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Some Subordination Theorems of Univalent Functions Defined by Linear Operators *

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Abstract

In this paper we study some applications of the theory of differential subordination defined on the space of univalent functions which are defined by linear operators. Also, some examples are given.

Keywords and Phrases: *Analytic functions, Convex functions, Linear operators, Differential subordination.*

1. Introduction

Let $\mathcal{H}(\mathbb{U})$ be the class of analytic functions in the open unit disc U=$\{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(\mathbb{U})$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{p+1} + ... \,(a \in \mathbb{C};\ z \in U)\,.$$

Also, let $\mathcal{A}$ be the subclass of the functions $f \in \mathcal{H}(\mathbb{U})$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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and set $\mathcal{A}_1 = \mathcal{A}$ the class of univalent functions. Let $\mathcal{K}$ denotes the class of all convex functions in $\mathcal{A}$ which are satisfy the condition

$$\mathcal{K} = \left\{ f \in \mathcal{A} : \operatorname{Re}\left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \ \ (z \in \mathbb{U}) \right\}.$$

For $f, g \in \mathcal{H}(\mathrm{U})$, we say that f is subordinate to g, or g is superordinate to f, written as $f \prec g$ or $f(z) \prec g(z)$, if there exists a Schwarz function $\omega(z)$, which (by definition) is analytic in U with $\omega(0) = 0$ and $|\omega(z)| < 1 \quad (z \in \mathrm{U})$ such that $f(z) = g(\omega(z)) \ \ (z \in \mathrm{U})$. Furthermore, if the function g is univalent in U, then we have the following equivalence:

$$f(z) \prec g(z) \quad \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let for $m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ and $\mu > 0, \lambda \geq 0, \ell > -1, a, c \in \mathbb{C}$ be such that $\operatorname{Re}(c - a) > 0$ and $\operatorname{Re}(a) > -\mu$, we consider the linear operator $\mathcal{J}^m_{\lambda,\ell}(a, c, \mu) : \mathcal{A} \longrightarrow \mathcal{A}$ was introduced by Raina and Sharma [13], where

$$\mathcal{J}^m_{\lambda,\ell}(a, c, \mu)f(z) = z + \frac{\Gamma(c+\mu)}{\Gamma(a+\mu)} \sum_{n=2}^{\infty} \left(1 + \frac{\lambda(n-1)}{1+\ell}\right)^m \frac{\Gamma(a+n\mu)}{\Gamma(c+n\mu)} a_n z^n. \quad (1.2)$$

It is readily verified from (1.2) that

$$\mathcal{J}^m_{\lambda,\ell}(a+1, c, \mu)f(z) = \frac{a}{a+\mu} \mathcal{J}^m_{\lambda,\ell}(a, c, \mu)f(z) + \frac{\mu}{a+\mu} z \left(\mathcal{J}^m_{\lambda,\ell}(a, c, \mu)f(z)\right)' \quad (1.3)$$

and

$$\mathcal{J}^{m+1}_{\lambda,\ell}(a, c, \mu)f(z) = (1 - \frac{\lambda}{1+\ell}) \mathcal{J}^m_{\lambda,\ell}(a, c, \mu)f(z) + \frac{\lambda}{1+\ell} z \left(\mathcal{J}^{m,1}_{\lambda,\ell}(a, c, \mu)f(z)\right)'. \quad (1.4)$$

We may point out here that some of the special cases of the operator defined by (1.2) can be found in [1], [2], [3], [5], [6], [7], [9], [14], [15].

Suppose that $\psi : \mathbb{C}^3 \times U \longrightarrow \mathbb{C}$ and let h be univalent in U. If $p(z)$ is analytic in U and satisfies the following second-order differential subordination

$$\psi\left(p(z), zp'(z), z^2 p''(z)\right) \prec h(z) \ \ (z \in U), \quad (1.5)$$

then $p(z)$ is called a solution of the differential subordination (1.5). The univalent function $q(z)$ is called a dominant of the solution of the differential subordination (1.5) or more simply, a dominant if $p(z) \prec q(z)$ for all $p(z)$ satisfying (1.5). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.5) is said to be the best dominant of (1.5).

2. The Main Results

To prove main results, we need the following lemmas:

Lemma 1. *(Hallenbeck and Ruscheweyh [4]; see also [10, Theorem 3.1.6, p.71]). Let h be a convex function in U with $h(0) = a, 0 \neq \gamma \in \mathbb{C}$ and* $\operatorname{Re}\gamma \geq 0$. *If $p(z) \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z),$$

then

$$p(z) \prec q(z) \prec h(z),$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt.$$

The function q is convex and is the best dominant.

Lemma 2. *([11, Lemma 13.5.1, p.375]). Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha zg'(z) \quad (z \in U),$$

where $\alpha > 0$ and n a positive integer. If

$$p(z) = g(0) + p_n z^n + p_{n+1}z^{n+1} + ...(z \in U),$$

is holomorphic in U, and

$$p(z) + \alpha zp'(z) \prec h(z) \quad (z \in U),$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

Lemma 3. *([10, p.7], see also [12]). For real or complex numbers a, b and c $(c \neq 0, -1, -2, ...)$, we have*
(i)

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a,b;c;z);$$

(ii)

$$ {}_2F_1\left(a,b;c;z\right) = (1-z)^{-a}\ {}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right); $$

(iii)

$$ {}_2F_1\left(1,1;3;\frac{z}{z-1}\right) = \frac{2(z-1)}{z}\left[1+\frac{\ln(1-z)}{z}\right]. $$

Unless otherwise mentioned we assume throughout the paper that $m \in \mathbb{Z}, \ell > -1, \lambda \geq 0, \mu > 0, a, c \in \mathbb{C}$ *be such that* $\mathrm{Re}(c-a) > 0$ *and* $\mathrm{Re}(a) > -\mu$.

Theorem 1. *Let*

$$ h(z) = \left(\frac{1+Az}{1+Bz}\right)^r \quad (|A| \leq 1, |B| < 1, 0 < r \leq 1). \tag{2.1} $$

If $f \in \mathcal{A}$ *satisfies the differential subordination:*

$$ \left(\mathcal{J}_{\lambda,\ell}^{m+1}(a,c,\mu)f(z)\right)' \prec h(z) \quad (z \in U), \tag{2.2} $$

then

$$ \left(\mathcal{J}_{\lambda,\ell}^{m}(a,c,\mu)f(z)\right)' \prec q(z), \tag{2.3} $$

where

$$ q(z) = \begin{cases} \left(\frac{A}{B}\right)^r \sum_{i\geq 0} \frac{(-r)i}{i!}\left(\frac{A-B}{A}\right)^i (1+Bz)^{-i} 2F_1\left(i,1,1+\frac{1+\ell}{\lambda};\frac{Bz}{1+Bz}\right) & (B \neq 0), \\ 2F_1(-r,\frac{1+\ell}{\lambda},1+\frac{1+\ell}{\lambda};-Az) & (B = 0) \end{cases} \tag{2.4} $$

and the function $q(z)$ *is convex and is the best dominant.*

Proof. We first observe ([16, p.16]; see also [12, p.132]) that the function defined by (2.1) is analytic and convex univalent in U, since

$$ \begin{aligned} \mathrm{Re}\left(1+\frac{zh''(z)}{h'(z)}\right) &= -1+(1-r)\,\mathrm{Re}\left(\frac{1}{1+Az}\right)+(1+r)\,\mathrm{Re}\left(\frac{1}{1+Bz}\right) \\ &> -1+\frac{1-r}{1+|A|}+\frac{1+r}{1+|B|} \geq 0 \quad (z \in U). \end{aligned} $$

Differentiating the recurrence relation (1.5) with respect to z, we get

$$ \left(\mathcal{J}_{\lambda,\ell}^{m+1}(a,c,\mu)f(z)\right)' = \left(\mathcal{J}_{\lambda,\ell}^{m}(a,c,\mu)f(z)\right)' + \frac{\lambda z}{1+\ell}\left(\mathcal{J}_{\lambda,\ell}^{m}(a,c,\mu)f(z)\right)'' \tag{2.5} $$

By using (2.5) in view of (2.2), we have

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' + \frac{\lambda z}{1+\ell}\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)'' \prec h(z) = \left(\frac{1+Az}{1+Bz}\right)^r \quad (z \in U) \tag{2.6}$$

Let

$$p(z) = \left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \tag{2.7}$$

then (2.6) and (2.7) yield the differential subordination

$$p(z) + \frac{\lambda z p'(z)}{1+\ell} \prec h(z) = \left(\frac{1+Az}{1+Bz}\right)^r \quad (z \in U).$$

Applying now Lemma 1, we conclude that

$$p(z) \prec q(z) = \frac{1}{(\frac{\lambda}{1+\ell})z^{\frac{1+\ell}{\lambda}}}\int_0^z h(t)t^{\frac{1+\ell}{\lambda}-1}dt = \frac{1}{(\frac{\lambda}{1+\ell})z^{\frac{1+\ell}{\lambda}}}\int_0^z \left(\frac{1+At}{1+Bt}\right)^r t^{\frac{1+\ell}{\lambda}-1}dt.$$

To evaluate the above integral, we first express the integrand in the form

$$t^{\frac{1+\ell}{\lambda}-1}\left(\frac{1+At}{1+Bt}\right)^r = \left(\frac{A}{B}\right)^r t^{\frac{1+\ell}{\lambda}-1}\left(1-\frac{A-B}{A(1+Bt)}\right)^r,$$

using Lemma 3 and some calculations, we have

$$q(z) = \left(\frac{A}{B}\right)^r \sum_{i\geq 0}\frac{(-r)i}{i!}\left(\frac{A-B}{A}\right)^i (1+Bz)^{-i} 2F_1\left(i,1,1+\frac{1+\ell}{\lambda};\frac{Bz}{1+Bz}\right)(B \neq 0)$$

On the other hand, if $B = 0$, we have

$$q(z) = \frac{1}{(\frac{\lambda}{1+\ell})z^{\frac{1+\ell}{\lambda}}}\int_0^z (1+At)^r t^{\frac{1+\ell}{\lambda}-1}dt,$$

which upon integrating similarly as above, we have

$$q(z) = 2F_1\left(-r,\frac{1+\ell}{\lambda},1+\frac{1+\ell}{\lambda};-Az\right). \tag{2.8}$$

In view of Lemma 1 (for $\gamma = \frac{1+\ell}{\lambda}, n = 1$), we assert that

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \prec q(z) \prec h(z),$$

where $q(z)$ is given by (2.4) is convex and the best dominant, which completes the proof of Theorem 1.

Remark. Putting $r = 1, A = 2\alpha - 1 \ \ (0 \leq \alpha < 1)$ and $B = 0$ in (2.8), we get

$$2F_1\left(-1, \frac{1+\ell}{\lambda}, 1 + \frac{1+\ell}{\lambda}; (1-2\alpha)z\right) = 1 + \frac{(2\alpha-1)(1+\ell)}{1+\ell+\lambda} z,$$

therefore, $q(z)$ given by (2.4) becomes

$$q(z) = \begin{cases} 2\alpha - 1 - 2(\alpha-1)(1+z)^{-1} 2F_1(1, 1, 1 + \frac{1+\ell}{\lambda}, \frac{z}{1+z}) & \text{for } B = 1, \\ 1 + \frac{(2\alpha-1)(1+\ell)}{1+\ell+\lambda} z & \text{for } B = 0. \end{cases}$$

Example 1. If $m = 0, r = 1, \lambda > 0, a, c \in \mathbb{C}, \mathrm{Re}(c-a) > 0, \mathrm{Re}(a) > -\mu, A = 1$ and $B = 0$, then from Theorem 1, we deduce the following assertion:

$$\left[(1 - \frac{\lambda}{\ell+1})\mathcal{I}_\mu^{a,c} f(z) + \frac{\lambda}{\ell+1} z \left(\mathcal{I}_\mu^{a,c} f(z)\right)'\right]' \prec 1 + z \ \ (f(z) \in \mathcal{A}\ , z \in U)$$

then

$$\left[\mathcal{I}_\mu^{a,c} f(z)\right]' \prec 1 + \frac{1+\ell}{1+\ell+\lambda} z \ \ (z \in U),$$

where $\mathcal{I}_\mu^{a,c}$ is the Erdelyi-Kober type integral operator [8].

Theorem 2. *Let q be a convex function in U with $q(0) = 1$ and let*

$$h(z) = q(z) + \frac{\lambda}{\ell+1} z q'(z) \ \ (z \in U).$$

If $f \in \mathcal{A}$ satisfies the differential subordination:

$$\left(\mathcal{J}_{\lambda,\ell}^{m+1}(a, c, \mu) f(z)\right)' \prec h(z), \tag{2.9}$$

then

$$\left(\mathcal{J}_{\lambda,\ell}^{m}(a, c, \mu) f(z)\right)' \prec q(z) \ \ (z \in U). \tag{2.10}$$

Proof. Making use of (2.7) in (2.5), then the differential subordination (2.9) becomes

$$p(z) + \frac{\lambda}{1+\ell} z p'(z) \prec h(z) = q(z) + \frac{\lambda}{1+\ell} z q'(z).$$

Applying Lemma 2, we have $p(z) \prec q(z)$, which implies that

$$\left(\mathcal{J}_{\lambda,\ell}^{m}(a,c,\mu)f(z)\right)' \prec q(z).$$

The proof of Theorem 2 is completed.

Example 2. If $m = 0, \lambda > 0, \ell \geq 0, a, c \in \mathbb{C}, \mathrm{Re}(c-a) > 0$, $\mathrm{Re}(a) > -\mu$ and $q(z) = \frac{1-z}{1+z}$ in Theorem 2, we obtain the following result:

$$\left[(1-\frac{\lambda}{\ell+1})\mathcal{I}_{\mu}^{a,c}f(z) + \frac{\lambda}{\ell+1}z\left(\mathcal{I}_{\mu}^{a,c}f(z)\right)'\right]' \prec \frac{1-\frac{2\lambda}{\ell+1}z - z^2}{(1+z)^2} \quad (f(z) \in \mathcal{A}\ , z \in U),$$

then

$$\left[\mathcal{I}_{\mu}^{a,c}f(z)\right]' \prec \frac{1-z}{1+z} \quad (z \in U).$$

Theorem 3. *Let q be a convex function in U with $q(0) = 1$ and let*

$$h(z) = q(z) + zq'(z) \quad (z \in U).$$

If $f \in \mathcal{A}$ satisfies the differential subordination:

$$\left(\mathcal{J}_{\lambda,\ell}^{m}(a,c,\mu)f(z)\right)' \prec h(z), \tag{2.11}$$

then

$$\frac{\mathcal{J}_{\lambda,\ell}^{m}(a,c,\mu)f(z)}{z} \prec q(z) \quad (z \in U). \tag{2.12}$$

Proof. Let

$$\frac{\mathcal{J}_{\lambda,\ell}^{m}(a,c,\mu)f(z)}{z} = \theta(z). \tag{2.13}$$

Differentiating with respect to z, we have

$$\left[\mathcal{J}_{\lambda,\ell}^{m}(a,c,\mu)f(z)\right]' = \theta(z) + z\theta'(z), \tag{2.14}$$

By using (2.11), we obtain the differential subordination relation

$$\theta(z) + z\theta'(z) \prec h(z) = q(z) + zq'(z).$$

Using Lemma 2, we deduce that $\theta(z) \prec q(z)$, which implies that

$$\frac{\mathcal{J}_{\lambda,\ell}^{m}(a,c,\mu)f(z)}{z} \prec q(z).$$

The proof of Theorem 3 is completed.

Example 3. If $m = 0, a = c \in \mathbb{C}$, and $q(z) = \frac{1-z}{1+z}$ in Theorem 3, we obtain the following result:

$$[f(z)]' \prec \frac{1-2z-z^2}{(1+z)^2} \quad (f(z) \in \mathcal{A}\ , z \in U),$$

then

$$\frac{f(z)}{z} \prec \frac{1-z}{1+z} \quad (z \in U).$$

Theorem 4. *Let $h(z)$ is given by (2.1). If $f \in \mathcal{A}$ satisfies the differential subordination:*

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \prec h(z) \quad (z \in U), \tag{2.15}$$

then

$$\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)}{z} \prec \phi(z),$$

where

$$\phi(z) = \begin{cases} (\frac{A}{B})^r \sum\limits_{i\geq 0}^{\infty} \frac{(-r)i}{i!} \left(\frac{A-B}{A}\right)^i (1+Bz)^{-i} 2F_1(i,1,2,\frac{Bz}{1+Bz}) & (B \neq 0), \\ 2F_1(-r,1,2,-Az) & (B = 0), \end{cases} \tag{2.16}$$

and the function $\phi(z)$ is the best dominant.

Proof. Using (2.14), the differential subordination (2.15) becomes

$$\theta(z) + z\theta'(z) \prec h(z) = \left(\frac{1+Az}{1+Bz}\right)^r,$$

and applying Lemma 1, we have

$$\theta(z) \prec \phi(z) = \frac{1}{z}\int_0^z \left(\frac{1+At}{1+Bt}\right)^r dt,$$

which upon integration gives (2.16), and hence it follows that

$$\frac{\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)}{z} \prec \phi(z).$$

The proof of Theorem 4 is completed.

Theorem 5. *Let h be a convex function in U with $h(0) = 1$. If $f \in \mathcal{A}$ satisfies the differential subordination:*

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \prec h(z), \tag{2.17}$$

then

$$\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)}{z} \prec \varphi(z) = \frac{1}{z}\int_0^z h(t)dt,$$

and φ is convex and is the best dominant.

Proof. Using (2.14) in (2.17), we have

$$p(z) + zp'(z) \prec h(z).$$

From Lemma 1, we obtain

$$p(z) \prec \varphi(z) = \frac{1}{z}\int_0^z h(t)dt,$$

and using (2.13), we conclude the result:

$$\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)}{z} \prec \varphi(z) = \frac{1}{z}\int_0^z h(t)dt.$$

This completes the proof of Theorem 5.

Example 4. If $m = \ell = 0, a = c \in \mathbb{C}$ and $h(z) = \frac{1-z}{1+z}$ in Theorem 5, we obtain the following result:

$$[f(z)]' \prec \frac{1-z}{1+z} \quad (f(z) \in \mathcal{A}\ , z \in U),$$

then

$$\frac{f(z)}{z} \prec \frac{2\log(1+z) - z}{z} \quad (z \in U).$$

Theorem 6. *Let $h(z)$ is given by (2.1). If $f \in \mathcal{A}$ satisfies the differential subordination:*

$$\left(\mathcal{J}^m_{\lambda,\ell}(a+1,c,\mu)f(z)\right)' \prec h(z) \quad (z \in U), \tag{2.18}$$

then

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \prec q(z), \tag{2.19}$$

where

$$q(z) = \begin{cases} \left(\frac{A}{B}\right)^r \sum\limits_{i\geq 0} \frac{(-r)i}{i!}\left(\frac{A-B}{A}\right)^i (1+Bz)^{-i} 2F_1\left(i,1,1+\frac{a+\mu}{\mu};\frac{Bz}{1+Bz}\right) & (B\neq 0), \\ 2F_1(-r,\frac{a+\mu}{\mu},1+\frac{a+\mu}{\mu};-Az) & (B=0), \end{cases} \tag{2.20}$$

and the function $q(z)$ is convex and is the best dominant.

Proof. Differentiating the recurrence relation (1.4) with respect to z, we get

$$\left(\mathcal{J}^m_{\lambda,\ell}(a+1,c,\mu)f(z)\right)' = \left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' + \frac{\mu z}{a+\mu}\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)'' \tag{2.21}$$

By using (2.21) in view of (2.18), we have

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' + \frac{\mu z}{a+\mu}\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)'' \prec h(z) = \left(\frac{1+Az}{1+Bz}\right)^r \quad (z\in U) \tag{2.22}$$

Let

$$p(z) = \left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \tag{2.23}$$

then (2.22) and (2.23) yield the differential subordination

$$p(z) + \frac{\mu}{a+\mu} zp'(z) \prec h(z) = \left(\frac{1+Az}{1+Bz}\right)^r \quad (z\in U).$$

Now, applying Lemma 1, we conclude that

$$p(z) \prec q(z) = \frac{1}{(\frac{\mu}{a+\mu})z^{\frac{a+\mu}{\mu}}}\int_0^z h(t)t^{\frac{a+\mu}{\mu}-1}dt = \frac{1}{(\frac{\mu}{a+\mu})z^{\frac{a+\mu}{\mu}}}\int_0^z \left(\frac{1+At}{1+Bt}\right)^r t^{\frac{a+\mu}{\mu}-1}dt,$$

where

$$t^{\frac{a+\mu}{\mu}-1}\left(\frac{1+At}{1+Bt}\right)^r = \left(\frac{A}{B}\right)^r t^{\frac{a+\mu}{\mu}-1}\left(1-\frac{A-B}{A(1+Bt)}\right)^r,$$

using Lemma 3 and some calculations, we have

$$q(z) = \left(\frac{A}{B}\right)^r \sum_{i\geq 0}\frac{(-r)i}{i!}\left(\frac{A-B}{A}\right)^i (1+Bz)^{-i} 2F_1\left(i,1,1+\frac{a+\mu}{\mu};\frac{Bz}{1+Bz}\right)(B\neq 0)$$

On the other hand, if $B = 0$, we have

$$q(z) = \frac{1}{(\frac{\mu}{a+\mu})z^{\frac{a+\mu}{\mu}}} \int_0^z (1 + At)^r t^{\frac{a+\mu}{\mu}-1} dt,$$

which upon integrating similarly as above, we have

$$q(z) = 2F_1\left(-r, \frac{a+\mu}{\mu}, 1 + \frac{a+\mu}{\mu}; -Az\right). \tag{2.24}$$

In view of Lemma 1 (for $\gamma = \frac{a+\mu}{\mu}, n = 1$), we assert that

$$\left(\mathcal{J}^m_{\lambda,\ell}(a, c, \mu)f(z)\right)' \prec q(z) \prec h(z),$$

where $q(z)$ is given by (2.20) is convex and the best dominant, which completes the proof of Theorem 6.

Example 5. If $m \in \mathbb{Z}, \ell > -1, \lambda \geq 0, r = 1, a = c = 1, \mu > -1, A = 1$ and $B = 0$, then from Theorem 6, we deduce the following assertion:

$$\left[(1 - \frac{\mu}{1+\mu})\mathcal{J}^m_{\lambda,\ell}f(z) + \frac{\mu}{1+\mu}z\left(\mathcal{J}^m_{\lambda,\ell}f(z)\right)'\right]' \prec 1 + z \quad (f(z) \in \mathcal{A}\ , z \in U)$$

then

$$\left[\mathcal{J}^m_{\lambda,\ell}f(z)\right]' \prec 1 + \frac{1+\mu}{1+2\mu}z \quad (z \in U).$$

Theorem 7. *Let q be a convex function in U with $q(0) = 1$ and let*

$$h(z) = q(z) + \frac{\mu}{a+\mu}zq'(z) \quad (z \in U).$$

If $f \in \mathcal{A}$ satisfies the differential subordination:

$$\left(\mathcal{J}^m_{\lambda,\ell}(a + 1, c, \mu)f(z)\right)' \prec h(z), \tag{2.25}$$

then

$$\left(\mathcal{J}^m_{\lambda,\ell}(a, c, \mu)f(z)\right)' \prec q(z) \quad (z \in U). \tag{2.26}$$

Proof. Making use of (2.23) in (2.21), then the differential subordination (2.25) becomes

$$p(z) + \frac{\mu}{a+\mu} zp'(z) \prec h(z) = q(z) + \frac{\mu}{a+\mu} zq'(z).$$

Applying Lemma 2, we have $p(z) \prec q(z)$, which implies that

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \prec q(z).$$

The proof is completed of Theorem 7.

Example 6. If $m \in \mathbb{Z}, \ell > -1, \lambda \geq 0, a = c = 1, \mu > -1$ and $q(z) = \frac{1-z}{1+z}$, then from Theorem 7, we obtain the following result:

$$\left[(1 - \frac{\mu}{1+\mu})\mathcal{J}^m_{\lambda,\ell} f(z) + \frac{\mu}{1+\mu} z \left(\mathcal{J}^m_{\lambda,\ell} f(z)\right)'\right]' \prec \frac{1 - \frac{2\mu}{1+\mu} z - z^2}{(1+z)^2} \quad (f(z) \in \mathcal{A}\ , z \in U)$$

then

$$\left[\mathcal{J}^m_{\lambda,\ell} f(z)\right]' \prec \frac{1-z}{1+z} \quad (z \in U).$$

Theorem 8. Let q be a convex function in U with $q(0) = 1$ and let

$$h(z) = q(z) + zq'(z) \quad (z \in U).$$

If $f \in \mathcal{A}$ satisfies the differential subordination:

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \prec h(z), \tag{2.27}$$

then

$$\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)}{z} \prec q(z) \quad (z \in U). \tag{2.28}$$

Proof. Let

$$\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)}{z} = \theta(z). \tag{2.29}$$

Differentiating with respect to z, we have

$$\left[\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right]' = \theta(z) + z\theta'(z), \tag{2.30}$$

By using (2.27), we obtain the differential subordination relation

$$\theta(z) + z\theta'(z) \prec h(z) = q(z) + zq'(z).$$

Using Lemma 2, we deduce that $\theta(z) \prec q(z)$, which implies that

$$\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)}{z} = q(z).$$

The proof of Theorem 8 is completed.

Example 7. If $m \in \mathbb{Z}, \ell > -1, \lambda \geq 0, a = c = 1$, and $q(z) = \frac{1-z}{1+z}$, then from Theorem 8, we obtain the following result:

$$\left[\left(\mathcal{J}^m_{\lambda,\ell}f(z)\right)\right]' \prec \frac{1-2z-z^2}{(1+z)^2} \quad (f(z) \in \mathcal{A}\ , z \in U),$$

then

$$\frac{\mathcal{J}^m_{\lambda,\ell}f(z)}{z} \prec \frac{1-z}{1+z} \quad (z \in U).$$

Theorem 9. *Let $h(z)$ is given by (2.1). If $f \in \mathcal{A}$ satisfies the differential subordination:*

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \prec h(z) \quad (z \in U), \tag{2.31}$$

then

$$\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)}{z} \prec \phi(z),$$

where

$$q(z) = \begin{cases} (\frac{A}{B})^r \sum\limits_{i\geq 0}^{\infty} \frac{(-r)i}{i!} \left(\frac{A-B}{A}\right)^i (1+Bz)^{-i} 2F_1(i,1,2,\frac{Bz}{1+Bz}) & (B \neq 0), \\ 2F_1(-r,1,2,-Az) & (B = 0), \end{cases} \tag{2.32}$$

and the function $\phi(z)$ is the best dominant.

Proof. Using (2.30), the differential subordination (2.31) becomes

$$\theta(z) + z\theta'(z) \prec h(z) = \left(\frac{1+Az}{1+Bz}\right)^r,$$

and applying Lemma 1, we have

$$\theta(z) \prec \phi(z) = \frac{1}{z}\int_0^z \left(\frac{1+At}{1+Bt}\right)^r dt,$$

which upon integration gives (2.32), and hence it follows that

$$\frac{\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)'}{z} \prec \phi(z).$$

The proof of Theorem 9 is completed.

Theorem 10. *Let h be a convex function in U with $h(0)=1$, If $f\in\mathcal{A}$ satisfies the differential subordination:*

$$\left(\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)\right)' \prec h(z), \tag{2.33}$$

then

$$\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)}{z} \prec \varphi(z) = \frac{1}{z}\int_0^z h(t)dt,$$

and φ is convex and is the best dominant.

Proof. Using (2.30) in (2.33), we have

$$p(z) + zp'(z) \prec h(z).$$

From Lemma 1, we obtain

$$p(z) \prec \varphi(z) = \frac{1}{z}\int_0^z h(t)dt,$$

and using (2.29), we conclude the result:

$$\frac{\mathcal{J}^m_{\lambda,\ell}(a,c,\mu)f(z)}{z} \prec \varphi(z) = \frac{1}{z}\int_0^z h(t)dt.$$

This completes the proof of Theorem 10.

Example 8. If $m\in\mathbb{Z}, \ell>-1, \lambda\geq 0, a=c=1$, and $h(z)=\frac{1-z}{1+z}$, then Theorem 10, yields the following result:

$$\left[\left(\mathcal{J}^m_{\lambda,\ell}f(z)\right)\right]' \prec \frac{1-z}{1+z} \quad (f(z)\in\mathcal{A}\ , z\in U),$$

then

$$\frac{\mathcal{J}^m_{\lambda,\ell}f(z)}{z} \prec \frac{2\log(1+z)-z}{z} \quad (z\in U).$$

References

[1] F. M. Al-Oboudi, On univalent functions defined by a generalized Salagean operator, *Indian J. Math. Math. Sci.*, **25-28** (2004), 1429–1436.

[2] A. Catas, On a certain differential sandwich theorem associated with a new generalized derivative operator, General Math., **17** (2009), no. 4, 83-95.

[3] Chun-Yi Gao, Shao-Mou Yuan and H. M. Srivastava, Some functional inequalities and inclusion relationships associated with certain families of integral operators, *Comput. Math. Appl.*, **49** (2005), 1787-1795.

[4] D. J. Hallenbeck and S. B. Ruscheweyh, Subordination by convex functionss, *Proc. Amer. Math. Soc.*, **52** (1975), 191–195.

[5] R. M. El-Ashwah and M. K. Aouf, Differential subordination and superordination for certain subclasses of analytic functions involving an extended integral operator, *Acta Univ. Apulensis*, **28** (2011), 341-350.

[6] R. M. El-Ashwah and M. K. Aouf, Some properties of new integral operator, *Acta Univ. Apulensis*, **24** (2010), 51-61.

[7] I. B. Jung, Y. C. Kim and H. M. Srivastava, The Hardy space of analytic functions associated with certain one-parameter families of integral operators,*J. Math. Anal. Appl.*, **176** (1993), 138-147.

[8] V. Kiryakova, Generalized Fractional Calculus and Applications, Pitman Research Notes in Mathematics Series, 301, John Willey & Sons, Inc. New York, (1994).

[9] Y. Komatu, On analytical prolongation of a family of operators, Math. (Cluj), **32** (1990), no. 55, 141-145.

[10] S. S. Miller and P. T. Mocanu, Differential Subordinations: Thoery and Applications, Monographs and Textbooks in Pure and Applied Mathematics Vol. 225, Marce Dekker Inc., New York/Basel, (2000).

[11] P. T. Mocanu, T. Bullboaca and G. S. Salagean, Geometric Theory of Univalent Functions (in Romanian), Casa Cartii di Stinta, Cluj, (1999).

[12] J. K. Prajapath and R. K. Raina, Subordination results for certain classes of multivalently analytic functions with a convolution stucture, *East Asian J. Math.*, **25** (2009), no. 2, 127–140.

[13] R.K Raina and P. Sharma, Subordination properties of univalent functions involving a new class of operators, preprint, submitted for publication.

[14] G. S. Salagean, Subclasses of univalent functions, in Complex Analysis: Fifth Romanian-Finnish Seminar, Part I (Bucharest, 1981), Lecture Notes in Mathematics 1013, Springer-Verlag, Berlin and New York (1983).

[15] C. Selvaraj and K. R. Karthikeyan, Differential subordination and superordination for analytic functions defined using a family of generalized differential operators,*An. St. Univ. Ovidus Constanta*, **17** (2009), no. 1, 201–210.

[16] H. M. Srivastava, Ding-Gong Yang and N-Eng Xu, Some subclasses of meromorphically multivalent functions associated with a linear operator, *Appl. Math. Comput.*, **195** (2008), 11–23.