence (3.63) follows from (3.80) and (3.82).

Invoking (3.3) into (3.17), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n \equiv \frac{f_2 f_3^6}{f_1^2 f_6^3} \pmod{8}. \tag{3.83}$$

Employing (2.13) into (3.83), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n \equiv \frac{f_6 f_9^6}{f_3^2 f_{18}^3} + 2q\frac{f_9^3}{f_3} + 4q^2\frac{f_{18}^3}{f_6} \pmod{8}. \tag{3.84}$$

Extracting the terms involving q^{3n+2} from (3.84), dividing by q^2 and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(9n+6)q^n \equiv 4\frac{f_6^3}{f_2} \pmod{8},$$

which implies

$$\overline{p}_3(18n+15) \equiv 0 \pmod{8} \tag{3.85}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_3(18n+6)q^n \equiv 4\frac{f_3^3}{f_1} \pmod{8}. \tag{3.86}$$

Again extracting the terms involving q^{3n+1} from (3.84), dividing by q and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(9n+3)q^n \equiv 2\frac{f_3^3}{f_1} \pmod{8}. \tag{3.87}$$

Congruence (3.61) follows from (3.86) and (3.87). □

Theorem 3.5. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_3(4^{\alpha}n) \equiv \overline{p}_3(n) \pmod{4}, \tag{3.88}$$

$$\overline{p}_3(6(4n+i)+1) \equiv 0 \pmod{4}, \tag{3.89}$$

where $i = 1, 2, 3$.

$$\overline{p}_3(24 \cdot 25^{\alpha+2}n + (120j+25) \cdot 25^{\alpha+1}) \equiv 0 \pmod{4}, \tag{3.90}$$

where $j = 1, 2, 3, 4$.

$$\overline{p}_3(2^{2\alpha+2}n + 2^{2\alpha+1}) \equiv 0 \pmod{4}, \tag{3.91}$$

$$\overline{p}_3(2 \cdot 3^{\alpha+2}n + 5 \cdot 3^{\alpha+1}) \equiv 0 \pmod{16}, \tag{3.92}$$

$$\overline{p}_3(6n+5) \equiv 0 \pmod{16}. \tag{3.93}$$

Proof. Employing (2.8) into (3.13), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(n)q^n = \frac{f_8^5 f_{24}^5}{f_2^4 f_6^4 f_{16}^2 f_{48}^2} + 2q\frac{f_4^4 f_{12}^4}{f_2^5 f_6^5} + 4q^4\frac{f_4^2 f_{12}^2 f_{16}^4 f_{48}^2}{f_2^4 f_6^4 f_8 f_{24}}. \tag{3.94}$$

Extracting the terms involving q^{2n+1} from (3.94), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(2n+1)q^n = 2\frac{f_2^4 f_6^4}{f_1^5 f_3^5}. \tag{3.95}$$

Invoking (3.3) into (3.95), we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_3(2n+1)q^n \equiv 2f_1^3 f_3^3 \pmod{16}. \tag{3.96}$$

Employing (2.10) into (3.96), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(2n+1)q^n \equiv 2\frac{f_3^2 f_6 f_9^6}{f_{18}^3} + 8q^3\frac{f_3^5 f_{18}^6}{f_6^2 f_9^3} - 6qf_3^3 f_9^3 \pmod{16}. \tag{3.97}$$

Congruence (3.93) follows by extracting the terms involving q^{3n+2} on both sides of (3.97).

Extracting the terms involving q^{3n+1} from (3.97), dividing by q and replacing q^3 by q, we get

$$\overline{p}_3(6n+3)q^n \equiv 10f_1^3 f_3^3 \pmod{16}. \tag{3.98}$$

Using (3.96) and (3.98), we have

$$\overline{p}_3(6n+3) \equiv 5\overline{p}_3(2n+1) \pmod{16}. \tag{3.99}$$

Utilizing (3.99) and by mathematical induction on α, we get

$$\overline{p}_3(6\cdot 3^{\alpha}n + 3^{\alpha+1}) \equiv 5^{\alpha+1}\overline{p}_3(2n+1) \pmod{16}. \tag{3.100}$$

Using (3.100) and (3.93), we get (3.92).

Extracting the terms involving q^{2n} from (3.94) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(2n)q^n = \frac{f_4^5 f_{12}^5}{f_1^4 f_3^4 f_8^2 f_{24}^2} + 4q^2\frac{f_2^2 f_6^2 f_8^4 f_{24}^2}{f_1^4 f_3^4 f_4 f_{12}}, \tag{3.101}$$

which implies that

$$\sum_{n=0}^{\infty}\overline{p}_3(2n)q^n \equiv \frac{f_4^5 f_{12}^5}{f_1^4 f_3^4 f_8^2 f_{24}^2} \pmod 4. \tag{3.102}$$

Invoking (3.2) into (3.102), we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_3(2n)q^n \equiv \frac{f_4 f_{12}}{f_2^2 f_6^2} \pmod 4. \tag{3.103}$$

Extracting the terms involving q^{2n+1} from (3.103), we obtain

$$\overline{p}_3(4n+2) \equiv 0 \pmod 4. \tag{3.104}$$

Again extracting the terms involving q^{2n} from (3.103) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(4n)q^n \equiv \frac{f_2 f_6}{f_1^2 f_3^2} \pmod 4. \tag{3.105}$$

In view of congruences (3.13) and (3.105), we have

$$\overline{p}_3(4n) \equiv \overline{p}_3(n) \pmod 4. \tag{3.106}$$

Utilizing (3.106) and by mathematical induction on α, we get (3.88). Using (3.104) in (3.88), we obtain (3.91).

Extracting the terms involving q^{3n} from (3.97) and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(6n+1)q^n \equiv 2\frac{f_1^2 f_2 f_3^6}{f_6^3} + 8q\frac{f_1^5 f_6^6}{f_2^2 f_3^3} \pmod{16}, \tag{3.107}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+1)q^n \equiv 2\frac{f_1^2 f_2 f_3^6}{f_6^3} \pmod{4}. \tag{3.108}$$

Invoking (3.1) into (3.108), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+1)q^n \equiv 2f_4 \pmod{4}. \tag{3.109}$$

Congruence (3.89) follows by extracting the terms involving q^{4n+i} on both sides of (3.109).

Extracting the terms involving q^{4n} from (3.109) and replacing q^4 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(24n+1)q^n \equiv 2f_1 \pmod{4}. \tag{3.110}$$

Employing (2.1) into (3.110) and extracting the term q^{5n+1}, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(120n+25)q^n \equiv 2f_5 \pmod{4}. \tag{3.111}$$

Extracting the terms involving q^{5n+i} from (3.111), we get

$$\overline{p}_3(600n+120i+25) \equiv 0 \pmod{4}, \quad i = 1, 2, 3, 4. \tag{3.112}$$

Extracting the terms involving q^{5n} from (3.111) and replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(600n+25)q^n \equiv 2f_1 \pmod{4}. \tag{3.113}$$

Using (3.110) and (3.113), we get

$$\overline{p}_3(600n+25) \equiv \overline{p}_3(24n+1) \pmod{4}. \tag{3.114}$$

Utilizing (3.114) and by mathematical induction on α, we get

$$\overline{p}_3(600 \cdot 25^{\alpha} n + 25^{\alpha+1}) \equiv \overline{p}_3(24n+1) \pmod{4}. \tag{3.115}$$

Utilizing (3.112) and (3.115), we get (3.90). □

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