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Decoding of Cluster Array Errors in Row-Cyclic Array Codes *

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Abstract

Row-cyclic array codes equipped with m-metric [13] suitable for parallel channel communication systems have been introduced by the first author in [10] and the notion of cluster/burst array errors were introduced by the first author in [6]. In this paper, we study cluster array errors detection and correction in row-cyclic array codes.

Keywords and Phrases: *Row-cyclic array codes , Cluster array errors*

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1. Introduction

Row-cyclic array codes equipped with m-metric [13] suitable for parallel channel communication systems have already been introduced by the first author in [10]. The first author also gave decoding algorithm for the correction of random array errors in row-cyclic array codes [10]. There are yet another kind of errors during parallel channel communication systems known as cluster errors or burst errors [6]. The errors in a burst error are not scattered through out the array but are confined to a subarray part of it. These errors arise, for example due to lightening, thundering etc. in deep space satellite communication. In this paper, we study burst error detection and correction in row-cyclic array codes.

2. Definitions and Notations

Let F_q be a finite field of q elements. Let $\mathrm{Mat}_{m\times s}(F_q)$ denote the linear space of all $m \times s$ matrices with entries from F_q. An m-metric array code is a subset of $\mathrm{Mat}_{m\times s}(F_q)$ and a linear m-metric array code is an F_q-linear subspace of $\mathrm{Mat}_{m\times s}(F_q)$. Note that the space $\mathrm{Mat}_{m\times s}(F_q)$ is identifiable with the space F_q^{ms}. Every matrix in $\mathrm{Mat}_{m\times s}(F_q)$ can be represented as a $1 \times ms$ vector by writing the first row of matrix followed by second row and so on. Similarly, every vector in F_q^{ms} can be represented as an $m \times s$ matrix in $\mathrm{Mat}_{m\times s}(F_q)$ by separating the co-ordinates of the vector into m groups of s-coordinates. The m-metric on $\mathrm{Mat}_{m\times s}(F_q)$ is defined as follows [13]:

Definition 2.1. Let $Y \in \mathrm{Mat}_{1\times s}(F_q)$ with $Y = (y_1, y_2, \cdots, y_s)$. Define row weight (or ρ-weight) of Y as

$$wt_\rho(Y) = \begin{cases} \max\ \{\ i \mid y_i \neq 0\} & \text{if } Y \neq 0 \\ 0 & \text{if } \ Y = 0. \end{cases}$$

Extending the definitions of wt_ρ to the class of $m \times s$ matrices as

$$wt_\rho(A) = \sum_{i=1}^{m} wt_\rho(R_i)$$

where $A = \begin{bmatrix} R_1 \\ R_2 \\ \cdots \\ R_m \end{bmatrix} \in \text{Mat}_{m\times s}(F_q)$ and R_i denotes the i^{th} row of A. Then wt_ρ satisfies $0 \leq wt_\rho(A) \leq n(= ms)\ \forall\ A \in \text{Mat}_{m\times s}(F_q)$ and determines a metric on $\text{Mat}_{m\times s}(F_q)$ known as m-metric (or ρ-metric).

Now we define burst errors in linear array codes [6]:

Definition 2.2. A burst of order pr(or $p \times r$)$(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\text{Mat}_{m\times s}(F_q)$ is an $m \times s$ matrix in which all the nonzero entries are confined to some $p \times r$ submatrix which has non-zero first and last rows as well as non-zero first and last columns.

Note. For $p = 1$, Definition 2.2 reduces to the definition of burst for classical codes [5].

Definition 2.3. A burst of order pr or less $(1 \leq p \leq m, 1 \leq r \leq s)$ in the space $\text{Mat}_{m\times s}(F_q)$ is a burst of order cd(or $c \times d$) where $1 \leq c \leq p \leq m$ and $1 \leq d \leq r \leq s$.

The following theorem gives a bound for the correction of burst array errors in linear m-metric array codes [6].

Theorem 2.1. *An (n, k) linear m-metric array code $V \subseteq Mat_{m\times s}(F_q)$ where $n = ms$ that corrects all bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ must satisfy*

$$q^{n-k} \geq 1 + B^{p\times r}_{m\times s}(F_q), \tag{2.1}$$

where $B^{p\times r}_{m\times s}(F_q)$ is the number of bursts of order $pr(1 \leq p \leq m, 1 \leq r \leq s)$ in

$Mat_{m\times s}(F_q)$ and is given by

$$B^{p\times r}_{m\times s}(F_q)=\begin{cases} ms(q-1) & \text{if } p=1,\ r=1,\\ m(s-r+1)(q-1)^2q^{r-2} & \text{if } p=1, r\geq 2,\\ (m-p+1)s(q-1)^2q^{p-2} & \text{if } p\geq 2, r=1\\ (m-p+1)(s-r+1)q^{r(p-2)} & \\ \times\Big[(q^r-1)^2-2(q^{r-1}-1)^2q^{2-p} & \\ +(q^{r-2}-1)^2q^{4-2p}\Big] & \text{if } \ p\geq 2,\ r\geq 2.\end{cases} \tag{2.2}$$

Now, we give the definition of row-cyclic array codes [10].

Definition 2.4. An $[m\times s,k]$ linear array codes $C\subseteq \mathrm{Mat}_{m\times s}(F_q)$ is said to be row-cyclic if

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1s}\\ a_{21} & a_{22} & \cdots & a_{2s}\\ \vdots & \vdots & \vdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{ms}\end{pmatrix}\in \mathbf{C}$$

$$\Longrightarrow \begin{pmatrix} a_{1s} & a_{11} & a_{12} & \cdots a_{1,s-1}\\ a_{2s} & a_{21} & a_{22} & \cdots a_{2,s-1}\\ \vdots & \vdots & \vdots & \vdots\\ a_{ms} & a_{m1} & a_{m2} & \cdots a_{m,s-1}\end{pmatrix}\in \mathbf{C}$$

i.e. the array obtained by shifting the columns of a code array cyclically by one position of the right and the last column occupying the first place is also a code array. In fact, a row-cyclic array code C of order $m\times s$ turns out to be $C=\bigoplus_{i=1}^{m} C_i$ where each C_i is a classical cyclic code of length s. Also, every matrix/array in $\mathrm{Mat}_{m\times s}(F_q)$ can be identified with an m-tuple in $A_s^{(m)}$ where $A_s^{(m)}$ is the direct product of algebra A_s taken m times and A_s is the algebra of all polynomials over F_q modulo the polynomial $x^s - 1$ and this identification is given by

$$\theta : \mathrm{Mat}_{m\times s}(F_q) \to A_s^{(m)}$$

$$\theta(A) = \theta \begin{pmatrix} R_1 \\ \vdots \\ R_m \end{pmatrix} = \begin{pmatrix} \theta' R_1 \\ \theta' R_2 \\ \vdots \\ \theta' R_m \end{pmatrix} = (\theta' R_1, \theta' R_2, \cdots, \theta' R_m)$$

where $R_i (i = 1 \text{ to } m)$ denotes the i^{th} row of A and $\theta' : F_q^s \longrightarrow A_s$ is given by

$$\theta'(a_0, a_1, \cdots, a_{s-1}) = a_0 + a_1 x + \cdots + a_{s-1} x^{s-1}.$$

An equivalent definition of row-cyclic array code is given by [10]:

Definition 2.5. An $m \times s$ linear array codes $C \subseteq \mathrm{Mat}_{m\times s}(F_q)$ is said to be row-cyclic if

$$C = \bigoplus_{i=1}^{m} C_i$$

where each C_i is an $[s, k_i, d_i]$ classical cyclic code equipped with m-metric. The parameters of row-cyclic array code C are given by $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$. If $g_i(x)$ is the generator polynomial of classical cyclic code C_i, then the m-tuple $(g_1(x) \cdots, g_m(x))$ is called the generator m-tuple of row cyclic code C.

3. Detection of Cluster/Burst Array Errors in Row-Cyclic Array Codes

In this section, we first obtain an upper bound on the order of burst array errors that can be detected by a row-cyclic array code and then obtain the ratio of bursts (of order exceeding the upper bound) that can go undetected. The upper bound on the order of bursts that can be detected in a row-cyclic array code is obtained in the following theorems:

Theorem 3.1 *Let* $C = \bigoplus_{i=1}^{m} C_i$ *be an* $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ *row-cyclic array code. Then no code array is a burst of order* $m \times r$ *or less where* $r = \min_{i=1}^{m}\{s - k_i\}$. *Therefore, every* $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ *row-cyclic array code detects every burst of order* $m \times \min_{i=1}^{m}\{s - k_i\}$ *or less.*

Proof. Let

$$\begin{aligned} A &= (0 \quad B \quad 0) = \begin{pmatrix} & b_1 & \\ 0 & b_2 & 0 \\ & \vdots & \\ & b_m & \end{pmatrix} \\ &= \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix} \in A_s^{(m)} \end{aligned}$$

denote a burst of order $m \times r$ or less where $r = \min_{i=1}^{m}\{s - k_i\}$ where B is a $m \times r$ submatrix of A such that B has a submatrix D with the first row, last row as well as first and the last column of D to be nonzero. Let $(g_1(x), g_2(x), \cdots, g_m(x))$ be the generator m-tuple of row-cyclic array code C. Then $deg(g_i(x)) = s - k_i$ for all $i = 1, 2, \cdots, m$. Choose $b_i(x)$ such that $b_i(x) \neq 0$. Then such a $b_i(x)$ is a classical burst of order r or less. Let the first nonzero component of the vector corresponding to $b_i(x)$ be the coefficient of x^j under the correspondence θ' i.e.

$$(a_0, a_1, \cdots, a_{s-1}) \longleftrightarrow a_0 + a_1 x + \cdots + a_{s-1} x^{s-1}.$$

Then, the polynomial $b_i(x)$ can be written as

$$\begin{aligned} b_i(x) &= a_j x^j + a_{j+1} x^{j+1} + \cdots + a_{j+r-1} x^{j+r-1} \\ &= x^j (a_j + a_{j+1} x + \cdots + a_{j+r-1} x^{r-1}) \\ &= x^j p(x) \end{aligned}$$

where deg $p(x) \leq r - 1 = \min_{i=1}^{m}\{s - k_i\} - 1 \leq s - k_i - 1 < s - k_i =$ deg $g_i(x)$.

Now $g_i(x)$ does not divide x^j and also $g_i(x)$ does not divide $p(x)$ as deg $(p(x)) <$ deg $g_i(x)$.

So, $g_i(x)$ does not divide $b_i(x)$.

This implies that $b_i(x)$ is not a code polynomial in classical code C_i which further implies that $A = \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix}$ is not an array of code polynomials in $C = \bigoplus_{i=1}^{m} C_i$. Hence, the row-cyclic array code C detects every burst of order $m \times \min_{i=1}^{m}\{s - k_i\}$ or less. $\square$

Another upper bound on the order of bursts that can be detected by a row-cyclic array code is obtained in the following theorem:

Theorem 3.2. *Let $C = \bigoplus_{i=1}^{m} C_i$ be an $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ row-cyclic array code. Then no code array is a burst of order $m \times r$ where $r \leq \max\{s - k_1, s - k_m\}$. Therefore, every $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ row-cyclic array code $C = \bigoplus_{i=1}^{m} C_i$ detects every burst of order $m \times r$ where $r \leq \max\{s - k_1, s - k_m\}$.*

Proof. Let

$$\begin{aligned} A &= (0 \quad B \quad 0) = \begin{pmatrix} & b_1 & \\ 0 & b_2 & 0 \\ & \vdots & \\ & b_m & \end{pmatrix} \\ &= \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix} \in A_s^{(m)} \end{aligned}$$

denote a burst of order $m \times r$ where $r \leq \max\{s - k_1, s - k_m\}$ and B is an $m \times r$ submatrix of A with first and last rows as well as the first and last columns of

B are nonzero. Let $(g_1(x), g_2(x), \cdots, g_m(x))$ be the generator m-tuple of row-cyclic array code C. Then $deg(g_i(x)) = s - k_i$ for all $i = 1, 2, \cdots, m$. Clearly $b_1(x), b_m(x) \neq 0$. Let $\max\{s-k_1, s-k_m\} = s-k_1$. Then as in Theorem 3.1, we have

$$\begin{aligned} b_1(x) &= a_j x^j + a_{j+1}x^{j+1} + \cdots + a_{j+r-1}x^{j+r-1} \\ &= x^j(a_j + a_{j+1}x + \cdots + a_{j+r-1}x^{r-1}) \\ &= x^j p(x) \end{aligned}$$

where

$$\begin{aligned} & \deg\, p(x) \leq r-1 \leq \max\{s-k_1, s-k_m\} - 1 \\ = \;& s-k_1-1 < s-k_1 = \deg\, g_1(x) \end{aligned}$$

Now $g_1(x)$ does not divide x^j and also $g_1(x)$ does not divide $p(x)$ as deg $(p(x)) <$ deg $g_1(x)$.

So $g_1(x)$ does not divide $b_1(x)$.

This implies that $b_1(x)$ is not a code polynomial in classical code C_1.

This implies that $A = \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix}$ is not an array of code polynomials in $C = \bigoplus_{i=1}^{m} C_i$. Again, when $\max\{s-k_1, s-k_m\} = s-k_m$, we arrive at the same conclusion by considering $b_m(x)$ to be a classical burst of length r or less in classical code C_m. $\square$

Remark 3.1. Clearly bound obtained in Theorem 3.2 is better than obtained in Theorem 3.1 as $\min_{i=1}^{m}\{s-k_i\} \leq \max\{s-k_1, s-k_m\}$ with the only constraint in Theorem 3.2 that order of nonzero submatrix in burst A is $m \times r$ and not $m \times r$ or less where $(r \leq \max\{s-k_1, s-k_m\})$. We may also take the order $m \times r$ or less in Theorem 3.2 but with the constraint that $b_1(x), b_m(x) \neq 0$ i.e. the first and last rows of burst A are nonzero.

Now, we obtain the ratio of bursts of orders $m \times r$ where $r > \max\{s-k_1, s-k_m\}$ that go undetected in row-cyclic array codes. We shall be using the notation $|J|$ for the cardinality of a set J.

Theorem 3.3. *Let* $C = \bigoplus_{i=1}^{m} C_i$ *be a row-cyclic array code over* F_q *where each* C_i *is a* $[s, k_i, d_i]$ *classical cyclic code equipped with* m*-metric and having generator polynomial* $g_i(x)$*. Then the ratio of bursts of order* $m \times r$ *(where* $r > \max\{s - k_1, s - k_m\}$*) that go undetected in a row-cyclic array code* C *is given by*

$$\frac{(s-r+1)(\alpha - 2\beta)}{B^{m\times r}_{m\times s}(F_q)} \tag{3.1}$$

where

$$\alpha = \left(q^{(|J|-2)(r-s)+\sum\limits_{i\in J/\{1,m\}} k_i}\right) \times (q^{r-s+k_1} - 1) \times (q^{r-s+k_m} - 1), \tag{3.2}$$

$$\beta = \left(q^{(|J|-2)(r-s-1)+\sum\limits_{i\in J/\{1,m\}} k_i}\right) \times (q^{(r-1)-(s-k_1)} - 1) \times \times (q^{(r-1)-(s-k_m)} - 1), \tag{3.3}$$

and J is a subset of $N = \{1, 2, \cdots, m\}$ such that $i \in J_1 \Leftrightarrow r - 1 \geq s - k_i$ and $B^{m\times r}_{m\times s}(F_q)$ is given by (2).

Proof. Consider a burst A of order $m \times r$ where $r > \max\{s - k_1, s - k_m\}$. We can write A as

$$A = \begin{pmatrix} 0 & B & 0 \end{pmatrix} = \begin{pmatrix} & b_1 & \\ 0 & b_2 & 0 \\ & \vdots & \\ & b_m & \end{pmatrix}$$

$$= \begin{pmatrix} & b_1(x) & \\ 0 & b_2(x) & 0 \\ & \vdots & \\ & b_m(x) & \end{pmatrix} \quad \text{(under the identification } \theta\text{).}$$

where $B = \begin{pmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_m(x) \end{pmatrix}$ is an $m \times r$ submatrix of A such that first row and first column as well as last row and last column of B are nonzero.

Now, the burst A will go undetected if $g_i(x)$ divides $b_i(x) \forall\ i \in N$.

Without any loss of generality, we may assume that deg $b_i(x) \leq r-1$ for all $i \in N$. Let $i \in N$. We find possible number of ways of choosing $b_i(x)$.

There are two mutually cases to consider:

Case 1. When $i \in N/J$.

In this case, $r-1 < s-k_i$. Since deg $b_i(x) \leq r-1$ and deg $g_i(x) = (s-k_i)$ and $r-1 < s-k_i$, therefore $g_i(x)$ divides $b_i(x)$ iff $b_i(x) = 0$.

So there is only one way of choosing $b_i(x)$ and hence the possible number of ways of choosing $b_i(x)$for all $i \in N/J$

$$= (1)^{|N/J|} = 1. \tag{3.4}$$

Case 2. When $i \in J$.

In this case, $r-1 \geq s-k_i$. Now $g_i(x)$ divides $b_i(x)$ iff $b_i(x) = g_i(x)q_i(x)$ for some $q_i(x)$.

Since deg $g_i(x) = s-k_i$ and deg $b_i(x) \leq r-1$, therefore deg $q_i(x) \leq (r-1) - (s-k_i)$.

Denote $(r-1)-(s-k_i)$ by P. Then deg $q_i(x) \leq P$.

Now, the number of possibilities for $q_i(x)$ for $i \in J/\{1,m\}$

$$\begin{aligned} &= \text{number of polynomials of dgree upto } P \\ &= q + (q-1)q + (q-1)q^2 + \cdots + (q-1)q^P \\ &= q^{P+1} = q^{r-s+k_i}. \end{aligned}$$

Also, the number of possible ways of choosing $q_1(x)$ and $q_m(x)$ are $(q^{r-s+k_1}-1)$ and $(q^{r-s+k_m}-1)$ respectively (because when $q_1(x)$ or $q_m(x)$ is a polynomial of degree zero then it has to be a nonzero constant).

So the total number of possible ways of choosing $g_i(x)$ and hence $b_i(x) \forall\ i \in J$

$$\begin{aligned} &= \prod_{i \in J/\{1,m\}} q^{r-s+k_i} \times (q^{r-s+k_1}-1) \times (q^{r-s+k_m}-1) \\ &= \left(q^{(|J|-2)(r-s)+\sum\limits_{i \in J/\{1,m\}} k_i} \right) \times (q^{(r-s+k_1)}-1) \times \times (q^{(r-s+k_m)}-1) \\ &= \alpha \end{aligned} \tag{3.5}$$

So the total number of possible ways of choosing $q_i(x)$ and hence $b_i(x)\ \forall\ i \in N$

$$\begin{aligned} &= (6)\times(7) \\ &= \alpha \times 1 = \alpha. \end{aligned} \tag{3.6}$$

Out of all these possible ways, we have to eliminate those possibilities which gives rise to either first column or last column of B to be zero.

Now, first column of B is zero when constant term of $q_i(x) = 0\ \forall i \in J$.

The number of ways in which constant term of $q_i(x) = 0$ for $i \in J/(\{1, m\}$

$$\begin{aligned} &= 1 + (q-1) + (q-1)q + \cdots + (q-1)q^{p-1} \\ &= 1 + (q-1)\left(\frac{q^P - 1}{q-1}\right) = q^P = q^{(r-1)-(s-k_i)}. \end{aligned}$$

Also, the number of ways in which constant term of $q_i(x)$ and $q_m(x)$ is zero are $(q^{(r-1)-(s-k_1)} - 1)$ and $(q^{(r-1)-(s-k_m)} - 1)$ respectively.

So the number of ways which give rise to first column of B to be zero are given by

$$\begin{aligned} &= \prod_{i \in J/\{1,m\}} q^{(r-1)-(s-k_i)} \times (q^{(r-1)-(s-k_1)} - 1) \times (q^{(r-1)-(s-k_m)} - 1) \\ &= \left(q^{(|J|-2)(r-1-s) + \sum\limits_{i \in J/\{1,m\}} k_i}\right) \times (q^{(r-1)-(s-k_1)} - 1) \times (q^{(r-1)-(s-k_m)} - 1) \\ &= \beta \end{aligned}$$

Similarly, the last column of submatrix B is zero where $q_i(x)$ is a polynomial of degree upto $P-1\ \forall\ i \in J$.

Now, the number of possible ways of choosing $q_i(x)$ $(i \in J/\{1, m\})$ such that $q_i(x)$ is a polynomial of degree upto $P-1$ is given by

$$\begin{aligned} & q + (q-1)q + \cdots + (q-1)q^{P-1} \\ &= q + (q-1)q(1 + q + \cdots + q^{P-2}) \\ &= q^P = q^{(r-1)-(s-k_i)} \end{aligned}$$

Also, for $i \in \{1, m\}$, the number of possible ways of choosing $q_i(x)$ such that $q_i(x)$ is a polynomial of degree upto $P-1$ is given by $(q^{(r-1)-(s-k_i)} - 1)$.

So the number of ways which give rise to the last column of B as zero are given by

$$\begin{aligned}
&= \prod_{i\in J/\{1,m\}} q^{(r-1)-(s-k_i)} \times (q^{(r-1)-(s-k_1)}-1) \times (q^{(r-1)-(s-k_m)}-1) \\
&= \left(q^{(|J|-2)(r-1-s)+\sum\limits_{i\in J/\{1,m\}} k_i}\right) \times (q^{(r-1)-(s-k_1)}-1) \times \\
&\quad \times (q^{(r-1)-(s-k_m)}-1) \\
&= \beta
\end{aligned}$$

So the number of ways which gives rise to either first column or last column of B to be zero

$$= \beta + \beta = 2\beta \tag{3.7}$$

Subtracting (9) and (8) and using the fact that the burst A of order $m \times r$ can have first $(s-r+1)$ positions as the starting positions, we get total number of bursts of order $m \times r (r > \max\{s-k_1, s-k_m\})$ that go undetected in the row-cyclic array code C and is given by

$$(s-r+1)(\alpha - 2\beta) \tag{3.8}$$

where α and β are given by (4) and (5) respectively.

Also, total number of bursts of order $m \times r$ viz. $B^{m\times r}_{m\times s}(F_q)$ is given by (2). Therefore, the required ratio is obtained on dividing (10) by (2). □

Example 3.1. Let C be the binary $[2\times 2, 1+1]$ row-cyclic array code of order 2×2 generated by $(g_1(x), g_2(x)) = (1+x, 1+x)$. Then $C = C_1 \oplus C_2$ where C_1 and C_2 are classical cyclic codes of length 2 each generated by $1+x$.

Here $k_1 = k_2 = 1$ and $s = 2$.

Therefore, $s-k_1 = s-k_2 = 1$. Let $r = 2$. Then $2 = r > \max\{= s-k_1, s-k_2\} = 1$.

Here $N = J = \{1, 2\}$.

The ratio computed in (3) for this example turns out to be 1/7. The ratio is justified by the fact that there are 7 bursts of order 2×2 in $\mathrm{Mat}_{2\times 2}(F_2)$ given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. and out of these 7 bursts, only one burst viz. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is undetected by the row-cyclic array code C.

Example 3.2. Let $C = C_1 \oplus C_2$ be a row-cyclic array code of order 2×2 generated by $(g_1(x), g_2(x)) = (1, 1+x)$. It is clear that C_1 and C_2 are classical cyclic codes of length 2 generated by 1 and $1 + x$ respectively.

Here $k_1 = 2, k_2 = 1$ and $s = 2$.

Therefore, $s - k_1 = 0$ and $s - k_2 = 1$.

Let $r = 2$. Then $2 = r > \max\{s - k_1, s - k_2\} = 1$.

Here $N = J = \{1, 2\}$.

The ratio computed in (3) for this example turns out to be 3/7 and is justified by the fact that out of 7 bursts of order 2×2 in $\mathrm{Mat}_{2\times2}(F_2)$ listed in Example 3.1, there are 3 burst viz.

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

that go undetected in the row-cyclic array code C. □

Example 3.3. Let $C = C_1 \oplus C_2$ be a row-cyclic array code of order 2×2 generated by $(g_1(x), g_2(x)) = (1, 1)$.

Here $k_1 = k_2 = 2$ and $s = 2$.

Therefore, $s - k_1 = s - k_2 = 0$.

Let $r = 1$. Then $1 = r > \max\{= s - k_1, s - k_2\} = 0$.

The ratio computed in (3) for this example turns out to be 2/2=1 and is justified by the fact that there are 2 bursts of order 2×1 in $\mathrm{Mat}_{2\times2}(F_2)$ viz.

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix},$$

and both of which go undetected by the row-cyclic array code C.

Similarly, on taking $r = 2$ in this example, the ratio compared in (3) turns out to be 7/7=1 (Note that for this case $\alpha = 9, \beta = 1$) and is justified by the fact that all the 7 bursts of order 2×2 in $\mathrm{Mat}_{2\times2}(F_2)$ listed in Example 3.1 go undetected.

4. Decoding Algorithm for Burst Error Correction

In this section, we give decoding algorithm for burst error correction in row-cyclic array codes.

Algorithm.

Let $C = \bigoplus_{i=1}^{m} C_i$ be a q-ary $[m \times s, \sum_{i=1}^{m} k_i, \min_{i=1}^{m} d_i]$ row-cyclic array code having generator m-tuple of polynomials $(g_1(x), g_2(x), \cdots, g_m(x))$ and correcting all burst errors of order mr or less $(1 \leq r \leq s)$. Let $w(x) = (w_1(x), w_2(x), \cdots, w_m(x))$ be a received array with an error pattern $e(x) = (e_1(x), e_2(x), \cdots, e_m(x))$ such that $e(x)$ is a burst of order mr or less $(1 \leq r \leq s)$. The goal is to determine $e(x)$. This is obtained in the following four steps:

Step 1. Compute the syndrome m-tuple $(S_j^{(1)}(x), S_j^{(2)}(x), \cdots, S_j^{(m)}(x))$ for $j = 0, 1, 2, \cdots$ where for all $i = i \ to \ m, S_j^{(i)}(x)$ is given by

$$S_j^{(i)}(x) = \text{syndrome of } x^j w_i(x).$$

Step 2. Find the m-tuple of nonnegative integers $(l_1, l_2, \cdots, l_m)$ such that syndrome for $x^{l_i} w_i(x)(1 \leq i \leq m)$ is a classical burst of length r or less. (Note that here less also means length of zero.)

Step 3. Compute the remainder m-tuple $e(x) = (e_1(x), \cdots, e_m(x))$ where for all $i = i \ to \ m, e_i(x)$ is given by

$$e_i(x) = x^{s-l_i} S_{l_i}^{(i)}(x)(\text{mod } (x^s - 1)).$$

Step 4. Decode $(w_1(x), \cdots, w_m(x))$ to $(w_1(x) - e_1(x), \cdots, w_m(x) - e_m(x))$.

Proof of Algorithm. First of all, we show the existence of m-tuple of nonnegative integers $(l_1, l_2, \cdots, l_m)$ in Step 2. By the assumption, there exists an error pattern $e(x) = (e_1(x), \cdots, e_m(x))$ such that $e(x)$ is a burst of order mr or less which in turn implies that each $e_i(x)(1 \leq i \leq m)$ has a cyclic run of zeros of length $s - r$. (A cyclic run of zeros of length l of an s-tuple is a succession of l cyclically consecutive zero components). Thus there exists an m-tuple $(l_1, l_2, \cdots, l_m)$ such that cyclic array shift of the error $(e_1(x), \cdots, e_m(x))$ through $(l_1, l_2, \cdots, l_m)$ positions (or equivalently, cyclic shift of error $e_i(x)$ through l_i positions ($1 \le i \le m$) in classical sense) has all its nonzero components confined to first r columns of e (Note that we are identifying $e(x) \leftrightarrow e$ under the map θ). The cyclic shift of error $e_i(x)$ through l_i positions ($1 \le i \le m$) is in fact the remainder of $x^{l_i} w_i(x) (\text{mod } (x^s - 1))$ divided by $g_i(x)$.

Also, for all $i = 1$ *to* m

$$\begin{aligned} S_{l_i}^{(i)}(x) &= (x^{l_i} w_i(x) (\text{mod } (x^s - 1)) (\text{mod } g_i(x)) \\ &= (x^{l_i} w_i(x) (\text{mod } g_i(x)). \end{aligned}$$

Therefore, each $S_{l_i}^{(i)}(x) (1 \le i \le m)$ is a classical burst of length r or less. Now, for all $i = 1$ *to* m, the word

$$t_i(x) = (x^{s-l_i} S_{l_i}^{(i)}(x)) (\text{mod } (x^s - 1))$$

is a cyclic shift of $(S_{l_i}^{(i)}, 0)$ through $s - l_i$ positions, where $S_{l_i}^{(i)}$ is a vector in $F_q^{s-k_i}$ corresponding to the polynomial $S_{l_i}^{(i)}$. It is clear that each $t_i(x)$ is a classical burst of order r or less. Also, for all $i = 1$ *to* m, we have

$$\begin{aligned} x^{l_i}(w_i(x) - t_i(x)) &= x^{l_i}(w_i(x) - x^{s-l_i} S_{l_i}^{(i)}(x)) \\ &= x^{l_i} w_i(x) - x^s S_{l_i}^{(i)}(x) \\ &= S_{l_i}^{(i)}(x) - x^s S_{l_i}^{(i)}(x) \\ &= (1 - x^s) S_{l_i}^{(i)}(x) \\ &\equiv 0 (\text{mod } (g_i(x))). \end{aligned} \tag{4.1}$$

Since $g_i(x)$ and x^{l_i} are coprime to each other, therefore from (11), we get

$$g_i(x) | (w_i(x) - t_i(x)) \quad \forall\ i = 1, 2, \cdots, m$$

$$\Rightarrow w_i(x) - t_i(x) \in C_i \quad i = 1 \text{ to } m.$$

Also $w_i(x) - e_i(x) \in C_i$ implies $e_i(x) - t_i(x) \in C_i$ which further implies that $e_i(x)$ and $t_i(x)$ belong to the same coset (mod $g_i(x)$). Since both $e_i(x)$ and $t_i(x)$ are the classical bursts of length r or less and each C_i is r burst error correcting classical cyclic code (since $C = \bigoplus_{i=1}^{m} C_i$ corrects all bursts of order $m \times r$, we get

$$e_i(x) = t_i(x) = (x^{s-l_i} S_{l_i}^{(i)}(x)) (\text{mod } (x^s - 1)).$$

□

Remark 4.1 The above algorithm also holds for the correction of all bursts of order pr or less $(1 \leq p \leq m, 1 \leq r \leq s)$.

Example 4.1. Consider the binary row-cyclic array code $C = \oplus_{i=1}^{2} C_i$ where C_1 and C_2 are [7,4,4] classical cyclic codes in F_2^7 equipped with m-metric and generated by $g_1(x) = 1 + x^2 + x^3$ and $g_2(x) = 1 + x + x^3$ respectively. Then parameters of row cyclic code C are $[2 \times 7, 4 + 4, 4]$. A simple calculation shows that code C satisfies the sufficient condition for burst error correction [7, Theorem 4.2] for $p = 2$ and $r = 1$. Alternatively, this can also be seen from the fact that syndromes 2-tuples of all burst array errors of order 2×1 or less are all distinct as shown in Table 4.1.

Table 4.1

Bursts of order 2×1 or less in $\text{Mat}_{2\times 7}(F_2)$	Syndrome 2-tuple
$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(100, 100)
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(010, 010)
$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	(001, 001)
$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$	(101, 110)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	(111, 011)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	(110, 111)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	(011, 101)
$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(100, 000)
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(010, 000)
$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(001, 000)
$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(101, 000)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(111, 000)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(110, 000)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(011, 000)

Table contd.

Bursts of order 2×1 or less in $\mathrm{Mat}_{2\times 7}(F_2)$	Syndrome 2-tuple
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(000, 100)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	(000, 010)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$	(000, 001)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$	(000, 110)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$	(000, 011)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$	(000, 111)
$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$	(000, 101)

The syndrome 2-tuple $S = (S_1, S_2)$ for a burst $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ of order 2×1 or less for the code C have been found by using the relation $S = bH^T$ where H is the parity check matrix of the code C and is given by

$$H = \begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

and

$$H_2 = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Now, consider the received array

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \in \mathrm{Mat}_{2\times 7}(F_2).$$

Under the identification $\theta : \mathrm{Mat}_{m\times s}(F_2) \longleftrightarrow A_s^{(m)}$, w can be identified as

$$w = \begin{pmatrix} 1+x^2+x^3+x^4 \\ 1+x+x^3+x^4 \end{pmatrix} = \begin{pmatrix} w_1(x) \\ w_2(x) \end{pmatrix}.$$

We compute the syndrome $S_j^{(i)}(x)$ of $x^j w_i(x)(1 \le i \le 2)$ until $S_j^{(i)}$ is a classical burst of length 1 or less.

Table 4.2

j	$S_j^{(1)}(x)$	$S_j^{(2)}(x)$
0	$1+x+x^2$	$x+x^2$
1	$1+x$	$1+x+x^2$
2	$x+x^2$	$1+x^2$
3	1	1

Therefore, $l_1 = l_2 = 3$ i.e. $(l_1, l_2) = (3, 3)$.
Decode $w_1(x) = (1011100) = 1 + x^2 + x^3 + x^4$ to $w_1(x) - t_1(x)$ where

$$\begin{aligned} t_1(x) = e_1(x) &= x^{s-l_1} S_{l_1}^{(1)}(x)(\mathrm{mod}\ (x^s - 1)) \\ &= x^{7-3} S_3^{(1)}(x)(\mathrm{mod}\ (x^7 - 1)) \\ &= x^4 \end{aligned}$$

Thus $w_1(x)$ is decoded to

$$w_1(x) - t_1(x) = 1 + x^2 + x^3 + x^4 - x^4 = 1 + x^2 + x^3 = 1011000$$

Similarly, decode $w_2(x) = 1101100 = 1 + x + x^3 + x^4$ to $w_2(x) - t_2(x)$ where

$$\begin{aligned} t_2(x) = e_2(x) &= x^{s-l_2} S_{l_2}^{(2)}(x)(\mathrm{mod}\ (x^s - 1)) \\ &= x^{7-3} S_3^{(2)}(x)(\mathrm{mod}\ (x^7 - 1)) \\ &= x^4 \end{aligned}$$

Therefore, $w_2(x)$ is decoded to

$$w_2(x) - t_2(x) = 1 + x + x^3 + x^4 - x^4 = 1 + x + x^3 = 1101000.$$

Hence

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

is decoded to $\begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$.

Remark 4.2. Since the $[2 \times 7, 4+4, 4]$ row-cyclic array code C of Example 4.1 corrects all bursts of order 2×1 or less, therefore the code C must satisfy the Rieger's bound for an $[m \times s, k]$ m-metric array code correcting all bursts or order pr or less ($1 \leq p, m, 1 \leq r \leq s$) obtained in [6] and is given by

$$ms - k \geq 2pr$$

or

$$ms - \sum_{i=1}^{m} k_i \geq 2pr \quad \left(\text{as } k = \sum_{i=1}^{m} k_i \ \text{ for row-cyclic array codes}\right)$$

which is true as

$$14 - 8 \geq 2 \times 2 \times 1$$

or

$$6 \geq 4.$$

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References

[1] M. Blaum, P.G. Farrell and H.C.A. van Tilborg, *Array Codes*, in Handbook of Coding Theory, (Ed.: V. Pless and Huffman), Vol. II, Elsevier, North-Holland, 1998, pp.1855-1909.

[2] C.N. Campopiano, *Bounds on Burst Error Correcting Codes*, IRE. Trans., IT-8 (1962), 257-259.

[3] S.T. Dougherty and M.M. Skriganov, *MacWilliams duality and the Rosenbloom-Tsfasman metric*, Moscow Mathematical Journal, **2** (2002), 83-99.

[4] E.M. Gabidulin and V.V. Zanin, *Matrix codes correcting array errors of size* 2×2, International Symp. on Communication Theory and Applications, Ambleside, U.K., 11-16 June, 1993.

[5] P. Fire, *A Class of Multiple-Error-Correcting Binary Codes for Non-Independent Errors*, Sylvania Reports RSL-E-2, 1959, Sylvania Reconnaissance Systems, Mountain View, California.

[6] S. Jain, *Bursts in m-Metric Array Codes*, Linear Algebra and Its Applications, **418** (2006), 130-141.

[7] S. Jain, *Campopiano-Type Bounds in Non-Hamming Array Coding*, Linear Algebra and Its Applications, **420** (2007), 135-159.

[8] S. Jain, *CT Bursts-From Classical to Array Coding*, Discrete Mathematics, **308-309** (2008), 1489-1499.

[9] S. Jain, *An Algorithmic Approach to Achieve Minimum ρ-Distance atleast d in Linear Array Codes*, Kyushu Journal of Mathematics, **62** (2008), 189-200.

[10] S. Jain, *Row-Cyclic Codes in Array Coding*, Algebras, Groups, Geometries, **25** (2008), 287-310.

[11] S. Jain, *On a Class of Blockwise-Bursts in Array Codes*, to appear in Ars Combinatoria.

[12] W.W. Peterson and E.J. Weldon, Jr., Error Correcting Codes, 2nd Edition, MIT Press, Cambridge, Massachusetts, 1972.

[13] M.Yu. Rosenbloom and M.A. Tsfasman, *Codes for m-metric*, Problems of Information Transmission, **33** (1997), 45-52.