Tamsui Oxford Journal of Information and Mathematical Sciences **30** (2014) 81-89
Aletheia University

Subordination Results for Classes of Functions of Reciprocal Order *

B.A. Frasin†

Faculty of Science, Department of Mathematics, Al al-Bayt University, P.O. Box: 130095 Mafraq, Jordan

Y. Talafha‡

Department of Mathematics and Computer Science, Ta la Technical University, P.O. Box 179, Ta la 66110, Jordan

and

Tariq Al-Hawary§

Department of Mathematics, Faculty of Science and Technology, Irbid National University, P.O. Box: 2600 Irbid, Jordan

Received October 13, 2014, Accepted December 30, 2014.

Abstract

In this paper, we introduce the subclasses $\mathcal{M}^{-1}(\alpha), \mathcal{N}^{-1}(\alpha), (\mathcal{M}^{*})^{-1}(\alpha)$ and $(\mathcal{N}^{*})^{-1}(\alpha)$ of analytic functions of reciprocal order α. Coefficient inequalities and an interesting subordination results are obtained.

Keywords and Phrases: *Analytic functions, Convex, Hadamard product, Subordination principle, Subordinating factor sequence, Univalent.*

---

*2000 *Mathematics Subject Classification.* Primary 30C45.

†E-mail: bafrasin@yahoo.com
‡E-mail: y.a.talafha@gmail.com
§E-mail: tariq_amh@yahoo.com

1. Introduction and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic and univalent in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. A function $f \in \mathcal{A}$ is said to be starlike of order α if it satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are starlike of order α in $\mathbb{U}$. Clearly $\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) = \mathcal{S}^*$, where $\mathcal{S}^*$is the class of functions that are starlike in $\mathbb{U}$. Also, a function $f \in \mathcal{A}$ is said to be convex of order α if it satisfies

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions which are convex of order α in $\mathbb{U}$. Clearly $\mathcal{K}(\alpha) \subseteq \mathcal{K}(0) = \mathcal{K}$, the class of functions that are convex in $\mathbb{U}$.

A function $f \in \mathcal{A}$ is said to be starlike of reciprocal order α if

$$\text{Re}\left\{\frac{f(z)}{zf'(z)}\right\} > \alpha \qquad (z \in \mathbb{U}) \tag{1.4}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote the class of such functions by $\mathcal{S}^{-1*}(\alpha)$. Also, a function $f \in \mathcal{A}$ is said to be convex of reciprocal order α if

$$\text{Re}\left\{\frac{1}{1 + \frac{zf''(z)}{f'(z)}}\right\} > \alpha \qquad (z \in \mathbb{U}) \tag{1.5}$$

for some $\alpha(0 \leq \alpha < 1)$. The class of all such convex functions of reciprocal order α is denoted by $\mathcal{K}^{-1}(\alpha)$.

We note that $\mathcal{S}^{-1*}(0) = \mathcal{S}^*$, $\mathcal{K}^{-1}(0) = \mathcal{K}$ and $f(z) \in \mathcal{K}^{-1}(\alpha)$ if and only if $zf'(z) \in \mathcal{S}^{-1*}(\alpha)$.

Example 1.1. *The function* $f(z) = ze^{(1-\alpha)z}$ *is a starlike function of reciprocal order* $1/(2-\alpha)$ *[9, Example 2].*

2. Coefficient Estimates

In this section, we introduce the subclasses $\mathcal{M}^{-1}(\alpha)$ and $\mathcal{N}^{-1}(\alpha)$ of analytic functions of reciprocal order α. The sufficient conditions for $f(z)$ to be in the class $\mathcal{M}^{-1}(\alpha)$ and $\mathcal{N}^{-1}(\alpha)$ are given by using coefficient inequalities.

Definition 2.1. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}^{-1}(\alpha)$of order α if and only if it satisfies the condition

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} < \alpha \qquad (z \in \mathbb{U}) \tag{2.1}$$

for some $\alpha > 1$.

Theorem 2.2. *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} \left[|\lambda n - 1| + 2n\alpha - n\lambda - 1\right] |a_n| \leq 2\,(\alpha - 1) \tag{2.2}$$

for some λ, $0 \leq \lambda \leq 1$and some $\alpha > 1$, then $f(z) \in \mathcal{M}^{-1}(\alpha)$.

Proof. To proceed, its sufficient to show that

$$\left|\frac{\frac{f(z)}{zf'(z)} - \lambda}{\frac{f(z)}{zf'(z)} - (2\alpha - \lambda)}\right| < 1.$$

By using the Cauchy-Schwarz inequality together with the use (2.2), we have

$$\begin{aligned}
\left|\frac{\frac{f(z)}{zf'(z)} - \lambda}{\frac{f(z)}{zf'(z)} - (2\alpha - \lambda)}\right| &= \left|\frac{1 - \lambda + \sum\limits_{n=2}^{\infty}(1 - \lambda n)\, a_n z^{n-1}}{1 - 2\alpha + \lambda + \sum\limits_{n=2}^{\infty}(1 - n\,(2\alpha - \lambda))\, a_n z^{n-1}}\right| \\
&\leq \frac{1 - \lambda + \sum\limits_{n=2}^{\infty}|\lambda n - 1|\,|a_n|\,|z|^{n-1}}{|1 - 2\alpha + \lambda| - \sum\limits_{n=2}^{\infty}|1 - n\,(2\alpha - \lambda)|\,|a_n|\,|z|^{n-1}} \\
&< \frac{1 - \lambda + \sum\limits_{n=2}^{\infty}|\lambda n - 1|\,|a_n|}{2\alpha - \lambda - 1 - \sum\limits_{n=2}^{\infty}(n\,(2\alpha - \lambda) - 1)\,|a_n|}.
\end{aligned}$$

It follows that the last term is bounded by 1 if

$$1-\lambda+\sum_{n=2}^{\infty}|\lambda n-1|\,|a_n|\leq 2\alpha-\lambda-1-\sum_{n=2}^{\infty}(2n\alpha-n\lambda-1)\,|a_n|$$

which is equivalent to (2.2). The desired result follows. □

Corollary 2.3. *If $f\in\mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}(\alpha n-1)\,|a_n|\leq\alpha-1, \tag{2.3}$$

for some $\alpha>1$, then $f(z)\in\mathcal{M}^{-1}(\alpha)$.

Example 2.4. *The function $f(z)$ given by*

$$f(z)=z+\sum_{n=2}^{\infty}\frac{2(\alpha-1)\varphi_n}{n(n-1)(|\lambda n-1|+2n\alpha-n\lambda-1)}z^n\in\mathcal{M}^{-1}(\alpha);\quad(|\varphi_n|=1)$$

Next, we introduce the class $\mathcal{N}^{-1}(\alpha)$ defined as follows.

Definition 2.5. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{N}^{-1}(\alpha)$of order α if and only if it satisfies the condition

$$\operatorname{Re}\left\{\frac{1}{1+\frac{zf''(z)}{f'(z)}}\right\}<\alpha\qquad(z\in\mathbb{U}) \tag{2.4}$$

for some $\alpha>1$.

It can be seen that from (2.1) and (2.4) that

$$f(z)\in\mathcal{N}^{-1}(\alpha)\text{ if and only if }zf'(z)\in\mathcal{M}^{-1}(\alpha). \tag{2.5}$$

In view of (2.5), we can conclude the following result.

Corollary 2.6. *If $f\in\mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty}n\left[|\lambda n-1|+2n\alpha-n\lambda-1\right]|a_n|\leq 2\left(\alpha-1\right) \tag{2.6}$$

for some λ, $0\leq\lambda\leq 1$and some $\alpha>1$, then $f(z)\in\mathcal{N}^{-1}(\alpha)$.

Corollary 2.7. *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n\,(\alpha n-1)\,|a_n| \leq \alpha-1, \tag{2.7}$$

for some $\alpha > 1$, then $f(z) \in \mathcal{N}^{-1}(\alpha)$.

Example 2.8. *The function $f(z)$ given by*

$$f(z) = z + \sum_{n=2}^{\infty} \frac{2(\alpha-1)\varphi_n}{n^2(n-1)(|\lambda n-1| + 2n\alpha - n\lambda - 1)} z^n \in \mathcal{N}^{-1}(\alpha); \quad (|\varphi_n| = 1)$$

3. Subordination Results

To proceed our main results, let us first recall the following definitions and lemma.

Definition 3.1. (Hadamard Product). For two functions $f(z)$, $g(z) \in \mathcal{A}$, where $f(z)$ is given by (1.1) and $g(z)$ is given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, then Hadamard product (convolution) $f * g$ is defined as follows

$$(f * g)(z) = \ z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{3.1}$$

Definition 3.2. (Subordination Principle). Given two functions $f(z)$, $g(z) \in \mathcal{A}$ in $\mathbb{U}$, g be univalent in $\mathbb{U}$, $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$, then we say that the function $f(z)$ is subordinate to $g(z)$ in $\mathbb{U}$, and write $f(z) \prec g(z)$, $z \in \mathbb{U}$. Moreover, we say that $g(z)$ is superordinate to $f(z)$ in $\mathbb{U}$.

Definition 3.3. A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form (1.1), $a_1 = 1$ is analytic, univalent and convex in $\mathbb{U}$, we have the subordination given by

$$\sum_{n=1}^{\infty} b_n a_n z^n \prec f(z), \quad z \in \mathbb{U}. \tag{3.2}$$

Lemma 3.4. *([8]). The sequence* $\{b_n\}_{n=1}^{\infty}$ *is subordinating factor sequence if and only if*

$$Re\left\{1+2\sum_{n=1}^{\infty} b_n z^n\right\} > 0 \qquad (z \in \mathbb{U}). \tag{3.3}$$

Let $(\mathcal{M}^*)^{-1}(\alpha) \subseteq \mathcal{M}^{-1}(\alpha)$ and $(\mathcal{N}^*)^{-1}(\alpha) \subseteq \mathcal{N}^{-1}(\alpha)$ denote the subclasses of functions $f \in \mathcal{A}$ whose coefficients a_n satisfy the inequalities (2.2) and (2.6)for all $\alpha > 1$, respectively. Employing the techniques used by Srivastava and Attiya [7], Attiya [2] and Singh [6](see also, [1], [3], [4] and [5]), we state and prove the following theorems.

Theorem 3.5. *Let the function* $f(z)$ *be in the class* $(\mathcal{M}^*)^{-1}(\alpha)$*. Then*

$$\frac{|2\lambda-1|+4\alpha-2\lambda-1}{2\left[6\alpha-2\lambda-3+|2\lambda-1|\right]}(f*g)(z) \prec g(z), \tag{3.4}$$

$$(0 \le \lambda \le 1; g \in \mathcal{K}),$$

and

$$Re\,(f(z)) > -\frac{6\alpha-2\lambda-3+|2\lambda-1|}{|2\lambda-1|+4\alpha-2\lambda-1}. \tag{3.5}$$

The constant $\frac{|2\lambda-1|+4\alpha-2\lambda-1}{2[6\alpha-2\lambda-3+|2\lambda-1|]}$ *is the best estimate.*

Proof. Now we can follow the same techniques in. Let $f(z) \in (\mathcal{M}^*)^{-1}(\alpha)$ and suppose that $g(z) = z + \sum\limits_{n=2}^{\infty} c_n z^n \in \mathcal{K}$. It follows that

$$\begin{aligned}&\frac{|2\lambda-1|+4\alpha-2\lambda-1}{2\left[6\alpha-2\lambda-3+|2\lambda-1|\right]}(f*g)(z)\\&=\frac{|2\lambda-1|+4\alpha-2\lambda-1}{2\left[6\alpha-2\lambda-3+|2\lambda-1|\right]}\left(z+\sum_{n=2}^{\infty} a_n c_n z^n\right).\end{aligned}$$

By using Subordinating Factor definition and the subordination result (3.4) will hold true if the sequence

$$\left\{\frac{|2\lambda-1|+4\alpha-2\lambda-1}{2\left[6\alpha-2\lambda-3+|2\lambda-1|\right]}a_n\right\}_{n=1}^{\infty} \tag{3.6}$$

is a subordinating factor sequence, with $a_1 = 1$. By virtue of Lemma 3.4, the sequence (3.6) is a subordinating factor sequence if and only if

$$\mathrm{Re}\left\{1+\sum_{n=1}^{\infty}\frac{|2\lambda-1|+4\alpha-2\lambda-1}{6\alpha-2\lambda-3+|2\lambda-1|}a_n z^n\right\}>0. \tag{3.7}$$

Since $\phi(n) = |\lambda n - 1| + 2n\alpha - n\lambda - 1$ is an increasing function of $n(n \geq 0)$ for $0 \leq \lambda \leq 1$ and $\alpha > 1$, we obtain

$$\begin{aligned}
&\mathrm{Re}\left\{1+\sum_{n=1}^{\infty}\frac{|2\lambda-1|+4\alpha-2\lambda-1}{6\alpha-2\lambda-3+|2\lambda-1|}a_n z^n\right\}\\
&=\mathrm{Re}\left\{1+\frac{|2\lambda-1|+4\alpha-2\lambda-1}{6\alpha-2\lambda-3+|2\lambda-1|}z+\right.\\
&\left.\frac{1}{6\alpha-2\lambda-3+|2\lambda-1|}\sum_{n=2}^{\infty}\left(|2\lambda-1|+4\alpha-2\lambda-1\right)a_n z^n\right\}\\
&\geq 1-\frac{|2\lambda-1|+4\alpha-2\lambda-1}{6\alpha-2\lambda-3+|2\lambda-1|}r-\\
&\frac{1}{6\alpha-2\lambda-3+|2\lambda-1|}\sum_{n=2}^{\infty}\left(|\lambda n-1|+2n\alpha-n\lambda-1\right)|a_n|\,r^n\\
&>1-\frac{|2\lambda-1|+4\alpha-2\lambda-1}{6\alpha-2\lambda-3+|2\lambda-1|}r-\frac{2(\alpha-1)}{6\alpha-2\lambda-3+|2\lambda-1|}r\\
&>0 \qquad (|z|=r<1),
\end{aligned}$$

where we have been used inequality (2.2) in Lemma 3.4. The inequality (3.7) is thus proved. We can also obtain the result (3.5) from (3.4) by setting

$$g(z)=\frac{z}{1-z}=z+\sum_{n=1}^{\infty}z^n.$$

To prove the sharpness, let us introduce $f_0(z) \in (\mathcal{M}^*)^{-1}(\alpha)$ by

$$f_0(z)=z-\frac{2(\alpha-1)}{|2\lambda-1|+4\alpha-2\lambda-1}z^2.$$

Then by using (3.4), we obtain

$$\frac{|2\lambda-1|+4\alpha-2\lambda-1}{2\left[6\alpha-2\lambda-3+|2\lambda-1|\right]}f_0(z)\prec\frac{z}{1-z}.$$

It follows that

$$\min_{z\in\mathbb{U}}\left\{\operatorname{Re}\left(\frac{|2\lambda-1|+4\alpha-2\lambda-1}{2\left[6\alpha-2\lambda-3+|2\lambda-1|\right]}f_0(z)\right)\right\}=\frac{-1}{2}.$$

Thus, the proof is completed. □

By using the similar argument in Theorem 3.5, we obtain that the following result.

Theorem 3.6. *Let the function $f(z)$ be in the class $(\mathcal{N}^*)^{-1}(\alpha)$. Then*

$$\frac{|2\lambda-1|+4\alpha-2\lambda-1}{2\left[5\alpha-2\lambda-2+|2\lambda-1|\right]}(f*g)(z)\prec g(z),$$
$$(0\leq\lambda\leq 1; g\in\mathcal{K}),$$

and

$$Re\left(f(z)\right)>-\frac{5\alpha-2\lambda-2+|2\lambda-1|}{|2\lambda-1|+4\alpha-2\lambda-1}.$$

The constant $\frac{|2\lambda-1|+4\alpha-2\lambda-1}{2[5\alpha-2\lambda-2+|2\lambda-1|]}$ is the best estimate.

Putting $\lambda=\frac{1}{2}$ in Theorems 3.5 and 3.6 , we have the following results.

Corollary 3.7. *Let the function $f(z)$ be in the class $(\mathcal{M}^*)^{-1}(\alpha)$. Then*

$$\frac{2\alpha-1}{6\alpha-4}(f*g)(z)\prec g(z), \textit{where } g\in\mathcal{K}$$

and

$$Ref(z)>-\frac{3\alpha-2}{2\alpha-1}.$$

The constant $\frac{2\alpha-1}{6\alpha-4}$ is the best estimate.

Corollary 3.8. *Let the function $f(z)$ be in the class $(\mathcal{N}^*)^{-1}(\alpha)$. Then*

$$\frac{2\alpha-1}{5\alpha-3}(f*g)(z)\prec g(z), \textit{where } g\in\mathcal{K}$$

and

$$Re\left(f(z)\right)>-\frac{5\alpha-3}{4\alpha-2}.$$

The constant $\frac{2\alpha-1}{5\alpha-3}$ is the best estimate.

Acknowledgements. The authors are grateful to the referee for careful reading of the article and suggested improvements.

References

[1] M. K. Aouf, A. Shamandy, A. O. Mostafa and E. A. Adwan, Subordination theorem of analytic functions defined by convolution, *Complex Anal. Oper. Theory*, **7** (2013),1117-1126.

[2] A.A. Attiya, On some application of a subordination theorems, *J. Math. Anal. Appl.* **311** (2005), 489-494.

[3] B. A. Frasin, A subordination result for a class of analytic functions, *Acta Univ. Apulensis Math. Inform.* No. **29** (2012), 99-103.

[4] B. A. Frasin, Subordination results for a class of analytic functions defined by a linear operator, *J. Inequal. Pure Appl. Math.* **7** (2006), no. 4, Article 134, 7 pp.

[5] B. A. Frasin and M. Darus, Subordination results on subclasses concerning Sakaguchi functions, *J. Inequal. Appl.* 2009, Art. ID 574014, 7 pp.

[6] S. Singh, A subordination theorems for spirallike functions, *Int. J. Math. and Math.* Sci., **24** (7) (2000), 433-435.

[7] H.M. Srivastava and A.A. Attiya, Some subordination results associated with certain subclasses of analytic functions, *J. Inequal. Pure Appl. Math.*, **5** (4) (2004), Article 82, 1-6.

[8] H.S. Wilf, Subordinating factor sequences for convex maps of the unit circle, *Proc. Amer. Math. Soc.*, **12** (1961), 689-693.

[9] M. Nunokawa, S. Owa, J. Nishiwaki, K. Kuroki and T. Hayami, Differential subordination and argumental property, *Comput. Math. Appl.* **56** (10) (2008) 2733–2736.