Tamsui Oxford Journal of Information and Mathematical Sciences **30** (2014) 91-100
Aletheia University

Class of modified Newton's method for solving nonlinear equations*

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Received October 9, 2014, Accepted January 22, 2015.

Abstract

A new classes of three-step Newton's methods based on power means Newton's method has been developed, where two existing numerical methods can be regarded as particular cases of the present method. It is shown that the order of convergence of the proposed methods is six. Also, the efficiency index of the present methods is 1.565, which is better than Newton's method 1.414. It is observed that our method takes less number of iterations than Newton's method. Few sixth order methods are compared with the present method where the numbers of

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* 2000 Mathematics Subject Classification. Primary 41A25, 65H05

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iterations for those methods are either same or more than the present methods. Some examples are given to illustrate the performance of the present methods.

Keywords and Phrases: *Non-linear equation, Iterative Methods, Newton's Method, order of convergence, power mean, Efficiency index.*

1. Introduction

Consider the problem of finding a simple root of a single non-linear equation of the form:

$$f(x) = 0, \tag{1}$$

where $f : I \subseteq R \to R$ for an open interval I is a scalar function. Solution of equation (1) has been given much attention because of its importance in many branches of engineering and science. One of familiar method to solve (1) is the following second order Newton's method:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \text{ where n = 0, 1, 2…} \tag{2}$$

In recent years, cubic convergence was established for solving (1) by different methods [1, 4, 5, 9, 10]. Among these, Xiaojian [10] considered a class of Newton's methods based on power means by using the trapezoidal formula and this method was extended to Simpson's formula by Jayakumar et al [5]. Recently, a number of sixth order methods have been developed for solving (1). In [2], Chun et al presented a three-step efficient sixth order method. Cordero et al [3] presented a three-step iterative methods using harmonic mean Newton's method with sixth order convergence. Parhi et al [7] gave a three-step sixth order method using arithmetic mean Newton's method. Sharma et al [8] developed one parameter family of sixth order methods based on Ostrowski's fourth order multipoint method.

In this paper, we have presented new classes of three-step sixth order modified Newton's methods using power mean Newton's method [10]. The present methods require per iteration, two function and two first derivative evaluations. Hence, the efficiency index of the present methods is 1.565, which is better than Newton's method (1.414). In Section 2, we have presented the new methods. Section 3 deals with the analysis of convergence for the methods. Finally, Section 4 and 5 gives numerical results and conclusions respectively.

2. Description of the new methods

A third order method called power means Newton's method [10] to solve (1) is given by

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \frac{2^{1/p} f(x_n)}{sign\left(f'(x_n)\right)\left(f'(x_n)^p + f'(y_n)^p\right)^{\frac{1}{p}}} \end{aligned}\right\} \qquad (3)$$

Let us consider as given below:

$$\left.\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\ z_n &= x_n - \frac{2^{1/p} f(x_n)}{sign\left(f'(x_n)\right)\left(f'(x_n)^p + f'(y_n)^p\right)^{\frac{1}{p}}} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)}. \end{aligned}\right\} \qquad (4)$$

This method (4) requires two function and three first derivative evaluations per iteration and have sixth order convergence. Efficiency index of this method (4) is $E^* = 1.431$ which is better than Newton's method ($E^* = 1.414$), but less than ($E^* = 1.442$) the power mean Newton's method (3). To improve the efficiency, we approximate $f'(z_n)$ by a combination of already computed function values. We use an approximation by linear interpolation of $f'(z_n)$ that does not introduce any new function evaluation. By using linear interpolation on two points $\left(x_n, f'(x_n)\right)$ and $\left(y_n, f'(y_n)\right)$, we get $f'(x) \approx \frac{x-x_n}{y_n-x_n} f'(y_n) + \frac{x-y_n}{x_n-y_n} f'(x_n)$. Thus, linear interpolation for $f'(z_n)$ is given by

$$f'(z_n) \approx \frac{z_n-x_n}{y_n-x_n} f'(y_n) + \frac{z_n-y_n}{x_n-y_n} f'(x_n) .$$

Simplifying the above equation using (3) using linear interpolation of $f'(z_n)$, we obtain

$$f'(z_n) \approx f'(x_n)\left[1+(z_n-x_n)\left(\frac{f'(x_n)-f'(y_n)}{f(x_n)}\right)\right] \tag{5}$$

Replacing the power mean Newton's method (3) for z_n in the right hand side of equation (5), we get

$$f'(z_n) \approx f'(x_n) - \frac{f'(x_n)^2 - f'(x_n)f'(y_n)}{sign\left(f'(x_n)\right)\left(\frac{f'(x_n)^p + f'(y_n)^p}{2}\right)^{\frac{1}{p}}} \tag{6}$$

Using (6) in the third step of (4), we get the sixth-order **Modified Newton's Method**:

$$\left.\begin{aligned} z_n &= x_n - \frac{2^{1/p} f(x_n)}{sign\left(f'(x_n)\right)\left(f'(x_n)^p + f'(y_n)^p\right)^{\frac{1}{p}}} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n) - \dfrac{f'(x_n)^2 - f'(x_n)f'(y_n)}{sign\left(f'(x_n)\right)\left(\frac{f'(x_n)^p + f'(y_n)^p}{2}\right)^{\frac{1}{p}}}} \end{aligned}\right\}, \text{ n =0, 1, 2 ...} \tag{7}$$

Where y_n is the Newton method and p ∈ R. Clearly, the present method requires only two evaluation function and two of its first derivatives and it also free from second derivative. The efficiency index for (7) is $E^* = 1.565$ and the new classes of modified Newton's method does not reach optimal but it has good efficiency.

3. Analysis of Convergence

Theorem 4.1 *Let $\alpha \in I$ be a simple zero of a sufficiently differentiable function $f: I \subseteq R \to R$ for an open interval I. If x_0 is sufficiently close to α, then the method (7) converge to α with sixth order.*

Proof Let α be a simple zero of the function $f(x) = 0$. (That is, $f(\alpha) = 0$ and $f'(\alpha) \neq 0$).

By expanding $f(x_n)$ and $f'(x_n)$ by Taylor series about α, we obtain

$$f(x_n) = f'(\alpha)\,[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + O(e_n^5)], \tag{8}$$

$$f'(x_n) = f'(\alpha)\,[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + O(e_n^4)], \tag{9}$$

where $c_k = \dfrac{f^{(k)}(\alpha)}{k!\, f'(\alpha)}$, k = 2, 3, 4 … and from equation (2) we get

$$y_n = \alpha + c_2 e_n^2 - 2(c_2^2 - c_3)e_n^3 + O(e_n^4). \tag{10}$$

Expanding $f'(y_n)$ by Tailor's series about α and using (10), we get

$$f'(y_n) = f'(\alpha)[1 + 2c_2^2 e_n^2 + 4(c_2 c_3 - c_2^3)e_n^3 + O(e_n^4)]. \tag{11}$$

Further, from equations (9) and (11) and upon simplification, we have

$$\left.\begin{aligned} & sign(f'(x_n))\left(\frac{f'(x_n)^p + f'(y_n)^p}{2}\right)^{\frac{1}{p}} = \\ & f'(\alpha)[1 + c_2 e_n + \tfrac{1}{2}(c_2^2 + p c_2^2 + 3c_3)e_n^2 + (2c_4 + \tfrac{1}{2}(3p+1)c_2 c_3 - \tfrac{1}{2}(3p+1)c_2^3)e_n^3 + O(e_n^4)]. \end{aligned}\right\} \tag{12}$$

and

$$f'(x_n) - \frac{f'(x_n)^2 - f'(x_n)f'(y_n)}{sign(f'(x_n))\left(\frac{f'(x_n)^p + f'(y_n)^p}{2}\right)^{\frac{1}{p}}} = f'(\alpha)[1-(2c_2c_3 - c_2^3 - pc_2^3)e_n^3 + O(e_n^4)]. \quad (13)$$

From equation (3), we have,

$$z_n = \alpha + \frac{1}{2}(c_2^2 + pc_2^2 + c_3)e_n^3 + O(e_n^4). \quad (14)$$

Also, using equation (13) and (14) in the proposed method (7), we have the following

$$x_{n+1} = \alpha + \frac{1}{2}(c_2^2 + pc_2^2 + c_3)e_n^3 - \frac{f(\alpha + \frac{1}{2}(c_2^2 + pc_2^2 + c_3)e_n^3)}{f'(\alpha)[1-(2c_2c_3 - c_2^3 - pc_2^3)e_n^3 + O(e_n^4)]} \quad (15)$$

Simplifying the above equation (15), we obtain

$$e_{n+1} = (\tfrac{1}{4}(1+2p+p^2)c_2^5 - \tfrac{5}{4}c_2c_3^2 - (1+p)c_2^3c_3)e_n^6 + o(e_n^7). \quad (16)$$

where $p \in$ R, and hence we obtain sixth order convergence for the method (7). □

4. Numerical Examples

In this section, we give numerical results on some test functions to compare the efficiency of proposed methods, with Newton's method (NM), and some existing sixth order methods. Numerical computations have been carried out in the MATLAB software rounding to 200 significant digits. Depending on the precision of the computer, we use the stopping criteria for the iterative process $|f(x_n)| < \varepsilon,$ where $\varepsilon = 10^{-100}$.

Table 1 Nonlinear equations

Test functions	Root
$f_1(x) = (x+2)e^x - 1$	$\alpha = -0.442854401002388 5831...$
$f_2(x) = e^{-x}\sin x + \log(x^2+1) - 2$	$\alpha = 2.4477482864524245021\ldots$

$f_3(x) = x^3 + 4x^2 - 10$	α = 1.3652300134140968457…
$f_4(x) = x^2 - (2 - x)^3$	α = 1
$f_5(x) = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$	α = − 1
$f_6(x) = e^x + x - 20$	α = 2.8424389537844470678…
$f_7(x) = \sin(x-1) + (x-1)^2$	α = 1
$f_8(x) = x^2 + \sin(\frac{x}{5}) - \frac{1}{4}$	α = 0.4099920179891371316…

Table 2 Comparison of different methods with same total number of evaluation (TNE = 12)

Functions	Guess	NM	Chun [2]	Sharma [8]	P = 1	P = − 1	P = 0	P = − 1/2
$f_1(x)$	− 0.2	0.7e-051	0.3e-155	0.1e-168	0.8e-182	0.1e-180	0.4e-175	0.3e-176
	− 0.3	0.7e-065	0.2e-200	Divergent	0	0	0	0
	− 0.6	0.5e-060	0.3e-178	0.1e-197	0	0	0.1e-207	0
$f_2(x)$	1.8	0.2e-081	Divergent	Divergent	0.2e-205	0	0	0
	2.6	0.4e-129	Divergent	Divergent	0	0	0	0
	2.8	0.2e-106	Divergent	Divergent	0	0	0	0
$f_3(x)$	1.8	0.1e-046	0.1e-136	0.1e-142	0.1e-173	0.1e-194	0	0.1e-193
	2.0	0.7e-038	0.8e-110	0.1e-114	0.2e-143	0.5e-163	0.1e-192	0.2e-164
	2.2	0.6e-032	0.7e-092	0.1e-095	0.9e-123	0.2e-141	0.7e-191	0.1e-144
$f_4(x)$	0.8	0.2e-069	Divergent	Divergent	0	0	0	0
	1.4	0.8e-054	Divergent	0.1e-174	0.1e-148	0.3e-159	0.2e-152	0.3e-155
	1.6	0.2e-047	0.5e-168	0.1e-158	0.1e-116	0.1e-125	0.3e-120	0.1e-122
	− 0.8	0.8e-086	Divergent	Divergent	0.3e-207	0	0	0
$f_5(x)$	− 1.3	0.3e-090	0.5e-187	0.5e-186	0.4e-187	0.2e-187	0.3e-187	0.2e-187

	− 1.5	0.9e-066	0.2e-136	0.1e-131	0.6e-126	0.1e-127	0.9e-127	0.3e-127
$f_6(x)$	2.4	0.2e-041	0.3e-117	0.2e-141	0.7e-139	0.1e-158	0.1e-140	0.7e-146
	2.6	0.9e-059	0.6e-177	0.4e-200	0.6e-198	0	0.8e-199	0.4e-203
	3.0	0.3e-072	Divergent	Divergent	0	0	0	0
$f_7(x)$	0.8	0.3e-037	0.7e-081	0.2e-090	0.4e-121	0.2e-200	0.1e-152	0.1e-205
	1.2	0.2e-050	0.2e-142	0.5e-145	0.1e-165	0	0.1e-183	0.3e-199
	1.4	0.5e-036	0.2e-098	0.3e-099	0.5e-117	0.1e-162	0.4e-132	0.1e-144
$f_8(x)$	0.2	0.2e-037	0.3e-084	0.9e-095	0.2e-126	0	0.9e-161	0
	0.6	0.3e-051	0.2e-147	0.2e-151	0.1e-173	0	0.1e-194	0
	0.8	0.1e-035	0.3e-099	0.2e-101	0.2e-120	0.6e-200	0.3e-138	0.7e-153

The absolute value of the given test functions after six iterations for NM and for methods Chun [2] and Sharma [8], proposed methods P = 1,−1, 0, −1/2 after three iterations are listed in Table 2. That is same Total Number of Evaluations (TNE = 12). In Table 3, shows that the classes of modified Newton's method have better efficiency index.

Table 3 Comparisons of efficiency index

Methods	Efficiency index (E^*)	f	f'	f''	Total Functions	Order of methods
NM (2)	1.414	1	1	0	2	2
Equ.(3)	1.442	1	2	0	3	3
Equ.(4)	1.431	2	3	0	5	6
Equ.(57) in [6]	1.316	1	3	0	4	3
Equ.(56) in [6]	1.414	1	3	0	4	4
Equ.(16) in [6]	1.442	1	1	1	3	3
Equ.(7)	1.565	2	2	0	4	6

Acknowledgement. The authors wish to thank the unknown referee for his comments.

5. Conclusions

Based on power mean Newton's method, we have presented a class of modified Newton's method MNM in this paper. This new methods have sixth order convergence and have efficiency index $E^* = 1.565$. As per efficiency is concerned, the method MNM performs equally well with few existing sixth order methods [2, 8] but better than Newton's method, power means Newton's method [10] and eqs. (57), (56), (16) in [6]. Also, by combining power means Newton method with the well-known Newton's method in a three step iteration process, we can get few existing sixth order methods as special cases of the present method and such method is p = 1 and -1 in [7, 3]. Hence, the present methods can be considered as a class of sixth order Newton's methods.

References

[1] D.K.R. Babajee, M.Z. Dauhoo, An analysis of the properties of the variants of Newton's method with third order convergence, *Applied Mathematics and Computation* **183** (2006) 659-684.

[2] C. Chun, B. Neta, A new sixth-order scheme for nonlinear equations, *Appl. Math. Lett.* (2011), doi:10.1016/j.aml.2011.08.012

[3] A. Cordero, J. L. Hueso, E. Martínez, J.R. Torregrosa, Efficient three-step iterative methods with sixth order convergence for nonlinear equations, *Numerical Algorithms* **53** (2010) 485 - 495.

[4] M.K. Jain, S.R.K. Iyengar, R.K. Jain, *Numerical Methods for Scientific and Engineering Computation*, New Age International, 6th edition, 2012.

[5] J. Jayakumar, Kalyanasundaram M., Generalized Power means Modification of Newton's Method for Simple Roots of Nonlinear Equation, *Int. J. Pure Appl. Sci. Technol.*, **18**(2) (2013) 45-51.

[6] V. Kanwar, S. K. Tomar, Sukhjit Singh, Sanjeev Kumar, Note on Super-Halley Method and its Variants, *Tamsui Oxford Journal of Information and Mathematical Sciences*, **28**(2) (2012) 191-216

[7] S.K. Parhi, D.K. Gupta, A sixth order method for nonlinear equations, *Applied Mathematics and Computation* **203** (2008) 50-55

[8] J.R. Sharma, R.K. Guha, A family of modified Ostrowski methods with accelerated sixth order convergence, *Applied Mathematics and Computation* **190** (2007) 111-115.

[9] S. Weerakoon. T.G.I. Fernando, A Variant of Newton's method with accelerated third-order convergence, *Appl. Math. Lett.* **13** (8), (2000) 87-93.

[10] Z. Xiaojian, A class of Newton's methods with third-order convergence, *Appl. Math. Lett*. **20** (2007) 1026 – 1030.