

Dunkl positive definite functions ^{*}

Jamel El Kamel[†]

*Department of Mathematics, FSM,
Monastir 5000, Tunisia.*

and

Khaled Mehrez[‡]

Department of Mathematics, IPEIM, Monastir 5000, Tunisia.

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Abstract

We introduce the notion of Dunkl positive definite and strictly positive definite functions on \mathbb{R}^d . This done by the use of the properties of Dunkl translation. We establish the analogue of Bochner's theorem in Dunkl setting. The case of radial functions is considered. We give a sufficient condition for a function to be Dunkl strictly positive definite on \mathbb{R}^d .

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1. Introduction

In classical analysis a complex valued continuous function is said positive definite (resp. strictly positive definite) on \mathbb{R} , if for every distinct real numbers

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[†]Corresponding author. E-mail: jamel.elkamel@fsm.rnu.tn

[‡]E-mail: sujoy.katwa@gmail.com

x_1, x_2, \dots, x_n and every complex numbers z_1, z_2, \dots, z_n not all zero, the inequality

$$\sum_{j=1}^n \sum_{k=1}^n z_j \overline{z_k} f(x_j - x_k) \geq 0 \text{ (resp. } > 0 \text{)}$$

hold true. (see [7])

In 1930, the class of positive definite functions is fully characterized by Bochner's theorem [1], the function f being positive definite if and only if it is the Fourier transform of a nonnegative finite Borel measure on the real line \mathbb{R} .

In this work, we introduce the analogue of positive definite function in Dunkl setting. This done by the use of the properties of the Dunkl translation. We establish a version of Bochner's theorem in Dunkl setting. We give a sufficient condition for a function in $A_\kappa(\mathbb{R}^d)$ to be Dunkl strictly positive definite.

Our paper is organized as follows: In section 2, we present some preliminaries results and notations that will be useful in the sequel. In section 3, we give some properties of the Dunkl transform, the Dunkl translation and the Dunkl convolution. In section 4, we introduce the notion of the Dunkl positive definite functions in studying their properties, some examples are given. We prove that if $\varphi \in A_\kappa(\mathbb{R}^d)$ is Dunkl positive definite, then the Dunkl transform of φ is nonnegative and φ is bounded. The case of radial function is considered. We state a version of Bochner's theorem in Dunkl setting. As application, we are interested with the Dunkl heat kernel, and we get a new equality for the modified Bessel function. The section 5 is devoted to Dunkl strictly positive definite functions.

2. Notations and preliminaries

Let R be a fixed root system in \mathbb{R}^d , G the associated finite reflexion group, and R_+ a fixed positive subsystem of R , normalized so that $\langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R_+$, where $\langle x, y \rangle$ denotes the usual Euclidean inner product.

For a non zero $\alpha \in \mathbb{R}^d$, let us define the reflexion σ_α by

$$\sigma_\alpha x = x - 2 \frac{\langle x, \alpha \rangle}{\langle \alpha, \alpha \rangle} \alpha, \quad x \in \mathbb{R}^d.$$

Let κ be a nonnegative multiplicity function $\alpha \mapsto \kappa_\alpha$ defined on R_+ with the property that $\kappa_\alpha = \kappa_\beta$ where σ_α is conjugate to σ_β in G . The weight function

h_κ est defined by

$$h_\kappa(x) = \prod_{\alpha \in R_+} |\langle x, \alpha \rangle|^{\kappa_\alpha}, \quad x \in \mathbb{R}^d. \quad (1)$$

This is a nonnegative homogeneous function of degree $\gamma_\kappa = \sum_{\alpha \in R_+} \kappa_\alpha$, which is invariant under the reflexion group G .

Let T_i denote Dunkl's differential-difference operator defined in [2] by

$$T_i f(x) = \partial_i f(x) + \sum_{\alpha \in R_+} \kappa_\alpha \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle} \langle \alpha, e_i \rangle, \quad 1 \leq i \leq d, \quad (2)$$

where ∂_i is the ordinary partial derivative with respect to x_i , and e_1, e_2, \dots, e_d are the standard unit vectors of \mathbb{R}^d . It was proved in [2] that T_1, T_2, \dots, T_d commute. Therefore we can naturally define $P(T)$ for any polynomial P , where $T = (T_1, T_2, \dots, T_d)$.

Let \mathcal{P}_n^d denote the space of homogeneous polynomials of degree n in d -variables. The operators T_i , $1 \leq i \leq d$ map \mathcal{P}_n^d to \mathcal{P}_{n-1}^d . The intertwining operator V_κ is linear and determined uniquely as

$$V_\kappa \mathcal{P}_n^d \subset \mathcal{P}_n^d, \quad V_\kappa 1 = 1, \quad T_i V_\kappa = V_\kappa \partial_i, \quad 1 \leq i \leq d. \quad (3)$$

The Dunkl kernel E_κ associated with G and κ is defined by

$$E_\kappa(x, y) = V_\kappa(e^{\langle \cdot, y \rangle})(x), \quad x, y \in \mathbb{R}^d. \quad (4)$$

Proposition 1. (see[8]) Let $y \in \mathbb{C}^d$. Then the function $f = E_\kappa(\cdot, y)$ is the unique solution of the system

$$T_i f = \langle e_i, y \rangle f, \quad \text{for all } 1 \leq i \leq d, \quad (5)$$

which is real-analytic in \mathbb{R}^d and satisfies $f(0) = 1$.

We collect some further properties of the Dunkl kernel E_κ

Proposition 2. (see[4],[8]) For $x, y \in \mathbb{C}^d$, $\lambda \in \mathbb{C}$

1. $E_\kappa(x, y) = E_\kappa(y, x)$,
2. $E_\kappa(\lambda x, y) = E_\kappa(x, \lambda y)$

3. $\overline{E_\kappa(x, y)} = E_\kappa(\overline{x}, \overline{y})$
4. $|E_\kappa(-ix, y)| \leq 1$.
5. $|E_\kappa(x, y)| \leq e^{\|x\| \cdot \|y\|}$,
6. Let $\nu(z) = z_1^2 + \dots + z_d^2$, $z_i \in \mathbb{C}$. For $z, \omega \in \mathbb{C}^d$,

$$c_\kappa \int_{\mathbb{R}^d} E_\kappa(x, z) E_\kappa(x, \omega) h_\kappa^2(x) e^{-\frac{\|x\|^2}{2}} dx = e^{\frac{(\nu(z) + \nu(\omega))}{2}} E_\kappa(z, \omega),$$

where c_κ denotes the Mehta-type constant defined by

$$c_\kappa^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(x) e^{-\frac{\|x\|^2}{2}} dx. \quad (6)$$

In particular, the function

$$E_\kappa(x, y) = V_\kappa^{(x)}(e^{<x, y>}), \quad x, y \in \mathbb{R}^d,$$

plays the role of $e^{i<x, y>}$ in the ordinary Fourier analysis. Trought this paper, we fix the values of γ and λ as

$$\gamma := \gamma_\kappa = \sum_{\alpha \in R_+} k_\alpha \quad \text{and} \quad \lambda := \gamma + \frac{d-2}{2}. \quad (7)$$

Let us recall some classical functional spaces:

- $C(\mathbb{R}^d)$ the set of continuous functions on \mathbb{R}^d and $C_0(\mathbb{R}^d)$ its subspace of continuous functions on \mathbb{R}^d vanishing at infinity.
- $S(\mathbb{R}^d)$ the Schwartz space of infinitely differentiable functions on \mathbb{R}^d which are rapidly decreasing as their derivatives.
- $L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \leq p < \infty$, the space of measurable functions on \mathbb{R}^d such that

$$\|f\|_{\kappa, p} = \left(\int_{\mathbb{R}^d} |f(x)|^p h_\kappa^2(x) dx \right)^{\frac{1}{p}} < \infty.$$

3. Harmonic analysis related to the Dunkl operator

In this section, we present some properties of the Dunkl transform, the Dunkl translation and the Dunkl convolution studied and developed in great detail in [4,6,10,11].

The Dunkl transform is defined for $f \in L^1(\mathbb{R}^d, h_\kappa^2)$ by

$$D_\kappa f(x) = c_\kappa \int_{\mathbb{R}^d} f(y) E_\kappa(-ix, y) h_\kappa^2(y) dy, \quad x \in \mathbb{R}^d. \quad (8)$$

If $\kappa = 0$, then $V_\kappa = id$ and the Dunkl transform coincides with the usual Fourier transform. If $d = 1$ and $G = \mathbb{Z}_2$, then the Dunkl transform is related closely to the Hankel transform on the real line.

In fact, in this case,

$$E_\kappa(x, -iy) = \Gamma\left(\kappa + \frac{1}{2}\right) \left(\frac{|xy|}{2}\right)^{-\kappa + \frac{1}{2}} \left[J_{\kappa - \frac{1}{2}}(|xy|) - i \operatorname{sign}(xy) J_{\kappa + \frac{1}{2}}(|xy|) \right],$$

where J_α denotes the usual Bessel function of first kind and order α .

Theorem 1. (see [10])

1. For $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, we have $D_\kappa f \in C_0(\mathbb{R}^d)$, and

$$\|D_\kappa f\|_{C_0} \leq \|f\|_{\kappa,1}.$$

2. When both f and $D_\kappa f \in L^1(\mathbb{R}^d, h_\kappa^2)$, we have the inversion formula

$$f(x) = c_\kappa \int_{\mathbb{R}^d} D_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y) dy \quad a.e.$$

3. The Dunkl transform D_κ is an isomorphism of the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ onto it self, and $D_\kappa^2 f(x) = f(-x)$.
4. The Dunkl transform D_κ on $\mathcal{S}(\mathbb{R}^d)$ extends uniquely to an isometry of $L^2(\mathbb{R}^d, h_\kappa^2)$.
5. If $f, g \in L^2(\mathbb{R}^d, h_\kappa^2)$ then

$$\int_{\mathbb{R}^d} D_\kappa f(y) g(y) h_\kappa^2(y) dy = \int_{\mathbb{R}^d} f(y) D_\kappa g(y) h_\kappa^2(y) dy.$$

Let $y \in \mathbb{R}^d$ be given. The Dunkl translation operator $f \mapsto \tau_y f$ is defined in $L^2(\mathbb{R}^d, h_k^2)$ by the equation

$$D_\kappa(\tau_y f)(x) = E_\kappa(iy, x) D_\kappa f(x), \quad x \in \mathbb{R}^d. \quad (9)$$

The above definition gives $\tau_y f$ as an L^2 function.

Let

$$\mathcal{A}_\kappa(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d, h_\kappa^2) : D_\kappa f \in L^1(\mathbb{R}^d, h_\kappa^2)\}. \quad (10)$$

Note that $\mathcal{A}_\kappa(\mathbb{R}^d)$ is contained in the intersection of $L^1(\mathbb{R}^d, h_\kappa^2)$ and L^∞ and hence is a subspace of $L^2(\mathbb{R}^d, h_\kappa^2)$. For $f \in \mathcal{A}_\kappa(\mathbb{R}^d)$ we have

$$\tau_y f(x) = \int_{\mathbb{R}^d} E_\kappa(ix, y) E_\kappa(-iy, \xi) D_\kappa f(\xi) h_\kappa^2(\xi) d\xi, \quad \forall x \in \mathbb{R}^d. \quad (11)$$

Theorem 2. (see [10]) Assume that $f \in \mathcal{A}_\kappa(\mathbb{R}^d)$ and $g \in L^1(\mathbb{R}^d, h_\kappa^2)$ is bounded. Then

$$1. \int_{\mathbb{R}^d} \tau_y f(\xi) g(\xi) h_\kappa^2(\xi) d\xi = \int_{\mathbb{R}^d} f(\xi) \tau_{-y} f(\xi) h(\xi) d\xi.$$

$$2. \tau_y f(x) = \tau_{-x} f(-y).$$

Theorem 3. (see[10]) Let $f \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be a radial and nonnegative function. Then $T_y f \geq 0$, $T_y f \in L_\alpha^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} T_y f(x) h_\kappa^2(x) dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(y) dx. \quad (12)$$

The Dunkl convolution operator is defined on $L^2(\mathbb{R}^d, h_k^2)$ by: for $f, g \in L^2(\mathbb{R}^d, h_k^2)$,

$$f \star_\kappa g(x) = \int_{\mathbb{R}^d} f(y) \tau_x g^\vee(y) h_\kappa^2(y) dy, \quad (13)$$

where $g^\vee(y) = g(-y)$.

Note that as $\tau_x g^\vee \in L^2(\mathbb{R}^d, h_k^2)$, the above convolution is well defined. We can also write the definition as

$$f \star_\kappa g(x) = \int_{\mathbb{R}^d} D_\kappa f(\xi) D_\kappa g(\xi) E_\kappa(ix, \xi) h_\kappa^2(\xi) d\xi. \quad (14)$$

Theorem 4. (see [9,10,11])

1. Let $f, g \in L^2(\mathbb{R}^d, h_\kappa^2)$, then

$$(a) \quad D_\kappa(f \star_\kappa g) = D_\kappa f \cdot D_\kappa g.$$

$$(b) \quad f \star_\kappa g = g \star_\kappa f.$$

2. Let $f \in L^2(\mathbb{R}^d, h_\kappa^2)$ and $g \in L^1 \cap L^2(\mathbb{R}^d, h_\kappa^2)$, then $f \star_\kappa g \in L^2(\mathbb{R}^d, h_\kappa^2)$ and

$$\|f \star_\kappa g\|_{\kappa,2} \leq \|g\|_{\kappa,1} \|f\|_{\kappa,2}. \quad (15)$$

4. Dunkl Positive definite Functions

Definition 1. A continuous function φ of $L^2(\mathbb{R}^d, h_\kappa^2)$ is said Dunkl positive definite (resp. strictly Dunkl positive definite) if for every finite distinct real numbers x_1, \dots, x_n , and every complex numbers $\alpha_1, \dots, \alpha_n$, not all zero, the inequality

$$\sum_{j=1}^n \sum_{k=1}^n \alpha_j \overline{\alpha_k} \tau_{x_j}(\varphi)(x_k) \geq 0, \quad (\text{resp. } > 0)$$

holds true. Where τ_x denotes the Dunkl translation.

From definition.1 we can read of the elementary properties of a Dunkl positive definite function.

Proposition 3. (Properties of Dunkl positive definite functions)

1. A nonnegative finite linear combination of Dunkl positive definite functions is Dunkl positive definite.
2. Let φ be a Dunkl positive definite function, then

$$(a) \quad \text{The function } \tau_x \varphi(x) \geq 0, \text{ for all } x \in \mathbb{R}^d. \text{ In particular, } \varphi(0) \geq 0.$$

$$(b) \quad \overline{\varphi(-x)} = \varphi(x), \text{ for all } x \in \mathbb{R}^d.$$

Proof. 1. The first property is immediate consequence of the definition 1.

2. The second property follows by choosing $n = 1$, $\alpha_1 = 1$ and $x_1 = x$ in the definition 1.

3. In the definition 1, let $n = 2$, $x_1 = 0$, $\alpha_1 = 1$, $\alpha_2 = c$ and $x_2 = x$, then

$$\varphi(0) + |c|^2 \tau_x \varphi(x) + c\varphi(-x) + \bar{c}\varphi(x) \geq 0.$$

Setting $c = 1$ and $c = i$, respectively, we deduce that $\varphi(x) + \varphi(-x)$ and $i(\varphi(-x) - \varphi(x))$ must be reals. This can only be satisfied when $\overline{\varphi(-x)} = \varphi(x)$. ■

Corollary 1. *Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be a Dunkl positive definite function, then $D_\kappa(\varphi)$ is real.*

Proof. For $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$, we have

$$D_\kappa(\varphi)(x) = c_\kappa \int_{\mathbb{R}^d} E_\kappa(y, -ix) \varphi(y) h_\kappa^2(y) dy.$$

Hence

$$\overline{D_\kappa(\varphi)(x)} = c_\kappa \int_{\mathbb{R}^d} \overline{E_\kappa(y, -ix)} \overline{\varphi(y)} h_\kappa^2(y) dy.$$

Since $\overline{E_\kappa(x, y)} = E_\kappa(\bar{x}, \bar{y})$ for $x, y \in \mathbb{C}^d$, we obtain

$$\begin{aligned} \overline{D_\kappa(\varphi)(x)} &= c_\kappa \int_{\mathbb{R}^d} E_\kappa(y, ix) \overline{\varphi(y)} h_\kappa^2(y) dy \\ &= c_\kappa \int_{\mathbb{R}^d} E_\kappa(-y, ix) \overline{\varphi(-y)} h_\kappa^2(y) dy. \end{aligned}$$

So, by proposition 3, we have

$$\overline{\varphi(-x)} = \varphi(x),$$

and $E_\kappa(\lambda x, y) = E_\kappa(x, \lambda y)$, for any $\lambda \in \mathbb{C}$ we obtain

$$\overline{D_\kappa(\varphi)(x)} = D_\kappa(\varphi)(x). ■$$

We begin by seeking sufficient conditions for a function to be Dunkl positive definite.

Theorem 5. *Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be a nonnegative function, then $D_\kappa(\varphi)$ is Dunkl positive definite.*

Proof. For $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$, we have

$$\tau_y(D_\kappa(\varphi))(x) = \int_{\mathbb{R}^d} E_\kappa(-iy, \xi) E_\kappa(ix, \xi) \varphi(-\xi) h_\kappa^2(\xi) d\xi.$$

Thus,

$$\begin{aligned} & \sum_{j=1}^n \sum_{l=1}^n \alpha_j \overline{\alpha_l} \tau_{x_j}(D_\kappa(\varphi))(x_l) \\ &= \int_{\mathbb{R}^d} \left[\sum_{j=1}^n \sum_{l=1}^n \alpha_j \overline{\alpha_l} (E_\kappa(-ix_j, \xi) E_\kappa(ix_l, \xi)) \right] \varphi(-\xi) h_\kappa^2(\xi) d\xi \\ &= \int_{\mathbb{R}^d} \left[\sum_{j=1}^N \alpha_j E_\kappa(-ix_j, \xi) \right] \overline{\left[\sum_{l=1}^N \alpha_l E_\kappa(-ix_l, \xi) \right]} \varphi(-\xi) h_\kappa^2(\xi) d\xi \\ &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^N \alpha_j E_\kappa(-ix_j, \xi) \right|^2 \varphi(-\xi) h_\kappa^2(\xi) d\xi \geq 0. \end{aligned}$$

Which completes the proof. ■

Example 1. *For $t > 0$, the function*

$$F_t(x) = e^{-t\|x\|^2}$$

is Dunkl positive definite.

Indeed, put

$$G_t(x) = \frac{c_\kappa}{(4t)^{\gamma + \frac{d}{2}}} e^{-\frac{\|x\|^2}{4t}}.$$

Thus, G_t is nonnegative function of $L^1(\mathbb{R}^d, h_\kappa^2)$. Moreover, (see [9])

$$F_t(x) = D_\kappa(G_t)(x). \tag{16}$$

Since $F_t(x) \in L^1(\mathbb{R}^d, h_\kappa^2)$, we conclude by theorem 5.

Example 2. Consider the modified Bessel function of the second kind of order α defined by

$$K_\alpha(x) = \int_0^{+\infty} e^{-x \cosh(t)} \cosh(\alpha t) dt, \quad x > 0.$$

Using the integral representation [5,(7.12.24)]

$$2K_\alpha(ax) = a^\alpha \int_0^{+\infty} t^{-1-\alpha} e^{-\frac{x}{2}(t+\frac{a^2}{t})} dt,$$

by setting $a = r$, $x = 1$ and substituting $u = 2t$, and using $K_\alpha = K_{-\alpha}$, we have

$$K_\alpha(r) = r^{-\alpha} 2^{\alpha-1} \int_0^{+\infty} u^{\alpha-1} e^{-u} e^{-\frac{r^2}{4u}} du, . \quad (17)$$

Now, putting

$$\Phi(y) = \frac{1}{(1+ \|y\|_2^2)^p}, \quad y \in \mathbb{R}^d,$$

with $p \in \mathbb{N}$, such that $p > \frac{d}{2} + \gamma + 1$.

Since $p > \frac{d}{2} + \gamma + 1$, the function Φ is in $(L^1 \cap L^2)(\mathbb{R}^d, h_\kappa^2)$. From, the integral representation of the gamma function, for $p > 0$, we have

$$\begin{aligned} \Gamma(p) &= \int_0^\infty t^{p-1} e^{-t} dt \\ &= s^p \int_0^\infty u^{p-1} e^{-su} du. \end{aligned}$$

Let $s = 1+ \|y\|_2^2$, then we get

$$\Phi(y) = \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-u} e^{-u\|y\|_2^2} du.$$

Thus

$$\begin{aligned}
D_\kappa(\Phi)(\omega) &= c_\kappa \int_{\mathbb{R}^d} \Phi(x) E_\kappa(x, -i\omega) h_\kappa^2(x) dx \\
&= \frac{c_\kappa}{\Gamma(p)} \int_{\mathbb{R}^d} \int_0^\infty u^{p-1} e^{-u} e^{-u\|y\|_2^2} E_\kappa(y, -i\omega) h_\kappa^2(y) dy du \\
&= \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-u} \left[c_\kappa \int_{\mathbb{R}^d} e^{-u\|y\|_2^2} E_\kappa(y, -i\omega) h_\kappa^2(y) dy \right] du \\
&= \frac{1}{\Gamma(p)} \int_0^\infty u^{p-1} e^{-u} D_\kappa(F_u(\cdot))(\omega) du \\
&= \frac{c_\kappa}{\Gamma(p) 2^{\gamma+\frac{d}{2}}} \int_0^\infty u^{p-\gamma-\frac{d}{2}-1} e^{-u} e^{-\frac{\|\omega\|_2^2}{4u}} du
\end{aligned}$$

Using the relation (17) we obtain

$$D_\kappa(\Phi)(\omega) = \frac{c_\kappa}{2^{p-1}} \|\omega\|_2^{p-\gamma-\frac{d}{2}} K_{p-\gamma-\frac{d}{2}}(\|\omega\|_2). \quad (18)$$

Since for $\alpha > 0$, the even function $x^\alpha K_\alpha(x)$ is positive and belongs to $L^1([0, +\infty[, x^{2\alpha+1} dx)$, by the inversion formula and theorem 5 we deduce that $\Phi(y) = \frac{1}{(1+\|y\|_2^2)^p}$ is a Dunkl positive definite function.

Example 3. Let $\varphi \in L^2(\mathbb{R}^d, h_\kappa^2)$ be a continuous function. We consider the functions γ_t , $t > 0$, defined by

$$\gamma_t(y) = \sum_{j=1}^N \alpha_j \tau_{x_j}(G_t(y)), \quad y \in \mathbb{R}^d$$

where $\alpha_j \in \mathbb{C}$, $x_j \in \mathbb{R}^d$ for all $1 \leq j \leq N$ and G_t is the function definite in example 1.

If $\langle \varphi \star_\kappa \gamma_t, \gamma_t \rangle \geq 0$, then φ is Dunkl positive definite.

Indeed, by the definition of the generalized translation operator and equation (16), we have

$$D_\kappa(\gamma_t)(\omega) = \frac{1}{c_\kappa} \sum_{j=1}^N \alpha_j E_\kappa(-ix_j, \omega) e^{-2t\|\omega\|^2},$$

which leads to

$$\begin{aligned}
D_\kappa (\gamma_t \star_\kappa \overline{\gamma_t^\vee}) (\omega) &= |D_\kappa(\gamma_t)|^2 (\omega) \\
&= \frac{1}{c_k^2} \left| \sum_{j=1}^N \alpha_j E_\kappa(-ix_j, \omega) \right|^2 e^{-2t\|\omega\|^2} \\
&= \frac{1}{c_k} \sum_{j,l=1}^N \alpha_j \overline{\alpha_l} E_\kappa(-ix_j, \omega) \overline{E_\kappa(-ix_l, \omega)} e^{-2t\|\omega\|^2} \quad (19) \\
&= D_\kappa \left(\sum_{j,l=1}^N \alpha_j \overline{\alpha_l} \tau_{x_j} [\tau_{-x_l} G_{2t}(\cdot)] \right) (\omega).
\end{aligned}$$

Since the Dunkl transform is a topological automorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, then

$$\gamma_t \star_\kappa \overline{\gamma_t^\vee}(\gamma) = \sum_{j,l=1}^N \alpha_j \overline{\alpha_l} \tau_{x_j} [\tau_{-x_l} G_{2t}(\cdot)] (\omega),$$

i.e

$$\gamma_t \star_\kappa \overline{\gamma_t^\vee}(\omega) = \sum_{j,l=1}^N \alpha_j \overline{\alpha_l} \tau_{x_j} (\Gamma_\kappa(2t, x_l, \cdot)) (\omega),$$

where Γ_κ is the Dunkl type heat kernel. Thus,

$$\begin{aligned}
\int_{\mathbb{R}^d} \varphi(y) \gamma_t \star_\kappa \overline{\gamma_t^\vee}(y) h_\kappa^2(y) dy &= \sum_{j,l=1}^N \alpha_j \overline{\alpha_l} \int_{\mathbb{R}^d} \varphi(y) \tau_{x_j} \Gamma_\kappa(2t, x_l, y) h_\kappa^2(y) dy, \\
&= \sum_{j,l=1}^N \alpha_j \overline{\alpha_l} \int_{\mathbb{R}^d} \tau_{x_j} \varphi(y) \Gamma_\kappa(2t, x_l, y) h_\kappa^2(y) dy.
\end{aligned}$$

By theorem 4.7 in [9], we have

$$\lim_{t \rightarrow 0} \int_{\mathbb{R}^d} \varphi(y) \gamma_t \star_\kappa \overline{\gamma_t^\vee}(y) h_\kappa^2(y) dy = \sum_{j,l=1}^N \alpha_j \overline{\alpha_l} \tau_{x_j} \varphi(x_l).$$

Which completes the proof.

Proposition 4. *Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$. If φ is Dunkl positive definite function and $f \in L^2(\mathbb{R}^d, h_\kappa^2)$, then*

$$\langle \varphi \star_\kappa f, f \rangle \geq 0. \quad (20)$$

Proof. Since $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$, and $f \in L^2(\mathbb{R}^d, h_\kappa^2)$, then $\varphi \star_\kappa f \in L^2(\mathbb{R}^d, h_\kappa^2)$, and

$$\varphi \star_\kappa f(x) = \int_{\mathbb{R}^d} \tau_x \check{\varphi}(y) f(y) h_\kappa^2(y) dy.$$

Since φ is Dunkl positive definite function, then $\overline{\varphi} = \check{\varphi}$, so

$$\varphi \star_\kappa f(x) = \int_{\mathbb{R}^d} \tau_x(\overline{\varphi})(y) f(y) h_\kappa^2(y) dy.$$

Thus, for $f \in L^2(\mathbb{R}^d, h_\kappa^2)$,

$$\langle \varphi \star_\kappa f, f \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_x(\overline{\varphi})(y) f(y) \overline{f(x)} h_\kappa^2(y) dy h_\kappa^2(x) dx.$$

Let $f \in \mathcal{S}(\mathbb{R}^d)$, its known that, for $\epsilon > 0$, there exists a closed cube $W \subseteq \mathbb{R}^d$, such that

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_x(\overline{\varphi})(y) f(y) \overline{f(x)} h_\kappa^2(y) dy h_\kappa^2(x) dx - \int_W \int_W \tau_x(\overline{\varphi})(y) f(y) \overline{f(x)} h_\kappa^2(y) dy h_\kappa^2(x) dx \right| < \frac{\epsilon}{2}.$$

But the double integral over the cubes is the limit of Riemannian sums. Hence, we can find $x_1, \dots, x_N \in \mathbb{R}^d$ and weights $\omega_1, \dots, \omega_N$ such that

$$\left| \int_W \int_W \tau_x(\overline{\varphi})(y) f(y) \overline{f(x)} h_\kappa^2(y) dy h_\kappa^2(x) dx - \sum_{j,l=1}^N \tau_{x_j} \varphi(x_l) f(x_j) \omega_j \overline{f(x_l) \omega_l} \right| < \frac{\epsilon}{2}.$$

This means that

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \tau_x(\overline{\varphi})(y) f(y) \overline{f(x)} h_\kappa^2(y) dy h_\kappa^2(x) dx > \sum_{j,l=1}^N \tau_{x_j} \varphi(x_l) f(x_j) \omega_j \overline{f(x_l) \omega_l} - \epsilon.$$

Letting ϵ tend to zero and using that φ is Dunkl positive definite function shows that (20) is true for all $f \in L^2(\mathbb{R}^d, h_\kappa^2)$. \blacksquare

Corollary 2. Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be a Dunkl positive definite function, we define $\lambda : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$ by

$$\lambda(\gamma) = \int_{\mathbb{R}^d} \varphi(x) D_\kappa^{-1}(\gamma)(x) h_\kappa^2(x) dx. \quad (21)$$

If $\gamma = |\psi|^2$ with $\psi \in \mathcal{S}(\mathbb{R}^d)$ and even, then $\lambda(\gamma)$ is nonnegative.

Proof. Put $f = D_\kappa^{-1}(\psi)$. Since ψ is even, then f and $D_\kappa(f)$ are even, and $D_\kappa(f)(x) = \psi(x)$.

Thus,

$$\gamma(x) = \psi(x)\overline{\psi(x)} = D_\kappa f(x).\overline{D_\kappa f(x)} = D_\kappa f(x).D_\kappa \bar{f}(x) = D_\kappa (f \star_\kappa \bar{f})(x).$$

Then,

$$\begin{aligned} \lambda(\gamma) &= \int_{\mathbb{R}^d} \varphi(x) D_\kappa^{-1}(\gamma)(x) h_\kappa^2(x) dx \\ &= \int_{\mathbb{R}^d} \varphi(x) (f \star_\kappa \bar{f})(-x) h_\kappa^2(x) dx \\ &= \int_{\mathbb{R}^d} \varphi(x) \left[\int_{\mathbb{R}^d} \tau_{-x} f(y) \overline{f(y)} h_\kappa^2(y) dy \right] h_\kappa^2(x) dx \\ &= \int_{\mathbb{R}^d} \overline{f(y)} \left[\int_{\mathbb{R}^d} \varphi(x) \tau_{-x} f(y) h_\kappa^2(x) dx \right] h_\kappa^2(y) dy \\ &= \int_{\mathbb{R}^d} \overline{f(y)} \left[\int_{\mathbb{R}^d} \varphi(x) \tau_{-y} f(x) h_\kappa^2(x) dx \right] h_\kappa^2(y) dy \\ &= \int_{\mathbb{R}^d} \overline{f(y)} (\varphi \star_\kappa f)(-y) h_\kappa^2(y) dy \\ &= \int_{\mathbb{R}^d} \overline{f(-y)} (\varphi \star_\kappa f)(y) h_\kappa^2(y) dy \\ &= \int_{\mathbb{R}^d} \overline{f(y)} (\varphi \star_\kappa f)(y) h_\kappa^2(y) dy \\ &= \langle \varphi \star_\kappa f, f \rangle \geq 0. \end{aligned}$$

■

Proposition 5. Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$. If φ is a Dunkl positive definite function, then

$$D_\kappa \varphi(x) \geq 0, \quad \forall x \in \mathbb{R}^d.$$

Proof. For $\varphi, f \in \mathcal{A}_\kappa(\mathbb{R}^d)$, we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |D_\kappa(f)|^2(\xi) D_\kappa(\varphi)(\xi) h_\kappa^2(\xi) d\xi &= \int_{\mathbb{R}^d} D_\kappa(f)(\xi) \overline{D_\kappa(f)(\xi)} D_\kappa(\varphi)(\xi) h_\kappa^2(\xi) d\xi \\
&= c_\kappa^2 \int_{\mathbb{R}^d} D_\kappa(\varphi)(\xi) \left[\int_{\mathbb{R}^d} f(x) E_\kappa(x, -i\xi) h_\kappa^2(x) dx \right] \\
&\quad \times \left[\int_{\mathbb{R}^d} \overline{f(y)} E_\kappa(y, i\xi) h_\kappa^2(y) dy \right] h_\kappa^2(\xi) d\xi \\
&= c_\kappa^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \overline{f(y)} \left[\int_{\mathbb{R}^d} E_\kappa(x, -i\xi) E_\kappa(y, i\xi) \right. \\
&\quad \left. D_\kappa(\varphi)(\xi) h_\kappa^2(\xi) d\xi \right] \times h_\kappa^2(x) dx h_\kappa^2(y) dy \\
&= c_\kappa^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x) \overline{f(y)} \tau_x \varphi(y) h_\kappa^2(x) dx h_\kappa^2(y) dy \\
&= c_\kappa^2 \langle \overline{\varphi} \star_\kappa f, f \rangle \geq 0.
\end{aligned}$$

Since φ is Dunkl positive definite then $D_\kappa \varphi$ is real. Since the last inequality holds for an arbitrary function $f \in \mathcal{A}_\kappa(\mathbb{R}^d)$, we conclude. \blacksquare

Corollary 3. *Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$. If φ is a Dunkl positive definite function, then φ is bounded and*

$$|\varphi(x)| \leq \varphi(0), \quad \forall x \in \mathbb{R}^d.$$

Proof. In definition 1, let $n = 2, \alpha_1 = |\varphi(x)|, \alpha_2 = -\overline{\varphi(x)}, x_1 = 0$ and $x_2 = x$, we have

$$\varphi(0)|\varphi(x)|^2 - \varphi(-x)|\varphi(x)|\varphi(x) - \overline{\varphi(x)}|\varphi(x)|\varphi(x) + |\varphi(x)|^2 \tau_x \varphi(x) \geq 0.$$

Since $\overline{\varphi(-x)} = \varphi(x)$, we obtain

$$|\varphi(x)|^2 [\varphi(0) - 2|\varphi(x)| + \tau_x \varphi(x)] \geq 0$$

i.e

$$|\varphi(x)| \leq \frac{1}{2}(\varphi(0) + \tau_x \varphi(x)). \quad (22)$$

Furthermore, by the definition of Dunkl translation, and since φ is Dunkl positive definite function, we have

$$\begin{aligned}
 \tau_x \varphi(x) &= |\tau_x \varphi(x)| = \left| \int_{\mathbb{R}^d} |E_\kappa(ix, \xi)|^2 D_\kappa \varphi(\xi) h_\kappa^2(\xi) d\xi \right| \\
 &\leq \int_{\mathbb{R}^d} |D_\kappa \varphi(\xi)| h_\kappa^2(\xi) d\xi \\
 &= \int_{\mathbb{R}^d} D_\kappa \varphi(\xi) h_\kappa^2(\xi) d\xi \\
 &= \varphi(0).
 \end{aligned} \tag{23}$$

The relations (22) and (23) lead to

$$|\varphi(x)| \leq \varphi(0).$$

■

Corollary 4. *Let $\varphi_1, \varphi_2 \in \mathcal{A}_\kappa(\mathbb{R}^d)$. If φ_1, φ_2 are Dunkl positive definite functions, then the convolution product $\varphi_1 \star_\kappa \varphi_2$ is also.*

Proof. For $\varphi_1, \varphi_2 \in \mathcal{A}_\kappa(\mathbb{R}^d)$, we have

$$\varphi_1 \star_\kappa \varphi_2 \in L^1(\mathbb{R}^d, h_\kappa^2),$$

and

$$D_\kappa(\varphi_1 \star_\kappa \varphi_2) = D_\kappa \varphi_1 \cdot D_\kappa \varphi_2 \in L^1(\mathbb{R}^d, h_\kappa^2).$$

Now, for every complex numbers $\alpha_1, \dots, \alpha_n$ and for every distinct real numbers x_1, \dots, x_n , we have

$$\begin{aligned}
 &\sum_{j=1}^n \sum_{l=1}^n \alpha_j \overline{\alpha_l} \tau_{x_j}(\varphi_1 \star_\kappa \varphi_2)(x_l) \\
 &= \int_{\mathbb{R}^d} \sum_{j=1}^N \sum_{l=1}^N \alpha_j \overline{\alpha_l} E_\kappa(ix_j, \xi) E_\kappa(-ix_l, \xi) D_\kappa(\varphi_1 \star_\kappa \varphi_2)(\xi) h_\kappa^2(\xi) d\xi \\
 &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^n \alpha_j E_\kappa(ix_j, \xi) \right|^2 D_\kappa(\varphi_1)(\xi) D_\kappa(\varphi_2)(\xi) h_\kappa^2(\xi) d\xi \\
 &\geq 0,
 \end{aligned}$$

where the last inequality follows from Proposition 5. ■

Proposition 6. *Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be a radial and Dunkl positive definite function. If $f \in \mathcal{A}_\kappa(\mathbb{R}^d)$ is a positive radial function, then the product $\varphi D_\kappa f$ is a Dunkl positive definite function.*

Proof. Since $\varphi, f \in \mathcal{A}_\kappa(\mathbb{R}^d)$, and radials we have

$$\varphi D_\kappa f \in \mathcal{A}_\kappa(\mathbb{R}^d),$$

and radial function. Thus,

$$\begin{aligned} \sum_{j=1}^n \sum_{l=1}^n \alpha_j \overline{\alpha_l} \tau_{x_j}(\varphi D_\kappa f)(x_l) &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^n \alpha_j E_\kappa(ix_j, \xi) \right|^2 D_\kappa(\varphi D_\kappa f)(\xi) h_\kappa^2(\xi) d\xi \\ &= \int_{\mathbb{R}^d} \left| \sum_{j=1}^n \alpha_j E_\kappa(ix_j, \xi) \right|^2 D_\kappa \varphi \star_\kappa f(\xi) h_\kappa^2(\xi) d\xi. \end{aligned}$$

Moreover, by the definition of Dunkl convolution, we can write

$$D_\kappa \varphi \star_\kappa f(x) = \int_{\mathbb{R}^d} D_\kappa \varphi(t) \tau_x \check{f}(t) h_\kappa^2(t) dt. \quad (24)$$

From proposition 5 and theorem 3.4 in [10], we have

$$D_\kappa \varphi \star_\kappa f(x) \geq 0.$$

Which completes the proof. ■

Corollary 5. *Let $\varphi_1, \varphi_2 \in \mathcal{A}_\kappa(\mathbb{R}^d)$ are radials. If φ_1, φ_2 are Dunkl positive definite functions, then the product $\varphi_1 \varphi_2$ is also.*

Proof. Let $\psi = D_\kappa \varphi_2$, then $D_\kappa \psi = \varphi_2$, and since φ_2 is radial, we have ψ is radial. So, by proposition 5, we have

$$\psi \geq 0.$$

By proposition 6, we conclude. ■

In the following we state a version of Bochner's theorem in Dunkl setting and we establish a necessary and sufficient condition for a function to be a Dunkl positive definite.

Theorem 6. (Bochner) *Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$. Then, φ is Dunkl positive definite, if and only if, there exist a nonnegative function $\psi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ such that*

$$\varphi = D_\kappa \psi. \quad (25)$$

Proof. Since φ is Dunkl positive definite function, we have $\overline{\varphi(-x)} = \varphi(x)$, and $D_\kappa \varphi$ is even real function (see corollary 1). By the inversion formula we have

$$\varphi(x) = D_\kappa^2 \varphi(x), \quad \forall x \in \mathbb{R}^d.$$

Let

$$\psi(x) = D_\kappa \varphi(x).$$

By proposition 5, we deduce that ψ is nonnegative function of $\mathcal{A}_\kappa(\mathbb{R}^d)$. Inversely, since ψ is nonnegative function and belong to $\mathcal{A}_\kappa(\mathbb{R}^d)$, by theorem 5 we deduce that $\varphi = D_\kappa \psi$ is Dunkl positive definite function. ■

4.1 Applications

Proposition 7. *Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be a radial function. If φ is Dunkl positive definite function, then there exist a nonnegative radial function $\psi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ such that*

$$1. \quad \tau_y \psi \geq 0,$$

$$2. \quad \tau_y \psi \in L^1(\mathbb{R}^d, h_\kappa^2),$$

and

$$\| \tau_y \psi \|_{1,\kappa} = \| \psi \|_{1,\kappa} = \varphi(0).$$

Proof. Bochner's theorem asserts that there exist a nonnegative function ψ such that

$$\varphi = D_\kappa \psi.$$

Since φ is radial, then $\psi = D_\kappa \varphi$ is radial, nonnegative and belongs to $\mathcal{A}_\kappa(\mathbb{R}^d)$. Using theorem 3.4 in [10], we get

$$(i) \quad \tau_y \psi \geq 0, \quad \tau_y \psi \in L^1(\mathbb{R}^d, h_\kappa^2).$$

$$(ii) \quad \| \tau_y \psi \|_{1,\kappa} = \| \psi \|_{1,\kappa} = \varphi(0).$$

■

Corollary 6. *For $t > 0$ and $x \in \mathbb{R}^d$, we have*

$$\Gamma_\kappa(t, x, y) \geq 0, \quad \forall y \in \mathbb{R}^d$$

$$\Gamma_\kappa(t, x, y) \in L^1(\mathbb{R}^d, h_\kappa^2(y)),$$

and

$$\int_{\mathbb{R}^d} \Gamma_\kappa(t, x, y) h_\kappa^2(y) dy = 1. \quad (26)$$

Where $\Gamma_\kappa(t, x, y)$ is the Dunkl type heat kernel defined by

$$\Gamma_\kappa(t, x, y) = \frac{c_\kappa}{(4t)^{\gamma + \frac{d}{2}}} e^{-\left(\frac{\|x\|^2 + \|y\|^2}{4t}\right)} E_\kappa\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right).$$

Proof. For $t > 0$, the function $\varphi(x) = e^{-t\|x\|^2} = F_t(x)$ is Dunkl positive definite function (see example 1), radial and belongs to $\mathcal{A}_\kappa(\mathbb{R}^d)$. Moreover,

$$\psi(x) = D_\kappa(\varphi)(x) = \frac{c_\kappa}{(4t)^{\gamma + \frac{d}{2}}} e^{-\frac{\|x\|^2}{4t}}.$$

Then

$$\Gamma_\kappa(t, x, y) = \tau_x(\psi)(y) = \tau_x(D_\kappa\varphi)(y).$$

By the last proposition $\Gamma_\kappa(t, x, y)$ is nonnegative, belongs to $L^1(\mathbb{R}^d, h_\kappa^2)$, and we have

$$\begin{aligned} \int_{\mathbb{R}^d} \Gamma_\kappa(t, x, y) h_\kappa^2(y) dy &= \int_{\mathbb{R}^d} \tau_x(\psi)(y) h_\kappa^2(y) dy \\ &= \int_{\mathbb{R}^d} \psi(y) h_\kappa^2(y) dy \\ &= \int_{\mathbb{R}^d} D_\kappa\varphi(y) h_\kappa^2(y) dy \\ &= \varphi(0) = 1. \end{aligned}$$

■

Corollary 7. For $p \geq \gamma + \frac{d}{2} + 1$, let K_α be the modified Bessel of the second kind and order α , then

$$\begin{aligned} & \int_{\mathbb{R}^d} \tau_y \left(\|x\|^{p-\gamma-\frac{d}{2}} K_{p-\gamma-\frac{d}{2}}(\|x\|) \right) (\xi) h_\kappa^2(\xi) d\xi \\ &= \int_{\mathbb{R}^d} \|x\|^{p-\gamma-\frac{d}{2}} K_{p-\gamma-\frac{d}{2}}(\|x\|) h_\kappa^2(x) dx = \frac{\Gamma(p)}{c_\kappa 2^{p-1}}. \end{aligned}$$

Proof. Let $p \geq \gamma + \frac{d}{2} + 1$, be an integer. Put $\varphi(y) = \frac{1}{(1+\|y\|^2)^p}$, then φ is Dunkl positive definite function (see example 2), we have

$$\psi(\lambda) = D_\kappa(\varphi)(\lambda) = \frac{c_\kappa}{\Gamma(p)2^{p-1}} \| \lambda \|^{p-\gamma-\frac{d}{2}} K_{p-\gamma-\frac{d}{2}}(\| \lambda \|).$$

By the last proposition we have

$$\begin{aligned} \int_{\mathbb{R}^d} \tau_x(\psi)(y) h_\kappa^2(y) dy &= \int_{\mathbb{R}^d} \psi(y) h_\kappa^2(y) dy \\ &= \int_{\mathbb{R}^d} D_\kappa \varphi(y) h_\kappa^2(y) dy \\ &= \varphi(0) = 1. \end{aligned}$$

■

5. Strictly Dunkl Positive Definite Functions

Lemma 1. Let $U \subseteq \mathbb{R}^d$ is open. Suppose that $x_1, \dots, x_n \in \mathbb{R}^d$, are pairwise distinct and that $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$. If $\sum_{j=1}^n \alpha_j E_\kappa(ix_j, \omega) = 0$, for all $\omega \in U$, then $\alpha \equiv 0$.

Proof. Suppose that

$$\sum_{j=1}^n \alpha_j E_\kappa(ix_j, \omega) = 0, \quad \forall \omega \in U.$$

Since $z \longrightarrow E_\kappa(y, z)$ is analytic on \mathbb{C} , by analytic continuation, we get

$$\sum_{j=1}^n \alpha_j E_\kappa(ix_j, \omega) = 0, \quad \forall \omega \in \mathbb{R}^d.$$

Let f be a C^∞ function with compactly supported, we know that

$$D_\kappa(\tau_x f)(\lambda) = E_\kappa(-ix, \lambda) D_\kappa f(\lambda).$$

Then

$$\sum_{j=1}^n \alpha_j E_\kappa(ix_j, \omega) = D_\kappa \left(\sum_{j=1}^n \alpha_j \tau_{x_j} f \right) (\lambda) = 0.$$

Since for all $j \in \{1, \dots, n\}$, τ_{x_j} is C^∞ function with compactly supported, then we get

$$\sum_{j=1}^n \alpha_j \tau_{x_j} f(\lambda) = 0, \quad \forall \lambda \in \mathbb{R}^d. \quad (27)$$

If the support of f is contained in the ball around zero with radius $\epsilon < \min_{j \neq k} \left| \|x_k\| - \|x_j\| \right|$, we have (see [10] proposition 3.13), $\tau_{x_j} f$ is supported in $\{x, \|x\| \leq \epsilon + \|x_j\|\}$.

Thus

$$\tau_{x_j} f(x_k) = 0, \quad \forall k \neq j; \quad \tau_{x_j} f(x_j) \neq 0, \quad \forall j, k \in \{1, \dots, n\}$$

Using (27), we obtain

$$\alpha_j \tau_{x_j} f(x_j) = 0, \quad \forall j \in \{1, \dots, n\}.$$

We conclude. ■

Theorem 7. Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$, be a nonidentically zero and Dunkl positive definite function. Then φ is Dunkl strictly positive definite.

Proof. Let $\varphi \in \mathcal{A}_\kappa(\mathbb{R}^d)$ be nonidentically zero and Dunkl positive definite function. Suppose that there exist distinct real points x_1, \dots, x_n and complex numbers $\alpha_1, \dots, \alpha_n$ not all zero, such that

$$\sum_{j=1}^n \sum_{l=1}^n \alpha_j \overline{\alpha_l} \tau_{x_j} \varphi(x_l) = 0.$$

By (11), we get

$$\int_{\mathbb{R}^d} \left| \sum_{j=1}^n \alpha_j E_{\kappa}(-ix_j, \xi) \right|^2 D_{\kappa} \varphi(\xi) h_{\kappa}^2(\xi) d\xi = 0.$$

Since φ is Dunkl positive definite and belongs to $\mathcal{A}_{\kappa}(\mathbb{R}^d)$, we have $D_{\kappa} \varphi$ is nonnegative continuous function. Then

$$\left| \sum_{j=1}^n \alpha_j E_{\kappa}(-ix_j, \xi) \right|^2 D_{\kappa} \varphi(\xi) = 0, \quad \forall \xi \in \mathbb{R}^d.$$

Since $D_{\kappa} \varphi$ is nonidentically zero, then there exist an open $U \subset \mathbb{R}^d$ such that

$$D_{\kappa} \varphi(\xi) \neq 0, \quad \forall \xi \in U.$$

Thus

$$\left| \sum_{j=1}^n \alpha_j E_{\kappa}(-ix_j, \xi) \right| = 0, \quad \forall \xi \in U.$$

Using, lemma 1, we get

$$\alpha_j = 0, \quad \forall j \in \{1, \dots, n\}.$$

We conclude. ■

Example 4. The functions $\varphi(x) = e^{-t\|x\|^2}$, $t > 0$, and $\psi(x) = \frac{1}{(1+\|x\|^2)^p}$, $p \geq \gamma + \frac{d}{2} + 1$ are Dunkl strictly positive definite functions.

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