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On a Sum Form Functional Equation Related to Various Nonadditive Entropies in Information Theory *

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Abstract

In this paper, the general solutions of a sum form functional equation containing four unknown mappings have been obtained without imposing any regularity condition on any of the four mappings. Some of the solutions obtained are related to nonadditive entropies of degree α and of type (α, β).

Keywords and Phrases: *Functional equation, Additive mapping, Multiplicative mapping, Logarithmic mapping, Entropy of degree* α*, Entropy of type* (α, β)

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1. Introduction

For $n = 1, 2, \ldots$; let $\Gamma_n = \left\{(p_1, \ldots, p_n) : p_i \geq 0, i = 1, \ldots, n; \sum_{i=1}^{n} p_i = 1\right\}$ denote the set of all n-component discrete probability distributions with nonnegative elements.

A mapping $a : I \to \mathbb{R}$ is said to be additive on I or on the unit triangle

$$\Delta = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq x + y \leq 1\}$$

if it satisfies the equation $a(x + y) = a(x) + a(y)$ for all $(x, y) \in \Delta$; $I = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$, $\mathbb{R}$ denoting the set of all the real numbers. It is known [3] that if a mapping $a : I \to \mathbb{R}$ is additive on the unit triangle Δ, then there exists one and only one mapping $A : \mathbb{R} \to \mathbb{R}$ which is an extension of $a : I \to \mathbb{R}$ in the sense that $A(x) = a(x)$ for all $x \in I$ and is additive on $\mathbb{R}$, that is, $A(x + y) = A(x) + A(y)$ for all $x \in \mathbb{R}$, $y \in \mathbb{R}$.

A mapping $M : I \to \mathbb{R}$ is said to be multiplicative on I if $M(pq) = M(p)M(q)$ for all $p \in I$, $q \in I$.

A mapping $\ell : I \to \mathbb{R}$ is said to be logarithmic if $\ell(0) = 0$ and $\ell(pq) = \ell(p) + \ell(q)$ for all $p \in]0, 1]$ and $q \in]0, 1]$ where $]0, 1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$.

This paper is devoted to the study of the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} G(p_i) \sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} K(p_i) \sum_{j=1}^{m} q_j^{\beta} \tag{A}$$

in which $F : I \to \mathbb{R}$, $G : I \to \mathbb{R}$, $H : I \to \mathbb{R}$, $K : I \to \mathbb{R}$ are unknown mappings; $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ are fixed integers; and β is fixed positive real power satisfying the conventions $0^\beta := 0$, $1^\beta := 1$. Assuming the continuity (or the Lebesgue measurability) of only two mappings G and H, the solutions of (A) have been obtained for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n, m = 1, 2, \ldots$ in [7].

If $\beta = 1$, then (A) reduces to the functional equation

$$\sum_{i=1}^{n} \sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} G(p_i) \sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} K(p_i). \tag{1.1}$$

The functional equation (1.1) has been considered by Taneja [16] assuming each of the mappings $F : I \to \mathbb{R}$, $G : I \to \mathbb{R}$, $H : I \to \mathbb{R}$, $K : I \to \mathbb{R}$

to be continuous on I. He determined the continuous solutions of (1.1) for $n = 1, 2, \ldots$; $m = 1, 2, \ldots$ and characterized the Shannon [15] entropy

$$H_n(p_1, \ldots, p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i \tag{1.2}$$

with $H_n : \Gamma_n \to \mathbb{R}$; $n = 1, 2, \ldots$; $0 \log_2 0 := 0$; and the nonadditive entropies $H_n^\alpha(p_1, \ldots, p_n)$ (see Havrda and Charvát [5]) of degree α, $0 < \alpha \in \mathbb{R}$, $\alpha \neq 1$, where

$$H_n^\alpha(p_1, \ldots, p_n) = (1 - 2^{1-\alpha})^{-1} \left[1 - \sum_{i=1}^{n} p_i^\alpha\right] \tag{1.3}$$

with $H_n^\alpha : \Gamma_n \to \mathbb{R}$; $n = 1, 2, \ldots$ and $0^\alpha := 0$, $1^\alpha := 1$ for all $0 < \alpha \in \mathbb{R}$. Dial [4] has also discussed the functional equation (1.1) for $n = m = 2$ by assuming each of the mappings F, G, H and K to be measurable on I in the sense of Lebesgue. Nath and Singh [12] have obtained the general solutions of (1.1) for fixed integers $n \geq 3$, $m \geq 3$ without imposing any regularity condition on any of the mappings appearing in (1.1). Some of the solutions obtained in this case are related to the entropies (1.2) and (1.3).

If $G(x) = x^\alpha$ for all $x \in I$; α being a fixed positive real power satisfying the conventions $0^\alpha := 0$, $1^\alpha := 1$; $H(x) = F(x)$ and $K(x) = F(x)$; then (A) reduces to the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} p_i^\alpha \sum_{j=1}^{m} F(q_j) + \sum_{i=1}^{n} F(p_i) \sum_{j=1}^{m} q_j^\beta \tag{1.4}$$

in which $F : I \to \mathbb{R}$ is an unknown mapping, $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; and α, β are fixed positive real powers satisfying the conventions

$$0^\alpha := 0, \quad 0^\beta := 0, \quad 1^\alpha := 1, \quad 1^\beta := 1. \tag{1.5}$$

The functional equation (1.4) plays an important role in the characterization of the entropy $H_n^{(\alpha,\beta)}(p_1, \ldots, p_n)$ (Behara and Nath [1]) of type (α, β) where $H_n^{(\alpha,\beta)} : \Gamma_n \to \mathbb{R}$, $n = 1, 2, \ldots$ as

$$H_n^{(\alpha,\beta)}(p_1, \ldots, p_n) = \begin{cases} (2^{1-\alpha} - 2^{1-\beta})^{-1} \left(\sum\limits_{i=1}^{n} p_i^\alpha - \sum\limits_{i=1}^{n} p_i^\beta \right) & \text{if } \alpha \neq \beta \\ -\, 2^{\beta-1} \sum\limits_{i=1}^{n} p_i^\beta \log_2 p_i & \text{if } \alpha = \beta \end{cases} \tag{1.6}$$

where $0^{\beta}\log_2 0 := 0$ and α, β are fixed positive real powers which satisfy the conventions stated in (1.5).

If (i) $\beta = 1$; $K(p) = F(p)$ and (ii) $\beta = 1$; $H(p) = F(p) = K(p)$, then (A) reduces respectively to the functional equations

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} G(p_i)\sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} F(p_i) \tag{1.7}$$

and

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} G(p_i)\sum_{j=1}^{m} F(q_j) + \sum_{i=1}^{n} F(p_i). \tag{1.8}$$

The general solutions of (1.7) and (1.8) for fixed integers $n \geq 3$, $m \geq 3$ have been discussed by Nath and Singh [11], [10] respectively.

The object of this paper is to determine the general solutions of (A) (without imposing any regularity condition on any of the mappings F, G, H and K) for fixed integers $n \geq 3$, $m \geq 3$ and for all probability distributions $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $\beta \neq 1$, a fixed positive real power. During the process of finding the general solutions of (A) for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers, we have come across the functional equations

$$\sum_{i=1}^{n}\sum_{j=1}^{m} F(p_i q_j) = \sum_{i=1}^{n} K(p_i)\sum_{j=1}^{m} q_j^{\beta}, \tag{B}$$

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} h(p_i q_j) = \sum_{i=1}^{n} g(p_i)\sum_{j=1}^{m} h(q_j) + \sum_{i=1}^{n} h(p_i)\sum_{j=1}^{m} q_j^{\beta}& \\ + (m-n)h(0)\sum_{j=1}^{m} q_j^{\beta} + m(n-1)h(0)&\end{aligned} \tag{C}$$

and

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{m} h(p_i q_j) = \sum_{i=1}^{n} p_i^{\beta}\sum_{j=1}^{m} h(q_j) + \sum_{i=1}^{n} h(p_i)\sum_{j=1}^{m} q_j^{\beta}& \\ + (m-n)h(0)\sum_{j=1}^{m} q_j^{\beta} + m(n-1)h(0)&\end{aligned} \tag{D}$$

valid for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; $n \geq 3$, $m \geq 3$ being fixed integers; β being a fixed positive real power, $\beta \neq 1$, satisfying the conventions $0^\beta := 0$, $1^\beta := 1$; $F : I \to \mathbb{R}$, $K : I \to \mathbb{R}$, $h : I \to \mathbb{R}$, $g : I \to \mathbb{R}$ being unknown mappings. Under the conditions specified above, the general solutions of (B), (C) and (D) have been investigated in this paper.

The functional equation (C) is an extended form of the functional equation

$$\sum_{i=1}^{n}\sum_{j=1}^{m} h(p_i q_j) = \sum_{i=1}^{n} g(p_i) \sum_{j=1}^{m} h(q_j) + \sum_{i=1}^{n} h(p_i) \sum_{j=1}^{m} q_j^\beta$$

recently considered by Nath and Singh [14]; the mappings h, g and the fixed real power β as described in (C)

Now, we mention below some results needed for the development of subsequent sections of this paper.

Result 1.1 ([6])**.** *Suppose a mapping $\phi : I \to \mathbb{R}$ satisfies the functional equation $\sum_{i=1}^{n} \phi(p_i) = c$ for all $(p_1, \ldots, p_n) \in \Gamma_n$, $n \geq 3$ a fixed integer and c a given real constant. Then there exists an additive mapping $A : \mathbb{R} \to \mathbb{R}$ such that $\phi(p) = A(p) - \frac{1}{n}A(1) + \frac{c}{n}$ for all $p \in I$.*

Result 1.2 ([8])**.** *Let $n \geq 3$, $m \geq 3$ be fixed integers. If a mapping $T : I \to \mathbb{R}$ satisfies the equation*

$$\sum_{i=1}^{n}\sum_{j=1}^{m} T(p_i q_j) = \sum_{i=1}^{n} T(p_i) \sum_{j=1}^{m} T(q_j) + (m-n)T(0) \sum_{j=1}^{m} T(q_j) + m(n-1)T(0) \tag{1.9}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$ and $(q_1, \ldots, q_m) \in \Gamma_m$, then either

$$T(p) = a(p) + T(0) \tag{1.10}$$

where $a : \mathbb{R} \to \mathbb{R}$ is an additive mapping with

$$a(1) = \begin{cases} -\,mT(0) & \text{if } \; T(1) + (m-1)T(0) \neq 1 \\ 1 - mT(0) & \text{if } \; T(1) + (m-1)T(0) = 1 \end{cases} \tag{1.11}$$

or

$$T(p) = M(p) - b(p) + T(0) \tag{1.12}$$

where $b : \mathbb{R} \to \mathbb{R}$ is an additive mapping with $b(1) = mT(0)$ and $M : I \to \mathbb{R}$ is a nonconstant nonadditive multiplicative mapping with $M(0) = 0$ and $M(1) = 1$.

Note. The functional equation (1.9) seems to be important because of its occurrence also in the papers [8] to [13].

Result 1.3 ([14])**.** *Let $m \geq 3$ be fixed integer. Suppose a mapping $h_1 : I \to \mathbb{R}$ satisfies the equation*

$$\sum_{j=1}^{m}[h_1(pq_j) - h_1(p)q_j^\beta - p^\beta h_1(q_j)] = 0 \tag{1.13}$$

for all $p \in I$, $(q_1, \ldots, q_m) \in \Gamma_m$; $\beta \neq 1$ being a fixed positive real power such that $0^\beta := 0$, $1^\beta := 1$. If $h_1(0) = 0$, then any general solution of (1.13) *is of the form*

$$h_1(p) = p^\beta \ell(p) \tag{1.14}$$

for all $p \in I$, $\ell : I \to \mathbb{R}$ being a logarithmic mapping.

2. The general solutions of functional equations (D) and (B)

In this section, we obtain:

Theorem 2.1. *Let $n \geq 3$, $m \geq 3$ be fixed integers. If a mapping $h : I \to \mathbb{R}$ satisfies the functional equation* (D) *for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$; β being a fixed positive real power such that $0^\beta := 0$, $1^\beta := 1$ and $\beta \neq 1$, then any general solution of it is of the form*

$$h(p) = p^\beta \ell(p) - b_2(p) + h(0) \tag{2.1}$$

for all $p \in I$; $\ell : I \to \mathbb{R}$ being a logarithmic mapping; $b_2 : \mathbb{R} \to \mathbb{R}$ being an additive mapping with $b_2(1) = mh(0)$.

Proof. Proceeding as in the proof of Lemma 3.1 [14] and using Result 1.3, the solution (2.1) of (D) can be obtained. The details are omitted for the sake of brevity.

Theorem 2.2. *Let $n \geq 3$, $m \geq 3$ be fixed integers and $F : I \to \mathbb{R}$, $K : I \to \mathbb{R}$ be mappings which satisfy the functional equation* (B) *for all $(p_1, \ldots, p_n) \in \Gamma_n$,*

$(q_1,\dots,q_m)\in\Gamma_m$; β a fixed positive real number such that $\beta\neq 1$ and $0^\beta:=0$, $1^\beta:=1$. Then any general solution (F,K) of (B) *is of the form*

$$\left.\begin{array}{ll}\text{(i)} & F(p)=A_1(p)+[K(1)+(n-1)K(0)]p^\beta+F(0)\\ \text{(ii)} & K(p)=A_2(p)+[K(1)+(n-1)K(0)]p^\beta+K(0)\end{array}\right\}\qquad(\alpha_1)$$

where $A_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ are additive mappings with $A_1(1)=-\,nmF(0)$ and $A_2(1)=-\,nK(0)$.

Proof. Putting $p_1=1$, $p_2=\cdots=p_n=0$ in equation (B), we obtain

$$\sum_{j=1}^{m}\{F(q_j)-[K(1)+(n-1)K(0)]q_j^\beta\}=-\,m(n-1)F(0).$$

By Result 1.1, there exists an additive mapping $A_1:\mathbb{R}\to\mathbb{R}$ with $A_1(1)=-\,nmF(0)$ such that (α_1)(i) follows. From (B), (α_1)(i), and the fact that $\sum_{j=1}^{m}q_j^\beta>0$ for all $(q_1,\dots,q_m)\in\Gamma_m$, it follows that

$$\sum_{i=1}^{n}\{K(p_i)-[K(1)+(n-1)K(0)]p_i^\beta\}=0.$$

By Result 1.1, there exists an additive mapping $A_2:\mathbb{R}\to\mathbb{R}$ with $A_2(1)=-\,nK(0)$ such that (α_1)(ii) follows.

3. The general solutions of functional equation (C)

We prove:

Theorem 3.1. *Let $n\geq 3$, $m\geq 3$ be fixed integers and $h:I\to\mathbb{R}$, $g:I\to\mathbb{R}$ be mappings which satisfy the functional equation* (C) *for all $(p_1,\dots,p_n)\in\Gamma_n$, $(q_1,\dots,q_m)\in\Gamma_m$; β a fixed positive real power such that $\beta\neq 1$ and $0^\beta:=0$, $1^\beta:=1$. Then, any general solution (h,g) of* (C) *is of the form*

$$\left.\begin{array}{ll}\text{(i)} & h(p)=[h(1)+(m-1)h(0)]p^\beta+b_1(p)+h(0);\quad b_1(1)=-\,mh(0)\\ \text{(ii)} & g(p)=a_1(p)+g(0);\quad a_1(1)=-\,ng(0)\end{array}\right\}\qquad(3.1)$$

or

$$\left.\begin{array}{rl}\text{(i)} & h(p)=p^{\beta}\ell(p)-b_2(p)+h(0);\quad b_2(1)=mh(0)\\ \text{(ii)} & g(p)=p^{\beta}+a_2(p)+g(0);\quad a_2(1)=-\,ng(0)\end{array}\right\}\tag{3.2}$$

or

$$\left.\begin{array}{rl}\text{(i)} & h(p)=b_3(p)+h(0);\quad b_3(1)=-\,mh(0)\\ \text{(ii)} & g\textit{ an arbitrary real-valued mapping}\end{array}\right\}\tag{3.3}$$

or

$$\left.\begin{array}{rl}\text{(i)} & h(p)=d[a(p)-p^{\beta}]+h(0);\quad a(1)=1-\dfrac{m}{d}h(0)\\ \text{(ii)} & g(p)=a(p)+\bar{B}(p)+g(0)\end{array}\right\}\tag{3.4}$$

or

$$\left.\begin{array}{rl}\text{(i)} & h(p)=d[M(p)-b(p)-p^{\beta}]+h(0);\quad b(1)=\dfrac{m}{d}h(0)\\ \text{(ii)} & g(p)=M(p)-b(p)+\bar{B}(p)+g(0)\end{array}\right\}\tag{3.5}$$

where $M : I \to \mathbb{R}$ is a nonconstant nonadditive multiplicative mapping with $M(0) = 0$, $M(1) = 1$; $\ell : I \to \mathbb{R}$ is a logarithmic mapping; d is an arbitrary nonzero real constant; $a_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2)$, $b_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, 2, 3)$, $a : \mathbb{R} \to \mathbb{R}$, $b : \mathbb{R} \to \mathbb{R}$, $\bar{B} : \mathbb{R} \to \mathbb{R}$ are additive mappings with $\bar{B}(1) = -\,ng(0)+\dfrac{m}{d}h(0)$.

Proof. Putting $q_1 = 1$, $q_2 = \cdots = q_m = 0$ in equation (C), we obtain

$$[h(1)+(m-1)h(0)]\sum_{i=1}^{n} g(p_i)=0. \tag{3.6}$$

Case 1. $h(1) + (m - 1)h(0) \neq 0$.

Then $\sum_{i=1}^{n} g(p_i) = 0$. By Result 1.1, there exists an additive mapping $a_1 : \mathbb{R} \to \mathbb{R}$ with $a_1(1) = -\,ng(0)$ such that (3.1)(ii) holds.

Also, by putting $p_1 = 1$, $p_2 = \cdots = p_n = 0$ in equation (C) and using (3.1)(ii), we obtain

$$\sum_{j=1}^{m}\{h(q_j)-[h(1)+(m-1)h(0)]q_j^{\beta}\}=0.$$

By Result 1.1, there exists an additive mapping $b_1 : \mathbb{R} \to \mathbb{R}$ with $b_1(1) = -\,mh(0)$ such that (3.1)(i) holds.

Case 2. $h(1) + (m-1)h(0) = 0$.

Let us write equation (C) in the form

$$\sum_{j=1}^{m} \left\{ \sum_{i=1}^{n} h(p_i q_j) - h(q_j) \sum_{i=1}^{n} g(p_i) - q_j^{\beta} \sum_{i=1}^{n} h(p_i) - (m-n)h(0)q_j^{\beta} \right\} \\ = m(n-1)h(0).$$

By Result 1.1, there exists a mapping $A_5 : \Gamma_n \times \mathbb{R} \to \mathbb{R}$, additive in second variable, such that

$$\sum_{i=1}^{n} h(p_i q) - h(q) \sum_{i=1}^{n} g(p_i) - q^{\beta} \sum_{i=1}^{n} h(p_i) - (m-n)h(0)q^{\beta} \\ = A_5(p_1, \ldots, p_n; q) - h(0) \sum_{i=1}^{n} g(p_i) + nh(0) \tag{3.7}$$

with $A_5(p_1, \ldots, p_n; 1) = mh(0) \sum\limits_{i=1}^{n} g(p_i) - mh(0)$.

Let $(r_1, \ldots, r_n) \in \Gamma_n$ be any probability distribution. Replacing q successively by $r_1, \ldots, r_n$ in (3.7) and adding the resulting n equations, we obtain the equation

$$\sum_{i=1}^{n} \sum_{t=1}^{n} h(p_i r_t) + (m-n^2)h(0) = \sum_{i=1}^{n} g(p_i) \sum_{t=1}^{n} h(r_t) + \sum_{i=1}^{n} h(p_i) \sum_{t=1}^{n} r_t^{\beta} \\ + (m-n)h(0) \sum_{t=1}^{n} r_t^{\beta} + (m-n)h(0) \sum_{i=1}^{n} g(p_i)\,.$$

The left hand side of the above equation is symmetric in p_i and r_t, $i = 1, \ldots, n$; $t = 1, \ldots, n$. So, the right hand side must also be symmetric in p_i and r_t, $i = 1, \ldots, n$; $t = 1, \ldots, n$. This gives us the equation

$$\left[\sum_{i=1}^{n} g(p_i) - \sum_{i=1}^{n} p_i^{\beta} \right] \left[\sum_{t=1}^{n} h(r_t) + (m-n)h(0) \right] \\ = \left[\sum_{t=1}^{n} g(r_t) - \sum_{t=1}^{n} r_t^{\beta} \right] \left[\sum_{i=1}^{n} h(p_i) + (m-n)h(0) \right]. \tag{3.8}$$

Case 2.1. $\sum\limits_{t=1}^{n} g(r_t) - \sum\limits_{t=1}^{n} r_t^{\beta}$ vanishes identically on Γ_n. This means that $\sum\limits_{t=1}^{n}\{g(r_t) - r_t^{\beta}\} = 0$ for all $(r_1, \ldots, r_n) \in \Gamma_n$.

By Result 1.1, there exists an additive mapping $a_2 : \mathbb{R} \to \mathbb{R}$ with $a_2(1) = -\,ng(0)$ such that (3.2)(ii) holds. Now, from equations (C) and (3.2)(ii); equation (D) follows. So, h is of the form (2.1) for all $p \in I$ with mappings $\ell : I \to \mathbb{R}$, $b_2 : \mathbb{R} \to \mathbb{R}$ as described in the statement of Theorem 2.1. So, (3.2)(i) holds.

Case 2.2. $\sum\limits_{t=1}^{n} g(r_t) - \sum\limits_{t=1}^{n} r_t^{\beta}$ does not vanish identically on Γ_n. In this case, there exists a probability distribution $(r_1^*, \ldots, r_n^*) \in \Gamma_n$ such that $\left(\sum\limits_{t=1}^{n} g(r_t^*) - \sum\limits_{t=1}^{n} r_t^{*\beta}\right) \neq 0$. Then from (3.8), we obtain

$$\sum_{i=1}^{n} h(p_i) = d\left[\sum_{i=1}^{n} g(p_i) - \sum_{i=1}^{n} p_i^{\beta}\right] - (m-n)h(0) \tag{3.9}$$

where

$$d = \left[\sum_{t=1}^{n} g(r_t^*) - \sum_{t=1}^{n} r_t^{*\beta}\right]^{-1}\left[\sum_{t=1}^{n} h(r_t^*) + (m-n)h(0)\right].$$

Case 2.2.1. $d = 0$.

In this case, (3.9) reduces to the equation $\sum\limits_{i=1}^{n} h(p_i) = -\,(m-n)h(0)$.

By Result 1.1, there exists an additive mapping $b_3 : \mathbb{R} \to \mathbb{R}$ with $b_3(1) = -\,mh(0)$ such that (3.3)(i) holds. Since $d = 0$, it follows from (3.9) that g is an arbitrary real-valued mapping. Thus, (3.3)(ii) holds.

Case 2.2.2. $d \neq 0$.

In this case, let us write (3.9) as

$$\sum_{i=1}^{n}\left\{g(p_i) - p_i^{\beta} - \frac{1}{d}h(p_i)\right\} = \frac{1}{d}(m-n)h(0). \tag{3.10}$$

By Result 1.1, there exists an additive mapping $\bar{B} : \mathbb{R} \to \mathbb{R}$, with $\bar{B}(1) = -\,ng(0) + \dfrac{m}{d}h(0)$, such that

$$g(p) = \frac{1}{d}[h(p) - h(0)] + p^{\beta} + \bar{B}(p) + g(0). \tag{3.11}$$

Let us define a mapping $T : I \to \mathbb{R}$ as

$$T(x) = \frac{1}{d}h(x) + x^{\beta}, \quad d \neq 0 \tag{3.12}$$

for all $x \in I$. Now, from (C), (3.10) and (3.12); equation (1.9) follows. From equation (3.12), we observe that $T(1) + (m-1)T(0) = 1$ because $h(1) + (m-1)h(0) = 0$. Making use of Result 1.2, it follows that T is of the form (1.10) with $a(1) = 1 - mT(0)$ or of the form (1.12) with $b(1) = mT(0)$, $a : \mathbb{R} \to \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ being additive mappings. In the former case, (3.12) gives the solution (3.4)(i) whereas in the latter case, it gives rise to the solution (3.5)(i). Notice that $T(0) = \frac{1}{d}h(0)$. Consequently, $a(1) = 1 - \frac{m}{d}h(0)$ and $b(1) = \frac{m}{d}h(0)$. The solution (3.4)(ii) follows from (3.4)(i) and (3.11). The solution (3.5)(ii) follows from (3.5)(i) and (3.11).

4. The general solutions of functional equation (A)

The main result of this paper is the following:

Theorem 4.1. *Let $n \geq 3$, $m \geq 3$ be fixed integers and $F : I \to \mathbb{R}$, $G : I \to \mathbb{R}$, $H : I \to \mathbb{R}$, $K : I \to \mathbb{R}$ be mappings which satisfy the functional equation* (A) *for all $(p_1, \dots, p_n) \in \Gamma_n$, $(q_1, \dots, q_m) \in \Gamma_m$; β a fixed positive real number such that $\beta \neq 1$ and $0^{\beta} := 0$, $1^{\beta} := 1$. Then any general solution (F, G, H, K) of* (A) *is of the form*

$$\left.\begin{array}{rl}
\text{(i)} & F(p) \textit{ of the form } (\alpha_1)\text{(i)} \textit{ with } A_1(1) = -\,nmF(0) \\
\text{(ii)} & G(p) = A_3(p) + G(0); \quad A_3(1) = -\,nG(0) \\
\text{(iii)} & H \textit{ an arbitrary real-valued mapping} \\
\text{(iv)} & K(p) \textit{ of the form } (\alpha_1)\text{(ii)} \textit{ with } A_2(1) = -\,nK(0)
\end{array}\right\} \tag{S_1}$$

or

$$\left.\begin{array}{rl}
\text{(i)} & F(p) \textit{ of the form } (\alpha_1)\text{(i)} \textit{ with } A_1(1) = -\,nmF(0) \\
\text{(ii)} & G \textit{ an arbitrary real-valued mapping} \\
\text{(iii)} & H(p) = A_4(p) + H(0); \quad A_4(1) = -\,mH(0) \\
\text{(iv)} & K(p) \textit{ of the form } (\alpha_1)\text{(ii)} \textit{ with } A_2(1) = -\,nK(0)
\end{array}\right\} \tag{S_2}$$

or

$$\left.\begin{array}{rl}
\text{(i)} & F(p) = [F(1) + (nm-1)F(0)]p^{\beta} + [G(1) + (n-1)G(0)]b_3(p) + B_1(p) + F(0) \\
\text{(ii)} & G \textit{ an arbitrary real-valued mapping} \\
\text{(iii)} & H(p) = [H(1) + (m-1)H(0)]p^{\beta} + b_3(p) + H(0); \quad b_3(1) = -\,mH(0) \\
\text{(iv)} & K(p) = [F(1) + (nm-1)F(0)]p^{\beta} + [G(1) + (n-1)G(0)]b_3(p) \\
& \qquad - [H(1) + (m-1)H(0)][G(p) - G(0)] + B_2(p) + K(0)
\end{array}\right\} \qquad (S_3)$$

or

$$\left.\begin{array}{rl}
\text{(i)} & F(p) = \{[G(1) + (n-1)G(0)]\ell(p) + [F(1) + (nm-1)F(0)]\}p^{\beta} \\
& \qquad -[G(1) + (n-1)G(0)]b_2(p) + B_1(p) + F(0) \\
\text{(ii)} & G(p) = [G(1) + (n-1)G(0)][p^{\beta} + a_2(p)] + G(0); [G(1) + (n-1)G(0)] \neq 0, \\
& \qquad a_2(1) = -\,n[G(1) + (n-1)G(0)]^{-1}G(0) \\
\text{(iii)} & H(p) = \{\ell(p) + [H(1) + (m-1)H(0)]\}p^{\beta} - b_2(p) + H(0); \quad b_2(1) = mH(0) \\
\text{(iv)} & K(p) = \{[G(1) + (n-1)G(0)]\ell(p) + [K(1) + (n-1)K(0)]\}p^{\beta} \\
& \qquad - [G(1) + (n-1)G(0)]\{b_2(p) + [H(1) + (m-1)H(0)]a_2(p)\} \\
& \qquad + B_2(p) + K(0)
\end{array}\right\} \qquad (S_4)$$

or

$$\left.\begin{array}{rl}
\text{(i)} & F(p) = \{[F(1) + (nm-1)F(0)] - d[G(1) + (n-1)G(0)]\}p^{\beta} \\
& \qquad + d[G(1) + (n-1)G(0)]a(p) + B_1(p) + F(0) \\
\text{(ii)} & G(p) = [G(1) + (n-1)G(0)][a(p) + \bar{B}(p)] + G(0); \quad [G(1) + (n-1)G(0)] \neq 0 \\
\text{(iii)} & H(p) = \{[H(1) + (m-1)H(0)] - d\}p^{\beta} - da(p) + H(0); \quad a(1) = 1 - \dfrac{m}{d}H(0) \\
\text{(iv)} & K(p) = \{[F(1) + (nm-1)F(0)] - d[G(1) + (n-1)G(0)]\}p^{\beta} \\
& \qquad + [G(1) + (n-1)G(0)]\{d - [H(1) + (m-1)H(0)]\}a(p) \\
& \qquad - [G(1) + (n-1)G(0)][H(1) + (m-1)H(0)]\bar{B}(p) \\
& \qquad + B_2(p) + K(0)
\end{array}\right\} \qquad (S_5)$$

or

$$\left.\begin{aligned}
&\text{(i)}\quad F(p)=d[G(1)+(n-1)G(0)][M(p)-b(p)]+\{[F(1)+(nm-1)F(0)]\\
&\qquad\qquad -d[G(1)+(n-1)G(0)]\}p^{\beta}+B_1(p)+F(0)\\
&\text{(ii)}\quad G(p)=[G(1)+(n-1)G(0)][M(p)-b(p)+\bar{B}(p)]+G(0);\\
&\qquad\qquad [G(1)+(n-1)G(0)]\neq 0\\
&\text{(iii)}\quad H(p)=d[M(p)-b(p)]+\{[H(1)+(m-1)H(0)]-d\}p^{\beta}+H(0);\\
&\qquad\qquad b(1)=\frac{m}{d}H(0)\\
&\text{(iv)}\quad K(p)=[G(1)+(n-1)G(0)]\{d-[H(1)+(m-1)H(0)]\}[M(p)-b(p)]\\
&\qquad\qquad +\{[F(1)+(nm-1)F(0)]-d[G(1)+(n-1)G(0)]\}p^{\beta}\\
&\qquad\qquad -[G(1)+(n-1)G(0)][H(1)+(m-1)H(0)]\bar{B}(p)+B_2(p)+K(0)
\end{aligned}\right\} \tag{S_6}$$

where

$$\begin{aligned}F(1)+(nm-1)F(0)=&[G(1)+(n-1)G(0)][H(1)+(m-1)H(0)]\\&+[K(1)+(n-1)K(0)].\end{aligned}$$

Also $M:I\to\mathbb{R}$ is a nonconstant nonadditive multiplicative mapping with $M(0)=0$ and $M(1)=1$; $\ell:I\to\mathbb{R}$ is a logarithmic mapping and $d\neq 0$ is an arbitrary real constant; $A_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2,3,4)$, $a_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2)$, $b_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2,3)$, $a:\mathbb{R}\to\mathbb{R}$, $b:\mathbb{R}\to\mathbb{R}$, $\bar{B}:\mathbb{R}\to\mathbb{R}$, $B_i:\mathbb{R}\to\mathbb{R}$ $(i=1,2)$ are additive mappings such that

$$\left.\begin{aligned}
&\text{(i)}\quad \bar{B}(1)=-n[G(1)+(n-1)G(0)]^{-1}G(0)+\frac{m}{d}H(0)\\
&\text{(ii)}\quad B_1(1)=m[G(1)+(n-1)G(0)]H(0)-nmF(0)\\
&\text{(iii)}\quad B_2(1)=-nK(0)-n[H(1)+(m-1)H(0)]G(0)\\
&\qquad\qquad +m[G(1)+(n-1)G(0)]H(0).
\end{aligned}\right\} \tag{S_7}$$

Proof. We divide our discussion into three cases:

Case 1. $\sum_{i=1}^{n} G(p_i)$ vanishes identically on Γ_n, that is

$$\sum_{i=1}^{n} G(p_i)=0 \tag{4.1}$$

for all $(p_1,\ldots,p_n)\in\Gamma_n$.

In this case, from equation (A), we observe that H is an arbitrary real-valued mapping. Thus, (S_1)(iii) stands proved. Applying Result 1.1 to equation (4.1), it follows that there exists an additive mapping $A_3:\mathbb{R}\to\mathbb{R}$ with $A_3(1)=-\,nG(0)$ such that $G(p)=A_3(p)+G(0)$. Thus (S_1)(ii) holds. Also, from (A) and (4.1); equation (B) follows. Thus, using Theorem 2.2, we obtain (S_1)(i) and (S_1)(iv).

Case 2. $\sum\limits_{j=1}^{m}H(q_j)$ vanishes identically on Γ_m, that is

$$\sum_{j=1}^{m}H(q_j)=0 \tag{4.2}$$

for all $(q_1,\ldots,q_m)\in\Gamma_m$.

In this case, from equation (A), it is obvious that G is an arbitrary real-valued mapping. So, (S_2)(ii) is true. Applying Result 1.1 to equation (4.2), it follows that there exists an additive mapping $A_4:\mathbb{R}\to\mathbb{R}$ with $A_4(1)=-\,mH(0)$ such that $H(p)=A_4(p)+H(0)$. Thus, (S_2)(iii) holds. Also, from (A) and (4.2), equation (B) follows. Now, using Theorem 2.2, we obtain (S_2)(i) and (S_2)(iv).

Case 3. Neither $\sum\limits_{i=1}^{n}G(p_i)$ vanishes identically on Γ_n nor $\sum\limits_{j=1}^{m}H(q_j)$ vanishes identically on Γ_m.

In this case, there exists a probability distribution $(p_1^*,\ldots,p_n^*)\in\Gamma_n$ and a probability distribution $(q_1^*,\ldots,q_m^*)\in\Gamma_m$ such that $\sum\limits_{i=1}^{n}G(p_i^*)\neq 0$ and $\sum\limits_{j=1}^{m}H(q_j^*)\neq 0$ hold respectively.

Putting $p_1=1$, $p_2=\cdots=p_n=0$ in equation (A), we obtain

$$\sum_{j=1}^{m}\{F(q_j)-[G(1)+(n-1)G(0)]H(q_j)-[K(1)+(n-1)K(0)]q_j^{\beta}\}$$
$$=-\,m(n-1)F(0)\,.$$

By Result 1.1, there exists an additive mapping $B_1:\mathbb{R}\to\mathbb{R}$ with (S_7)(ii) such that

$$F(p) = [G(1) + (n-1)G(0)][H(p) - H(0)] \\ + [K(1) + (n-1)K(0)]p^{\beta} + B_1(p) + F(0)\,. \tag{4.3}$$

From (A) and (4.3), it follows that

$$\begin{aligned} &[G(1) + (n-1)G(0)] \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} H(p_i q_j) - m(n-1)H(0) \right\} \\ &+ [K(1) + (n-1)K(0)] \sum_{i=1}^{n} \sum_{j=1}^{m} (p_i q_j)^{\beta} \\ &\quad = \sum_{i=1}^{n} G(p_i) \sum_{j=1}^{m} H(q_j) + \sum_{i=1}^{n} K(p_i) \sum_{j=1}^{m} q_j^{\beta}\,. \end{aligned} \tag{4.4}$$

The substitutions $q_1 = 1$, $q_2 = \cdots = q_m = 0$ in (4.4) yield the equation

$$\begin{aligned} &\sum_{i=1}^{n} \{K(p_i) + [H(1) + (m-1)H(0)]G(p_i) - [G(1) + (n-1)G(0)]H(p_i) \\ &- [K(1) + (n-1)K(0)]p_i^{\beta}\} = (m-n)[G(1) + (n-1)G(0)]H(0). \end{aligned} \tag{4.5}$$

By Result 1.1, there exists an additive mapping $B_2 : \mathbb{R} \to \mathbb{R}$ with (S_7)(iii) such that

$$\begin{aligned} K(p) = {} & [G(1)+(n-1)G(0)][H(p)-H(0)] - [H(1)+(m-1)H(0)][G(p)-G(0)] \\ & + [K(1) + (n-1)K(0)]p^{\beta} + B_2(p) + K(0). \end{aligned} \tag{4.6}$$

From (4.4), (4.5) and (4.6), we obtain

$$\begin{aligned}
&[G(1)+(n-1)G(0)]\sum_{i=1}^{n}\sum_{j=1}^{m}H(p_iq_j)\\
&=\sum_{i=1}^{n}G(p_i)\left\{\sum_{j=1}^{m}H(q_j)-[H(1)+(m-1)H(0)]\sum_{j=1}^{m}q_j^{\beta}\right\}\\
&\quad+[G(1)+(n-1)G(0)]\sum_{i=1}^{n}H(p_i)\sum_{j=1}^{m}q_j^{\beta}\\
&\quad+(m-n)[G(1)+(n-1)G(0)]H(0)\sum_{j=1}^{m}q_j^{\beta}\\
&\quad+m(n-1)[G(1)+(n-1)G(0)]H(0). \qquad (4.7)
\end{aligned}$$

Define a mapping $h : I \to \mathbb{R}$ as

$$h(x)=H(x)-[H(1)+(m-1)H(0)]x^{\beta} \qquad (4.8)$$

for all $x \in I$. From (4.8), $h(0)=H(0)$ and $h(1)+(m-1)h(0)=0$. Using (4.8), (4.7) can be written as

$$\begin{aligned}
&[G(1)+(n-1)G(0)]\sum_{i=1}^{n}\sum_{j=1}^{m}h(p_iq_j)\\
&=\sum_{i=1}^{n}G(p_i)\sum_{j=1}^{m}h(q_j)+[G(1)+(n-1)G(0)]\sum_{i=1}^{n}h(p_i)\sum_{j=1}^{m}q_j^{\beta}\\
&\quad+(m-n)[G(1)+(n-1)G(0)]h(0)\sum_{j=1}^{m}q_j^{\beta}\\
&\quad+m(n-1)[G(1)+(n-1)G(0)]h(0) \qquad (4.9)
\end{aligned}$$

for all $(p_1,\ldots,p_n)\in\Gamma_n$, $(q_1,\ldots,q_m)\in\Gamma_m$.

Case 3.1. $\sum\limits_{j=1}^{m}h(q_j)$ vanishes identically on Γ_m, that is

$$\sum_{j=1}^{m}h(q_j)=0. \qquad (4.10)$$

Then from (4.9), we observe that G is an arbitrary real-valued mapping. So, (S_3)(ii) holds. Applying Result 1.1 on (4.10), there exists an additive mapping $b_3 : \mathbb{R} \to \mathbb{R}$ with $b_3(1) = -mh(0)$ such that $h(p) = b_3(p) + h(0)$ holds. Solution (S_3)(i) follows from (S_3)(iii) and (4.3); whereas (S_3)(iv) follows from (S_3)(ii), (S_3)(iii) and (4.6).

Case 3.2. $\sum_{j=1}^{m} h(q_j)$ does not vanish identically on Γ_m. Then there exists a probability distribution $(q_1^*, \ldots, q_m^*) \in \Gamma_m$ such that $\sum_{j=1}^{m} h(q_j^*) \neq 0$.

We prove that $G(1) + (n-1)G(0) \neq 0$. To the contrary, suppose that $G(1) + (n-1)G(0) = 0$. Then, (4.9) reduces to $\sum_{i=1}^{n} G(p_i) \sum_{j=1}^{m} h(q_j) = 0$ for all $(p_1, \ldots, p_n) \in \Gamma_n$, $(q_1, \ldots, q_m) \in \Gamma_m$. In particular, $\sum_{i=1}^{n} G(p_i^*) \sum_{j=1}^{m} h(q_j^*) = 0$, which contradicts $\sum_{i=1}^{n} G(p_i^*) \sum_{j=1}^{m} h(q_j^*) \neq 0$. Hence $G(1) + (n-1)G(0) \neq 0$. Then (4.9) reduces to (C) in which $g : I \to \mathbb{R}$ is a mapping defined as

$$g(x) = [G(1) + (n-1)G(0)]^{-1} G(x) \tag{4.11}$$

for all $x \in I$. From (4.11), we observe that $g(0) = [G(1) + (n-1)G(0)]^{-1} G(0)$ and $g(1) + (n-1)g(0) = 1$. Therefore, we consider only those solutions from Theorem 3.1 which satisfy $h(1) + (m-1)h(0) = 0$ and $g(1) + (n-1)g(0) = 1$ and these are (3.2), (3.3), (3.4) and (3.5). Equation (4.11), (4.8), (4.3) and (4.6), taken together; respectively with each of (3.3), (3.2), (3.4) and (3.5), yield the solutions (S_3), (S_4), (S_5) and (S_6) together with the condition (S_7)(i). This completes the proof of the theorem.

5. Comments

The object of this section is to comment upon various solutions, mentioned in Theorem 4.1, from the point of view of information theory.

Keeping in view the form of the entropy of type (α, β) (1.6) for $\alpha = \beta$, it seems desirable to choose the logarithmic mapping $\ell : I \to \mathbb{R}$ defined as

$$\ell(p) = \begin{cases} \lambda \log_2 p & \text{if } 0 < p \leq 1 \\ 0 & \text{if } p = 0 \end{cases} \tag{5.1}$$

where $\lambda \neq 0$ is an arbitrary real constant.

Using the forms of the entropies (1.3) and (1.6), solution (S_4) gives

$$\begin{aligned}
\sum_{j=1}^{m} F(q_j) &= -\lambda[G(1)+(n-1)G(0)]2^{1-\beta} H_m^{(\beta,\beta)}(q_1,\ldots,q_m) \\
&\quad - [F(1)+(nm-1)F(0)](1-2^{1-\beta})H_m^{\beta}(q_1,\ldots,q_m) \\
&\quad + [F(1)+(m-1)F(0)], \\
\sum_{i=1}^{n} G(p_i) &= [G(1)+(n-1)G(0)][-(1-2^{1-\beta})H_n^{\beta}(p_1,\ldots,p_n)+1], \\
\sum_{j=1}^{m} H(q_j) &= -\lambda 2^{1-\beta} H_m^{(\beta,\beta)}(q_1,\ldots,q_m) + [H(1)+(m-1)H(0)] \\
&\quad \times [-(1-2^{1-\beta})H_m^{\beta}(q_1,\ldots,q_m)+1], \\
\sum_{i=1}^{n} K(p_i) &= -\lambda[G(1)+(n-1)G(0)]2^{1-\beta} H_n^{(\beta,\beta)}(p_1,\ldots,p_n) \\
&\quad + [K(1)+(n-1)K(0)][-(1-2^{1-\beta})H_n^{\beta}(p_1,\ldots,p_n)+1].
\end{aligned}$$

Thus we see that in the case of solution (S_4), the mapping G is connected to the entropy of degree β and the mappings F, H, K are connected to the entropy of type (β,β) as well as entropy of degree β.

Similarly, it can be observed that the three mappings F, H, K, appearing in the solutions (S_3) and (S_5) are related to nonadditive entropy of degree β but the summand $\sum_{i=1}^{n} G(p_i)$ in (S_5) is independent of the probabilities whereas, in (S_3), $\sum_{i=1}^{n} G(p_i)$ is arbitrary.

From the point of view of information theory, taking into consideration the nonadditive entropy of type (α,β) (1.6), it is desirable to choose the mapping $M : I \to \mathbb{R}$ defined as $M(p) = p^{\alpha}$, $p \in I$, $\alpha \in \mathbb{R}$, $\alpha > 0$, $\alpha \neq 1$, $0^{\alpha} := 0$ and

$1^\alpha := 1$. Then solution (S_6) gives

$$\begin{aligned}
\sum_{j=1}^{m} F(q_j) &= d[G(1)+(n-1)G(0)](2^{1-\alpha}-2^{1-\beta})H_m^{(\alpha,\beta)}(q_1,\ldots,q_m)\\
&\quad - [F(1)+(nm-1)F(0)](1-2^{1-\beta})H_m^{\beta}(q_1,\ldots,q_m),\\
\sum_{i=1}^{n} G(p_i) &= [G(1)+(n-1)G(0)][-(1-2^{1-\alpha})H_n^{\alpha}(p_1,\ldots,p_n)+1],\\
\sum_{j=1}^{m} H(q_j) &= d(2^{1-\alpha}-2^{1-\beta})H_m^{(\alpha,\beta)}(q_1,\ldots,q_m)\\
&\quad + [H(1)+(m-1)H(0)][-(1-2^{1-\beta})H_m^{\beta}(q_1,\ldots,q_m)+1],\\
\sum_{i=1}^{n} K(p_i) &= [G(1)+(n-1)G(0)]\{d-[H(1)+(m-1)H(0)]\}\\
&\quad \times (2^{1-\alpha}-2^{1-\beta})H_n^{(\alpha,\beta)}(p_1,\ldots,p_n)\\
&\quad + [K(1)+(n-1)K(0)][-(1-2^{1-\beta})H_n^{\beta}(p_1,\ldots,p_n)+1].
\end{aligned}$$

Thus we see that in the case of solution (S_6), mappings are related to entropy of type (α,β) when $\alpha\neq\beta$ and entropy of degree α and of degree β.

The summands of the mappings, F, G, K appearing in the solution (S_1) are independent of the probabilities $p_1,\ldots,p_n$ or $q_1,\ldots,q_m$. The solution (S_1) does not seem to be any importance from information theoretic point of view unless the mapping H is a suitable mapping of probability p, $p\in I$. Similar is the situation with the solution (S_2).

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