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Two new families of iterative methods for solving nonlinear equations *

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Abstract

In this paper, we have presented a family of fourth order iterative method and another family of sixth order iterative method without memory based on power mean using weight functions. The family of fourth order methods given here is optimal in the sense of Kung-Traub hypothesis. In terms of computational point of view, our first method require three evaluations (one function and two first derivatives) per iteration to get fourth order and the second method require four evaluations (two functions and two derivatives) per iteration to get sixth order. Hence, these methods have high efficiency indices 1.5874 and 1.5651 respectively. Few existing methods can be regarded as particular cases of our family of methods. Some numerical examples are tested

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to know the performance of the new methods which verifies the theoretical results.

Keywords and Phrases: *Non-linear equation, Multi-point iteration, Optimal order, Kung-Traub conjecture, Power mean.*

1. Introduction

It is known that a wide class of problems which arise in boundary value problems in Kinetic theory of gases, elasticity and other applied areas are mostly reduced to single variable nonlinear equations. One of the best root-finding methods for solving nonlinear scalar equation $f(x) = 0$ is Newton's iteration method. The local order of convergence of Newton's method is two and it is optimal with two function evaluations per iterative step. In recent years, numerous higher order iterative methods have been developed and analyzed for solving nonlinear equations that improve classical methods such as Newton's, Chebyshev, Chebyshev-Halley's, etc. As the order of convergence increases, so does the number of function evaluations per step. Hence, a new index to determine the efficiency called "Efficiency Index" (EI) is introduced in [8] to measure the balance between these quantities. Kung-Traub [7] conjectured that the order of convergence of any multi-point without memory method with d function evaluations cannot exceed the bound 2^{d-1}, the optimal order. Thus the optimal order for three evaluations per iteration would be four. Jarrat's method [4] is an example of an optimal fourth order method.

Recently, some optimal and non-optimal multi-point methods have been developed (see [1, 2, 3, 5, 6, 9, 10] and references therein). In [12], a third order method has been presented using the idea of power mean. In this paper, we have improved the order of the method given in [12] to four (optimal) and six using weight functions. In Section 2, some definitions are included which are required for our study. Section 3 presents the development of the new methods. Section 4 discusses the convergence analysis and section 5 presents few numerical examples and compare the results of the present methods with Newton's method, power mean Newton's method [12] and few optimal and non-optimal methods . Finally, section 5 gives conclusions on the present work.

2. Preliminaries

Definition 2.1. *[11] If the sequence $\{x_n\}$ tends to a limit x^* in such a way that*

$$\lim_{n\to\infty} \frac{x_{n+1}-x^*}{(x_n-x^*)^p} = C$$

for $p \geq 1$, then the order of convergence of the sequence is said to be p, and C is known as the asymptotic error constant. If $p=1$, $p=2$ or $p=3$, the convergence is said to be linear, quadratic or cubic, respectively.
Let $e_n = x_n - x^*$, then the relation

$$e_{n+1} = C\ e_n^p + O\left(e_n^{p+1}\right) = O\left(e_n^p\right). \tag{1}$$

is called the error equation. The value of p is called the order of convergence of the method.

Definition 2.2. *[8] The Efficiency Index of any iterative method is given by*

$$EI = p^{\frac{1}{d}}, \tag{2}$$

where d is the total number of new function evaluations (the values of f and its derivatives) per iteration and p is its order.

Let $x_{n+1} = \psi(x_n)$ define an Iterative Function (I.F.). Let x_{n+1} be determined by new information at $x_n, \phi_1(x_n), ..., \phi_{\mathrm{i}}(x_n), \mathrm{i} \geq 1$. No old information is reused. Thus,

$$x_{n+1} = \psi(x_n, \phi_1(x_n), ..., \phi_{\mathrm{i}}(x_n)). \tag{3}$$

Then ψ is called a multipoint I.F without memory.

Kung-Traub Conjecture [7]
Let ψ be an I.F. without memory with d evaluations. Then

$$p(\psi) \leq p_{opt} = 2^{d-1}, \tag{4}$$

where p_{opt} is the maximum order.

3. Development of the methods

The Newton (also called Newton-Raphson) I.F. ($2^{nd}NR$) is given by

$$\psi_{2^{nd}NR}(x) = x - u(x), \ \ where \ u(x) = \frac{f(x)}{f'(x)}. \tag{5}$$

The $2^{nd}NR$ has 2 function evaluations and satisfies the Kung-Traub conjecture with $d = 2$. A family of third-order I.F. based on power means ($3^{rd}PM$) considered in [12] is given by

$$\psi_{3^{rd}PM}(x) = x - \frac{2^{1/\beta} f(x)}{sign(f'(x))\Big(f'(x)^\beta + f'(\psi_{2^{nd}NR}(x))^\beta\Big)^{1/\beta}}, \tag{6}$$

where $\beta \in \mathbb{R} \setminus \{0\}$. The cases $\beta = 1, -1, 2$ correspond to arithmetic mean ($3^{rd}AM$), harmonic mean ($3^{rd}HM$) and square mean ($3^{rd}SM$) respectively. The case $\beta \longrightarrow 0$ is the geometric mean ($3^{rd}GM$) where
$\lim\limits_{\beta\longrightarrow 0} \left(\frac{f'(x)^\beta + f'(\psi_{2^{nd}NR}(x))^\beta}{2}\right)^{\frac{1}{\beta}} = \sqrt{f'(x) f'(\psi_{2^{nd}NR}(x))}.$

Fourth order family of power mean method (4^{th}*PM*):
Let us define the first new family of methods with 3 function evaluations which is optimal

$$\psi_{4^{th}PM}(x) = x - \frac{2^{1/\beta} f(x)}{sign(f'(x))\Big(f'(x)^\beta + f'(y)^\beta\Big)^{1/\beta}}[H(\tau) \times G(t)], \tag{7}$$

where

$$y = x - \frac{2}{3}u(x), \, \tau = \frac{f'(y)}{f'(x)}, \, t = u(x).$$

Expanding $H(\tau)$ about 1 and $G(t)$ about 0, we have

$$\begin{aligned} H(\tau) \times G(t) &= H(1)G(0) + (\tau-1)H'(1)G(0) + \frac{(\tau-1)^2}{2}H''(1)G(0) \\ &+ \frac{(\tau-1)^3}{6}H'''(1)G(0) + \frac{(t-0)^3}{6}H(1)G'''(0) + ... \end{aligned}$$

and choosing H, G and their derivatives as follows

$$H(1) = 1, \quad G(0) = 1, \quad H'(1) = -1/4, \quad G'(0) = 0 = G''(0),$$

$$H''(1) = \frac{\beta+5}{8}, \quad H'''(1) = G'''(0) = -1,$$

we get

$$H(\tau)\times G(t) = 1 - \frac{1}{4}(\tau-1) + \left(\frac{\beta+5}{8}\right)(\tau-1)^2 - \frac{1}{6}\Big((\tau-1)^3 + t^3\Big).$$

Sixth order family of power mean method (6^{th}*PM*):
Furthermore, we consider the second new family of method which is non-optimal

$$\psi_{6^{th}PM}(x) = \psi_{4^{th}PM}(x) - \frac{2^{1/\beta} f(\psi_{4^{th}PM}(x))}{sign(f'(x))\left(f'(x)^\beta + f'(y)^\beta\right)^{1/\beta}} K(\tau), \qquad (8)$$

where $K(\tau)$ is obtained by expanding it about $\tau = 1$ as follows:

$$K(\tau) = K(1) + (\tau-1)K'(1) + \frac{(\tau-1)^2}{2}K''(1) + ... \quad .$$

By choosing $K(1) = 1, K'(1) = -1, K''(1) = K'''(1) = ... = 0$, we get

$$K(\tau) = 2 - \tau$$

Note: In the above two methods, the cases $\beta = 1, -1, 2$ may be respectively called as arithmetic mean method ($4^{th}AM$ and $6^{th}AM$), harmonic mean method ($4^{th}HM$ and $6^{th}HM$) and square mean method ($4^{th}SM$ and $6^{th}SM$). The case $\beta = 0$ is called geometric mean method ($4^{th}GM$ and $6^{th}GM$) where it can be obtained as $\lim\limits_{\beta\longrightarrow 0}\left(\frac{f'(x)^\beta + f'(y)^\beta}{2}\right)^{\frac{1}{\beta}} = \sqrt{f'(x)f'(y)}$ [12].

4. Convergence Analysis

In this section, we analyze the convergence proof of the methods (7) and (8).

Theorem 4.1. *For sufficiently smooth function $f : D \subset \mathbb{R} \rightarrow \mathbb{R}$ having a simple root x^* in the open interval D, the $4^{th}PM$ family of I.F. (7) and the $6^{th}PM$ family of I.F. (8) are of local fourth-order and sixth-order convergent respectively.*

Proof. Let $c_j = \dfrac{f^{(j)}(x^*)}{j!f'(x^*)}$, $j = 2, 3, 4.$ and $e_n = x - x^*$. Taylor expansion of $f(x)$ and $f'(x)$ about x^* gives

$$f(x) = f'(x^*)\Big[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + \dots\Big] \tag{9}$$

and

$$f'(x) = f'(x^*)\Big[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + \dots\Big] \tag{10}$$

so that

$$t = u(x) = e_n - c_2e_n^2 + 2\Big(c_2^2 - c_3\Big)e_n^3 + \Big(7c_2c_3 - 4c_2^3 - 3c_4\Big)e_n^4 + \dots \tag{11}$$

and

$$y = x - \frac{2}{3}u(x) = x^* + \frac{e_n}{3} + \frac{2}{3}c_2e_n^2 - \frac{4}{3}\Big(c_2^2 - c_3\Big)e_n^3 + \frac{2}{3}\Big(4c_2^3 - 7c_2c_3 + 3c_4\Big)e_n^4 + \dots \quad . \tag{12}$$

Again, the Taylor expansion of $f'(y)$ about x^* gives

$$f'(y) = f'(x^*)\Big[1 + \frac{2}{3}c_2e_n + \frac{1}{3}\Big(4c_2^2 + c_3\Big)e_n^2 + \frac{4}{27}\Big(-18c_2^3 + 27c_2c_3 + c_4\Big)e_n^3 + \dots\Big], \tag{13}$$

$$\tau = 1 - \frac{4}{3}c_2e_n + \Big(4c_2^2 - \frac{8}{3}c_3\Big)e_n^2 + \Big(-\frac{32}{3}c_2^3 + \frac{40}{3}c_2c_3 - \frac{104}{27}c_4\Big)e_n^3 + \dots \quad . \tag{14}$$

Using equations (10) and (13), we have

$$\begin{aligned} sign(f'(x))\left(\frac{f'(x)^\beta + f'(y)^\beta}{2}\right)^{1/\beta} &= f'(x^*)\Big[1 + \frac{4}{3}c_2e_n + \left(\Big(\frac{2}{9}\beta + \frac{4}{9}\Big)c_2^2 + \frac{5}{3}c_3\right)e_n^2 \\ &+ \Big(\frac{56}{27}c_4 + \Big(\frac{8}{9}\beta + \frac{10}{9}\Big)c_2c_3 \\ &+ \Big(\frac{52}{81}\beta^2 - \frac{80}{27}\beta + \frac{80}{81}\Big)c_2^3\Big)e_n^3 + \dots\Big] \end{aligned} \tag{15}$$

From (9) and (15), we get

$$\begin{aligned} \frac{2^{1/\beta}f(x)}{sign(f'(x))\Big(f'(x)^\beta + f'(y)^\beta\Big)^{1/\beta}} &= e_n - \frac{1}{3}c_2e_n^2 - \frac{2}{9}\Big(\beta c_2^2 + 3c_3\Big)e_n^3 \\ &- \frac{1}{81}\Big(87c_4 + \Big(72\beta - 27\Big)c_2c_3 \\ &+ \Big(52\beta^2 - 270\beta + 68\Big)c_2^3\Big)e_n^4 + \dots \end{aligned} \tag{16}$$

$$H(\tau)\times G(t) = 1+\frac{1}{3}c_2e_n+\left(\frac{2}{3}c_3+\left(\left(\beta+5\right)\frac{2}{9}-1\right)c_2^2\right)e_n^2 \\ +\left(\frac{26}{27}c_4+\left(\frac{8}{3}-\left(\beta+5\right)\frac{4}{3}+\frac{64}{162}\right)c_2^3 \\ +\left(\frac{8}{9}(\beta+5)-\frac{10}{3}\right)c_2c_3-\frac{1}{6}\right)e_n^3+\dots \tag{17}$$

Using (16) and (17) in (7), we obtain

$$\psi_{4^{th}PM}(x)-x^* = \left(\frac{c_4}{9}-c_2c_3+\frac{1}{6}+\left(\frac{52}{81}\beta^2-\frac{50}{27}\beta+\frac{395}{81}-\frac{64}{162}\right)c_2^3\right)e_n^4+O(e_n^5). \tag{18}$$

Hence, we proved that $4^{th}PM$ family I.F. has fourth order convergence. Furthermore, using (14),(15) and (18) into (8), we have

$$\psi_{6^{th}PM}(x)-x^* = \Big(\Big(\frac{2}{81}\beta+\frac{40}{81}\Big)c_2^2c_4-\frac{1}{6}c_3+c_2c_3^2-\frac{1}{9}c_3c_4-\Big(-\frac{4}{9}-\frac{1}{27}\beta+\frac{8}{27}\Big)c_2^2 \\ +\Big(-\frac{156}{243}\beta^2+\frac{132}{81}\beta-\frac{1568}{4131}-\frac{2265}{243}\Big)c_2^3c_3 \\ +\Big(\frac{104}{729}\beta^3+\frac{1780}{729}\beta^2-\frac{5210}{729}\beta-\frac{1280}{729}-\frac{64}{729}\beta+\frac{6320}{729}\Big)c_2^5\Big)e_n^6 \\ +O(e_n^7). \tag{19}$$

Hence, we proved that $6^{th}PM$ family I.F. has sixth order convergence. □

5. Numerical examples

In this section, numerical results on some test functions are compared for the new methods ($4^{th}PM$ and $6^{th}PM$) with some existing fourth and sixth order methods. Numerical computations have been carried out in the MATLAB software with 500 significant digits. Depending on the precision of the computer, we have used the stopping criteria for the iterative process either $|f(x_N)| < \epsilon$ or $|x_N - x_{N-1}| < \epsilon$ where $\epsilon = 10^{-50}$ and N is the number of iterations required for convergence. The computational order of convergence is given by

$$\rho = \frac{\ln|(x_N - x_{N-1})/(x_{N-1}-x_{N-2})|}{\ln|(x_{N-1}-x_{N-2})/(x_{N-2}-x_{N-3})|}.$$

Table 1 shows the efficiency indices of the new methods with some known methods. It is clear that the $4^{th}PM$ and $6^{th}PM$ methods have better efficiency

indices when compared with $2^{nd}NR$ and $3^{rd}PM$.
The test functions and their simple zeros for our study are given below:

Table 1: Comparison of Efficiency Indices (EI) and Optimality

I.F	Order of convergence (p)	No. of function evaluations per iteration (d)	EI	Optimal / Non-optimal
$2^{nd}NR$	2	2	1.4142	Optimal
$3^{rd}PM$	3	3	1.4422	Non-optimal
(5) in (Jarratt method)[10]	4	3	1.5874	Optimal
(6) in [10]	4	3	1.5874	Optimal
$4^{th}PM$ (eq. (7))	4	3	1.5874	Optimal
(6) in [3]	6	4	1.5651	Non-optimal
(3) in [9]	6	4	1.5651	Non-optimal
$6^{th}PM$ (eq. (8))	6	4	1.5651	Non-optimal

$$f_1(x) = \sin(2\cos x) - 1 - x^2 + e^{\sin(x^3)}, \qquad x^* = -0.7848959876612125352...$$
$$f_2(x) = xe^{x^2} - \sin^2 x + 3\cos x + 5, \qquad x^* = -1.2076478271309189270...$$
$$f_3(x) = x^3 + 4x^2 - 10, \qquad x^* = 1.3652300134140968457...$$
$$f_4(x) = sin(x) + cos(x) + x, \qquad x^* = -0.4566247045676308244...$$
$$f_5(x) = \frac{x}{2} - \sin x, \qquad x^* = 1.8954942670339809471...$$
$$f_6(x) = (x+2)e^x - 1, \qquad x^* = -0.4428544010023885831...$$
$$f_7(x) = \sqrt{x} - \cos x, \qquad x^* = 0.6417143708728826583...$$
$$f_8(x) = x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}, \qquad x^* = 0.4099920179891371316...$$
$$f_9(x) = e^{-x}\sin x + \log(1 + x^2) - 2, \qquad x^* = 2.4477482864524245021...$$
$$f_{10}(x) = \sqrt{x^3} + \sin x - 30, \qquad x^* = 9.7165019933652005655...$$

Table 2: Comparison of results for $4^{th}PM$ and $6^{th}PM$ for $\beta = 1, -1, 0, 2$ with $3^{rd}PM$ and $2^{nd}NR$.

I.F.	$f_1(x)$				$f_2(x)$			
	x_0	N	$\lvert x_N - x_{N-1}\rvert$	ρ	x_0	N	$\lvert x_N - x_{N-1}\rvert$	ρ
$2^{nd}NR$	-0.9	7	7.7e-074	1.99	-1.7	9	4.3e-054	2.00
$3^{rd}AM$	-0.9	5	2.6e-109	3.00	-1.7	7	4.3e-124	3.00
$4^{th}AM$	-0.9	4	1.7e-063	3.99	-1.7	6	5.3e-154	4.00
$6^{th}AM$	-0.9	4	8.6e-197	6.00	-1.7	5	3.3e-159	5.99
$3^{rd}HM$	-0.9	5	5.3e-104	3.00	-1.7	6	1.2e-072	3.00
$4^{th}HM$	-0.9	4	3.1e-065	3.99	-1.7	6	8.3e-180	4.00
$6^{th}HM$	-0.9	4	1.5e-201	6.00	-1.7	5	5.1e-183	6.00
$3^{rd}GM$	-0.9	5	2.2e-131	3.00	-1.7	6	6.9e-054	2.99
$4^{th}GM$	-0.9	4	2.5e-064	3.99	-1.7	6	4.1e-166	4.00
$6^{th}GM$	-0.9	4	4.5e-199	6.00	-1.7	5	1.6e-170	5.99
$3^{rd}SM$	-0.9	5	1.6e-095	3.00	-1.7	7	1.4e-100	3.00
$4^{th}SM$	-0.9	4	9.9e-063	3.99	-1.7	6	3.6e-144	4.00
$6^{th}SM$	-0.9	4	1.1e-194	5.99	-1.7	5	9.0e-150	5.99

I.F.	$f_3(x)$			
	x_0	N	$\lvert x_N - x_{N-1}\rvert$	ρ
$2^{nd}NR$	1.6	7	7.7e-063	2.00
$3^{nd}AM$	1.6	5	1.8e-077	2.99
$4^{nd}AM$	1.6	5	1.1e-198	4.00
$6^{nd}AM$	1.6	4	1.8e-160	5.99
$3^{nd}HM$	1.6	5	9.3e-109	3.00
$4^{nd}HM$	1.6	4	1.4e-051	3.99
$6^{nd}HM$	1.6	4	3.8e-165	5.99
$3^{nd}GM$	1.6	5	9.3e-087	2.99
$4^{nd}GM$	1.6	4	7.9e-051	3.99
$6^{nd}GM$	1.6	4	1.0e-162	5.99
$3^{nd}SM$	1.6	5	1.6e-071	2.99
$4^{nd}SM$	1.6	5	4.1e-196	4.00
$6^{nd}SM$	1.6	4	2.4e-158	5.99

Table 3: Comparison of results for best value of β for $3^{rd}PM$, $4^{th}PM$ and $6^{th}PM$ along with $2^{nd}NR$

Functions	Guess	$2^{nd}NR$		$3^{rd}PM$			$4^{th}PM$		
		N	$\|x_N - x_{N-1}\|$	β	N	$\|x_N - x_{N-1}\|$	β	N	$\|x_N - x_{N-1}\|$
$f_1(x)$	-0.9	7	7.7e-074	0	5	2.2e-131	-9	4	9.6e-078
	-0.7	7	1.0e-074	0	5	1.2e-130	-6	4	7.5e-089
$f_2(x)$	-1.7	9	4.3e-054	-14	5	2.8e-068	20	5	6.0e-071
	-1.0	8	1.1e-064	-2	5	2.9e-071	-3	5	1.9e-157
$f_3(x)$	1.6	7	7.7e-063	-1	5	9.3e-109	-11	4	1.0e-070
	1.0	8	2.8e-088	-1	5	8.7e-101	-6	5	3.2e-194
$f_4(x)$	-0.2	7	6.8e-096	4	5	4.8e-148	20	4	1.9e-057
	-0.6	6	1.5e-061	16	4	9.2e-060	20	4	6.3e-072
$f_5(x)$	1.6	8	6.8e-087	-1	5	1.8e-083	-5	4	1.8e-058
	2.0	7	1.8e-080	-1	5	1.4e-147	-11	4	5.2e-082
$f_6(x)$	-0.3	7	7.7e-066	-2	5	1.2e-122	-8	4	1.0e-071
	-0.7	8	1.3e-092	-1	5	5.2e-086	-5	4	5.7e-071
$f_7(x)$	0.2	7	2.0e-074	-6	5	1.1e-088	11	5	2.1e-177
	0.9	7	3.0e-094	-1	5	8.1e-139	-17	4	1.4e-056
$f_8(x)$	0.2	8	8.2e-076	-1	5	1.8e-098	-4	4	2.1e-052
	1.5	9	2.7e-074	-3	5	1.3e-085	-8	5	1.8e-059
$f_9(x)$	1.9	7	2.9e-088	-6	5	1.9e-140	15	5	1.6e-139
	2.7	6	5.9e-058	-4	4	2.0e-055	20	4	3.4e-055
$f_{10}(x)$	9.9	6	9.5e-059	-7	4	7.0e-060	-20	4	7.6e-065
	9.2	6	3.1e-052	-9	5	1.6e-110	-20	5	1.5e-139

Functions	Guess	$6^{th}PM$		
		β	N	$\|x_N - x_{N-1}\|$
$f_1(x)$	-0.9	-9	4	6.5e-230
	-0.7	-6	4	2.2e-255
$f_2(x)$	-1.7	-9	5	4.6e-269
	-1.0	-3	4	4.6e-098
$f_3(x)$	1.6	-11	4	6.6e-202
	1.0	-6	4	6.2e-133
$f_4(x)$	-0.2	-20	4	4.5e-203
	-0.6	-20	4	1.0e-260
$f_5(x)$	1.6	-5	4	2.8e-149
	2.0	-20	4	1.1e-258
$f_6(x)$	-0.3	-20	4	1.7e-220
	-0.7	-5	4	1.1e-186
$f_7(x)$	0.2	0	4	8.2e-160
	0.9	-18	4	1.0e-277
$f_8(x)$	0.2	-4	4	2.1e-126
	1.5	-11	5	1.7e-270
$f_9(x)$	1.9	1	4	5.1e-230
	2.7	-5	3	7.3e-055
$f_{10}(x)$	9.9	7	3	1.6e-053
	9.2	18	4	2.2e-171

Table 4: Comparison of results for best value of β for $4^{th}PM$ along with some known fourth order methods

Functions	Guess	(5) in [10]		(6) in [10]		$4^{th}PM$ (eq. (7))		
		N	$\lvert x_N - x_{N-1}\rvert$	N	$\lvert x_N - x_{N-1}\rvert$	β	N	$\lvert x_N - x_{N-1}\rvert$
$f_1(x)$	-0.9	4	1.6e-067	4	4.0e-055	-9	4	9.6e-078
	-0.7	4	1.4e-070	4	8.0e-051	-6	4	7.5e-089
$f_2(x)$	-1.7	5	1.4e-085	6	2.5e-087	20	5	6.0e-071
	-1.0	5	2.0e-199	6	6.8e-087	-3	5	1.9e-157
$f_3(x)$	1.6	4	2.4e-065	5	2.0e-179	-11	4	1.0e-070
	1.0	5	1.4e-187	5	9.6e-068	-6	5	3.2e-194
$f_4(x)$	-0.2	4	2.1e-077	4	4.9e-068	20	4	1.9e-057
	-0.6	4	4.3e-100	4	2.3e-096	20	4	6.3e-072
$f_5(x)$	1.6	5	5.7e-169	5	2.6e-059	-5	4	1.8e-058
	2.0	4	7.4e-079	4	1.4e-061	-11	4	5.2e-082
$f_6(x)$	-0.3	4	1.9e-073	5	3.9e-190	-8	4	1.0e-071
	-0.7	4	3.3e-055	5	4.8e-083	-5	4	5.7e-071
$f_7(x)$	0.2	4	8.7e-063	4	8.5e-054	11	5	2.1e-177
	0.9	4	3.5e-079	4	8.6e-082	-17	4	1.4e-056
$f_8(x)$	0.2	5	7.4e-151	6	2.0e-137	-4	4	2.1e-052
	1.5	5	3.1e-074	6	1.3e-148	-8	5	1.8e-059
$f_9(x)$	1.9	4	1.0e-084	4	4.1e-067	15	5	1.6e-139
	2.7	4	5.8e-102	4	6.0e-093	20	4	3.4e-055
$f_{10}(x)$	9.9	4	3.3e-100	4	7.9e-117	-20	4	7.6e-065
	9.2	4	1.9e-078	4	1.4e-090	-20	5	1.5e-139

Table 5: Comparison of results for best value of β for $6^{th}PM$ along with some known sixth order methods

Functions	Guess	(6) in [3]		(3) in [9]		$6^{th}PM$ (eq. (8))		
		N	$\lvert x_N - x_{N-1}\rvert$	N	$\lvert x_N - x_{N-1}\rvert$	β	N	$\lvert x_N - x_{N-1}\rvert$
$f_1(x)$	-0.9	4	6.9e-204	4	4.6e-205	-9	4	6.5e-230
	-0.7	4	4.5e-204	4	3.3e-210	-6	4	2.2e-255
$f_2(x)$	-1.7	5	2.3e-245	4	5.4e-052	-9	5	4.6e-269
	-1.0	4	1.8e-090	4	2.4e-133	-3	4	4.6e-098
$f_3(x)$	1.6	4	4.2e-186	4	1.0e-193	-11	4	6.6e-202
	1.0	4	8.2e-113	4	7.7e-126	-6	4	6.2e-133
$f_4(x)$	-0.2	4	2.6e-237	4	6.0e-239	-20	4	4.5e-203
	-0.6	4	1.6e-052	4	1.7e-052	-20	4	1.0e-260
$f_5(x)$	1.6	4	2.7e-102	4	2.7e-112	-5	4	2.8e-149
	2.0	4	2.0e-236	4	4.1e-241	-20	4	1.1e-258
$f_6(x)$	-0.3	4	2.4e-201	4	1.8e-216	-20	4	1.7e-220
	-0.7	4	2.1e-129	4	1.7e-149	-5	4	1.1e-186
$f_7(x)$	0.2	4	3.3e-153	4	3.2e-152	0	4	8.2e-160
	0.9	4	5.0e-290	4	4.1e-300	-18	4	1.0e-277
$f_8(x)$	0.2	4	3.1e-085	4	9.7e-096	-4	4	2.1e-126
	1.5	5	8.4e-291	5	1.8e-293	-11	5	1.7e-270
$f_9(x)$	1.9	4	1.2e-255	4	8.4e-259	1	4	5.1e-230
	2.7	3	2.3e-066	3	6.4e-064	-5	3	7.3e-055
$f_{10}(x)$	9.9	3	1.5e-057	3	6.9e-058	7	3	1.6e-053
	9.2	4	2.4e-247	4	5.2e-248	18	4	2.2e-171

Table 2 shows the results for $f_1(x)$, $f_2(x)$ and $f_3(x)$ for some specific values of β for a given starting point. We observe that the computational order of convergence (ρ) agrees with theoretical results. The members of the $4^{th}PM$ family converge in less number of iterations with least error than that of the $3^{rd}PM$ family. Also, we see that $6^{th}HM$ method is the most efficient method with least number of iterations and least error among the methods compared in Table 2.
Table 3, 4 and 5 shows the results for $f_1(x)$ to $f_{10}(x)$ and list the best integer β in the interval $[-20, 20]$ for which N and $|x_N - x_{N-1}|$ are least. If the initial points are close to the root, then we obtain least number of iterations and lowest error. Table 3 shows that the present methods $4^{th}PM$ and $6^{th}PM$ are better than $2^{nd}NR$ and $3^{rd}PM$ in terms of error. Table 4 and 5 show that $4^{th}PM$ method and $6^{th}PM$ are better in terms of error when compared with similar fourth order and sixth order methods respectively.

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6. Conclusions

In this work, we have proposed two family of methods: first, a two-point fourth order method (optimal) and second a three-point sixth order method using weight functions. It is clear that the proposed first family requires only three evaluations per iterative step to obtain fourth order accuracy and the second family requires four evaluations per iterative step to get sixth order accuracy. We have thus increased the order of convergence to four and six compared to the $3^{rd}PM$ method found in [12]. The proposed new methods are also better than Newton's method and hence preferable. It is also noted that $4^{th}PM$ family has the best efficiency index among many higher order methods found in literature and hence preferable for application purpose.

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