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Generalization on uniqueness of non-linear differential polynomials sharing 1-points [*]

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Abstract

In this paper, we generalize two theorems on the uniqueness of non-linear differential polynomials sharing 1-points, which improves a result of Indrajit Lahiri and Pulak Sahoo[14].

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1. Introduction and main results

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. Let k be a positive integer or infinity and $a \in \{\infty\} \cup \mathbb{C}$. We denote by $E_{k)}(a; f)$ the set of all a-points of f with multiplicities not exceeding k, where an a-point is counted according to its multiplicity. If for some $a \in \{\infty\} \cup \mathbb{C}$, $E_{\infty)}(a; f) = E_{\infty)}(a; g)$, then we say that f, g share the value a CM(counting multiplicities).

In [3,5], the problem of uniqueness of meromorphic functions when two linear differential polynomials share the same 1-points was studied. Also in [3] the following question was asked: What can be said if two non-linear differential polynomials generated by two meromorphic functions share 1 CM?

In the meantime some works have been done in this direction (cf.[1,7]). In 2001, Fang and Hong [1] have proved the following result.

Theorem A([1]). Let f and g be two transcendental entire functions and $n(\geq 11)$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f \equiv g$.

In 2005, Indrajit Lahiri and Pulak Sahoo [14] have proved the following result.

Theorem B([14]). Let f and g be two transcendental entire functions and $n(\geq 7)$ be an integer. If $E_{3)}(1; f^n(f-1)f') = E_{3)}(1; g^n(g-1)g')$, then $f \equiv g$.

Theorem C([14]). Let f and g be two transcendental meromorphic functions such that $\Theta(\infty; f) > 0, \Theta(\infty; g) > 0, \Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$, and $n(\geq 11)$ be an integer. If $E_{3)}(1; f^n(f-1)f') = E_{3)}(1; g^n(g-1)g')$, then $f \equiv g$.

Definition 1.1([4]). For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting functions of simple a-points of f.

For a positive integer m we denote by $N(r,a;f \mid\leq m)(N(r,a;f \mid\geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m, where each a-point is counted according to its multiplicity.

$\overline{N}(r,a;f \mid\leq m)$ and $\overline{N}(r,a;f \mid\geq m)$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r,a;f \mid< m), N(r,a;f \mid> m)$, $\overline{N}(r,a;f \mid< m)$ and $\overline{N}(r,a;f \mid> m)$ are defined analogously.

Definition 1.2([cf. [12]]). For $a \in \mathbb{C} \cup \{\infty\}$ we put

$$N_2(r,a;f) = \overline{N}(r,a;f) + \overline{N}(r,a;f \mid\geq 2).$$

In this paper we prove the following theorems which improves Theorem B and Theorem C.

Theorem 1.1. Let f and g be two transcendental entire functions and $n(\geq 5+2m), m \geq 1$, are integers. If $E_{3)}(1; f^n(f^m-1)f') = E_{3)}(1; g^n(g^m-1)g')$, then $f \equiv g$.

Theorem 1.2. Let f and g be two transcendental meromorphic functions such that $\Theta(\infty; f) > 0, \Theta(\infty; g) > 0$, $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n+1}$, and $n(\geq 9+2m), m \geq 1$ are integers. If $E_{3)}(1; f^n(f^m-1)f') = E_{3)}(1; g^n(g^m-1)g')$, then $f \equiv g$.

2. Some Lemmas

We denote by h the function

$$h = (\frac{f''}{f'} - \frac{2f'}{f-1}) - (\frac{g''}{g'} - \frac{2g'}{g-1}).$$

Lemma 2.1([8,10]). If f and g share 1 CM, then one of the following cases holds:

(i)$T(r,f)+T(r,g)\leq 2\{N_2(r,0;f)+N_2(r,0;g)+N_2(r,\infty;f)+N_2(r,\infty;g)\}+S(r,f)+S(r,g)$,

(ii)$f\equiv g$, (iii)$fg\equiv 1$.

Lemma 2.2([14]). If $E_{3)}(1;f)=E_{3)}(1;g)$, then the conclusion of Lemma 2.1 holds.

Lemma 2.3([14]). Let f and g be two nonconstant meromorphic functions such that $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n+1}$, where $n(\geq 2)$ is an integer. Then $f^{n+1}(af+b)\equiv g^{n+1}(ag+b)$ implies $f\equiv g$, where a,b are finite nonzero constants and n is an integer.

Lemma 2.4. Let f and g be nonconstant meromorphic functions. Then $f^n(f^m-1)f'g^n(g^m-1)g'\not\equiv 1$, where n is an integer.

Proof. If possible, let

$$f^n(f^m-1)f'g^n(g^m-1)g'\equiv 1.$$

Let z_0 be an 1-point of f with multiplicity $p(\geq 1)$. Then z_0 is a pole of g with multiplicity $q(\geq 1)$, such that

$mp+p-1=(n+m+1)q+1$

$(m+1)p=(n+m+1)q+2\geq n+m+3.$

i.e., $p\geq\frac{n+m+3}{m+1}.$

Let z_0 be a zero of f with multiplicity $p(\geq 1)$ and it be a pole of g with multiplicity $q(\geq 1)$. Then

$np+p-1=nq+mq+q+1$

i.e, $(n+1)(p-q)=mq+2$

i.e, $p \geq \frac{n+m-1}{m}$.

Since a pole of f is either a zero of $g(g-1)$ or a zero of g', we see that

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \overline{N}(r,1;g) + \overline{N}_0(r,0;g') \\ &\leq \frac{m}{n+m-1}N(r,0;g) + \frac{m+1}{n+m+3}N(r,1;g) + \overline{N}_0(r,0;g') \\ &\leq (\frac{m}{n+m-1} + \frac{m+1}{n+m+3})T(r,g) + \overline{N}_0(r,0;g').\end{aligned}$$

Now by the second fundamental theorem we obtain

$$\begin{aligned}T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,1;f) + \overline{N}(r,\infty;f) - \overline{N}_0(r,0;f') + S(r,f) \\ &\leq \frac{m}{n+m-1}N(r,0;f) + \frac{m+1}{n+m+3}N(r,1;f) + \overline{N}(r,\infty;f) \\ &- \overline{N}_0(r,0;f') + S(r,f)\end{aligned}$$

i.e.,

$$\begin{aligned}(1 - \frac{m}{n+m-1} - \frac{m+1}{n+m+3})T(r,f) &\leq (\frac{m}{n+m-1} + \frac{m+1}{n+m+3})T(r,g) \\ &+ \overline{N}_0(r,0;g') - \overline{N}_0(r,0;f') + S(r,f).\end{aligned} \tag{1}$$

Similarly, we get

$$\begin{aligned}(1 - \frac{m}{n+m-1} - \frac{m+1}{n+m+3})T(r,g) &\leq (\frac{m}{n+m-1} + \frac{m+1}{n+m+3})T(r,f) \\ &+ \overline{N}_0(r,0;f') - \overline{N}_0(r,0;g') + S(r,g).\end{aligned} \tag{2}$$

Adding (1) and (2) we get

$$(1 - \frac{2m}{n+m-1} - \frac{2(m+1)}{n+m+3})\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g).$$

which is a contradiction because

$$1 - \frac{m}{n+m-1} - \frac{m+1}{n+m+3} > 0.$$

This proves the lemma.

Lemma 2.5([9]). Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + ... + a_n f^n$, where $a_0, a_1, ..., a_n$ are constants and $a_n \neq 0$. Then $T(r, P(f)) = nT(r, f) + O(1)$.

Lemma 2.6([11]). Let f be a nonconstant meromorphic function. Then $N(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty; f) + N(r, 0; f) + S(r, f)$.

Lemma 2.7. Let f and g be two nonconstant meromorphic functions and $F = f^{n+1}(\frac{f^m}{n+m+1} - \frac{1}{n+1})$ and $G = g^{n+1}(\frac{g^m}{n+m+1} - \frac{1}{n+1})$, where $n(\geq 5 - m)$ is an integer. Then $F' \equiv G'$ implies $F \equiv G$.

Proof. If $F' \equiv G'$ then $F = G + c$, where c is a constant. If possible, let $c \neq 0$. Then by the second fundamental theorem and Lemma 2.5 we get

$$\begin{aligned}(n+m+1)T(r,f) &\leq \overline{N}(r,\infty;F) + \overline{N}(r,0;F) + \overline{N}(r,c;F) + S(r,f) \\ &= \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \overline{N}(r,\frac{n+m+1}{n+1};f) \\ &+ \overline{N}(r,0;g) + \overline{N}(r,\frac{n+m+1}{n+1};g) + S(r,f) \\ &\leq 3T(r,f) + 2T(r,g) + S(r,f).\end{aligned}$$

i.e.,

$$(n+m-2)T(r,f) \leq 2T(r,g) + S(r,f).$$

Similarly, we get

$$(n+m-2)T(r,g) \leq 2T(r,f) + S(r,g).$$

This shows that

$$(n+m-4)T(r,f) + (n+m-4)T(r,g) \leq S(r,f) + S(r,g),$$

which is a contradiction.

Therefore $c = 0$ and so $F \equiv G$.

This proves the Lemma.

3. Proof of Theorems

Proof of Theorem 1.2. Let F and G be defined as in Lemma 2.7. Now in view of the first fundamental theorem and Lemma 2.5 we get

$$\begin{aligned}
T(r,F) &= T(r,\frac{1}{F}) + O(1) \\
&= N(r,\frac{1}{F}) + m(r,\frac{1}{F}) + O(1) \\
&\leq N(r,\frac{1}{F}) + m(r,\frac{F'}{F}) + m(r,\frac{1}{F'}) \\
&= T(r,F') + N(r,0;F) - N(r,0;F') + S(r,F) \\
&= T(r,F') + (n+1)N(r,0,f) + N(r,\frac{n+m+1}{n+1};f^m) \\
&- nN(r,0;f) - N(r,1;f^m) - N(r,0;f') + S(r,f) \\
&= T(r,F') + N(r,0;f) + N(r,\frac{n+m+1}{n+1};f^m) \\
&- N(r,1;f^m) - N(r,0;f') + S(r,f).
\end{aligned}$$

If possible, suppose that

$$\begin{aligned}
T(r,F') + T(r,G') \leq 2\{&N_2(r,0;F') + N_2(r,0;G') + N_2(r,\infty;F') \\
&+ N_2(r,\infty;G')\} + S(r,F') + S(r,G').
\end{aligned} \tag{3}$$

Then we get by Lemma 2.6,

$$
\begin{aligned}
T(r,F)+T(r,G) \leq\ & T(r,F')+T(r,G')+N(r,0;f)+N(r,\frac{n+m+1}{n+1};f^m)\\
& -N(r,1;f^m)-N(r,0;f')+N(r,0;g)+N(r,\frac{n+m+1}{n+1};g^m)\\
& -N(r,1;g^m)-N(r,0;g')+S(r,f)+S(r,g)\\
\leq\ & 2N_2(r,0;F')+2N_2(r,0;G')+2N_2(r,\infty;F')\\
& +2N_2(r,\infty;G')+N(r,0;f)+N(r,\frac{n+m+1}{n+1};f^m)\\
& -N(r,1;f^m)-N(r,0;f')+N(r,0;g)+N(r,\frac{n+m+1}{n+1};g^m)\\
& -N(r,1;g^m)-N(r,0;g')+S(r,f)+S(r,g)
\end{aligned}
$$

$$
\begin{aligned}
& \leq 4T(r,f)+2mT(r,f)+2T(r,f)+2\overline{N}(r,\infty;f)+4T(r,g)\\
& +2mT(r,g)+2\overline{N}(r,\infty;g)+2T(r,g)+3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)\\
& +S(r,f)+S(r,g)\\
& \leq (6+2m)T(r,f)+(6+2m)T(r,g)+5\overline{N}(r,\infty;f)\\
& +5\overline{N}(r,\infty;g)+S(r,f)+S(r,g).
\end{aligned}
$$

So by Lemma 2.5 we obtain

$$
\begin{aligned}
(n-4-2m)T(r,f)+(n-4-2m)T(r,g) \leq\ & 5\overline{N}(r,\infty;f)+5\overline{N}(r,\infty;g)\\
& +S(r,f)+S(r,g).
\end{aligned}
\tag{4}
$$

Let us choose ϵ such that

$0<\epsilon<n-9-2m+min\{\Theta(\infty,f),\Theta(\infty,g)\}$.

Then from (4) we get

$$(n-9-2m+\Theta(\infty;f)-\epsilon)T(r,f)+(n-9-2m+\Theta(\infty;g)-\epsilon)T(r,g) \leq S(r,f)+S(r,g),$$

which is a contradiction.

Therefore inequality (3) does not hold.

Since $E_{3)}(1; F') = E_{3)}(1; G')$ by Lemmas 2.2,2.3,2.4 and 2.7 we get $f \equiv g$.

This proves the theorem.

Proof of Theorem 1.1.

If (3) holds, then from (4) we get

$(n-4-2m)T(r,f) + (n-4-2m)T(r,g) \le S(r,f) + S(r,g)$

which is a contradiction.

Therefore inequality (3) does not hold.

Since

$E_{3)}(1; F') = E_{3)}(1; G')$, by Lemmas 2.2,2.3,2.4 and 2.7 we get $f \equiv g$.

This proves the theorem.

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