Tamsui Oxford Journal of Mathematical Sciences **23(4)** (2007) 473-478
Aletheia University

A Note on Unsteady Viscous Flow with General Free Stream Velocity*

R. Mohyuddin†

Department of Science in Engineering, Kulliyyah of Engineering, International Islamic University Malaysia, Jalan Gombak 53100 Kuala Lumpur

Received April 6, 2006, Accepted May 24, 2006.

Abstract

We have presented exact solutions of incompressible, Navier-Stokes equations when fluid at infinity is in general function of time. Laplace transform is used to find the general solution. Few special cases are discussed when the general free stream velocity is zero, oscillating (damped and undamped) and linearly increasing.

Keywords and Phrases: *Analytic solution; Laplace tranform, general free stream velocity*

1. Formulation of the Problem

A general class of non-steady solutions of the Navier-Stokes equations which possess boundary layer character is obtained in the special case when the velocity component are independent of the longitudinal coordinate, x. In this case we choose the velocity field of the form

$$\mathbf{V} = \left[u\left(y,t\right), v\left(t\right), 0\right]. \tag{1.1}$$

* *2000 AMS Subject Classification.* 76D05, 76D07, 74A05
†E-mail: m.raheel@gcisc.org.pk

Then the Navier-Stokes equations with negligible body force take the form [1]

$$\frac{\partial u}{\partial t}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}, \tag{1.2}$$

$$\frac{\partial v}{\partial t}=-\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{1.3}$$

where ν is the kinematic viscosity, p the pressure, ρ is the density.

We note that for constant velocity, $v=V_0<0$ (at the wall 'suction') the above equation is satisfied identically and pressure 'p' becomes independent of 'y'. We put

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{dU}{dt}, \tag{1.4}$$

where $U(t)$ denotes the free stream velocity at a very large distance from the wall, so that we obtain [2]

$$\frac{\partial u}{\partial t}+V_0\frac{\partial u}{\partial y}=\frac{dU}{dt}+\nu\frac{\partial^2 u}{\partial y^2}. \tag{1.5}$$

According to the Stuart [3] there exist an exact solution of equation (1.5) for the arbitrary velocity

$$U(t)=U_0\left[1+f(t)\right]. \tag{1.6}$$

The solution is

$$u(y,t)=U_0\left[\xi(y)+g(y,t)\right], \tag{1.7}$$

where

$$\xi(y)=1-e^{\frac{V_0}{\nu}y}. \tag{1.8}$$

The information from equations (1.6) – (1.8) reduces the equation (1.5) to the form

$$\frac{\partial g}{\partial t}+V_0\frac{\partial g}{\partial y}=\frac{df}{dt}+\nu\frac{\partial^2 g}{\partial y^2}. \tag{1.9}$$

Using the non-dimensional parameters

$$\eta=-\frac{yV_0}{\nu},\ T=\frac{tV_0^2}{4\nu} \tag{1.10}$$

the above equation reduces to the form

$$\frac{\partial g}{\partial T} - 4\frac{\partial g}{\partial \eta} = \frac{df}{dT} + 4\frac{\partial^2 g}{\partial \eta^2}. \tag{1.11}$$

For the problem under consideration we assume the following boundary conditions

$$g = 0 \text{ at } \eta = 0, \tag{1.12a}$$

$$g = f \text{ at } \eta = \infty. \tag{1.12b}$$

2. Solution

Applying Laplace transform, equations (1.11) and (1.12)a,b transform as

$$4\frac{d^2\overline{g}}{d\eta^2} + 4\frac{d\overline{g}}{d\eta} - s\overline{g}\left(\eta, s\right) = -sF\left(s\right), \tag{2.1}$$

$$\overline{g}\left(0, s\right) = 0, \tag{2.2a}$$

$$\overline{g}\left(\infty, s\right) = F\left(s\right), \tag{2.2b}$$

where $F(s)$ is the Laplace transform of $f(t)$ [4].

In the special case when the external flow is independent of time, $f(t) = 0$, equation (1.11) leads to a simple solution $g(\eta, T) = 0$.

For non-homogeneous case the solution of equation (2.1) is given by

$$\overline{g}\left(\eta, s\right) = Ae^{-m_1\eta} + Be^{m_2\eta} + F\left(s\right), \tag{2.3}$$

where

$$m_1 = \frac{1 + \sqrt{1+s}}{2}, \; m_2 = \frac{-1 + \sqrt{1+s}}{2}.$$

Using the boundary conditions (2.2)a,b in equation (2.3) we obtain

$$\overline{g}\left(\eta, s\right) = F\left(s\right) - F\left(s\right) e^{-m_1\eta}. \tag{2.4}$$

Finally, the Laplace inversion of equation (2.4) gives

$$g(\eta, T) = f(T) - \frac{\eta e^{-\frac{\eta}{2}}}{4\sqrt{\pi}} \int_0^T f(T-\tau)\, e^{-\tau} e^{-\frac{\eta^2}{16\tau}} \tau^{-\frac{3}{2}} d\tau. \tag{2.5}$$

In order to understand some physical aspects of the solution (2.5) we consider a few special cases.

3. Special Cases

3.1 Trivial case $f(T) = 0$

For $f(T) = 0$, equation (2.5) leads to the simple or trivial solution

$$g(\eta, T) = 0. \tag{3.1}$$

3.2 Free stream oscillations $f(T) = 1 + \epsilon e^{i\varpi T}$

For

$$f(T) = 1 + \epsilon e^{i\varpi T} \tag{3.2}$$

equation (2.5) takes the following integral form

$$g(\eta, T) = \left(1 + \epsilon e^{i\varpi T}\right) - \frac{\eta e^{-\frac{\eta}{2}}}{4\sqrt{\pi}} \left[I_1 + \epsilon e^{i\varpi T} I_2\right], \tag{3.3}$$

where $\varpi = 4\nu\omega/V_0^2$ and

$$I_1 = \int_0^T \tau^{-\frac{3}{2}} e^{-\tau - \frac{\eta^2}{16\tau}} d\tau, \tag{3.4}$$

$$I_2 = \int_0^T \tau^{-\frac{3}{2}} e^{-(1+i\varpi)\tau - \frac{\eta^2}{16\tau}} d\tau. \tag{3.5}$$

In order to get the complete solution the integrals are evaluated and the solution (3.3) is given as

$$g(\eta, T) = \left(1 + \epsilon e^{i\varpi T}\right) + \frac{\eta e^{-\frac{\eta}{2} - T - \frac{\eta^2}{16T}}}{4\sqrt{\pi T}} \left[\frac{1}{T - \frac{\eta^2}{16T}} + \frac{\varepsilon}{(i\varpi + 1)\, T - \frac{\eta^2}{16T}}\right]. \tag{3.6}$$

3.3 Linear increase $f(T) = T$

For linear increase i.e.,

$$f(T) = T \tag{3.7}$$

the integral form of equation (2.5) becomes

$$g(\eta, T) = T - \frac{\eta e^{-\frac{\eta}{2}}}{4\sqrt{\pi}} \left[T I_3 - I_4 \right], \tag{3.8}$$

where

$$I_3 = \int_0^T \tau^{-\frac{3}{2}} e^{-\tau - \frac{\eta^2}{16\tau}} d\tau, \tag{3.9}$$

$$I_4 = \int_0^T \tau^{-\frac{1}{2}} e^{-\tau - \frac{\eta^2}{16\tau}} d\tau. \tag{3.10}$$

The integrals (3.9) and (3.10) are also evaluated to have the following form of the solution (3.8)

$$g(\eta, T) = T. \tag{3.11}$$

which shows that the flow velocity field $g(\eta, T)$ increases linearly with the increase of free stream velocity $f(T)$.

3.4 Elliptic oscillations $f(T) = ae^{i\varpi T} + be^{-i\varpi T}$

In this case we consider damped and undamped (elliptic) oscillations by taking the free stream velocity of the form

$$f(T) = ae^{i\varpi T} + be^{-i\varpi T}, \tag{3.12}$$

where a, b are complex constants.

The integral form of equation (2.5) is given as

$$g(\eta, T) = ae^{i\varpi T} + be^{-i\varpi T} - \frac{\eta e^{-\frac{\eta}{2}}}{4\sqrt{\pi}} \left[ae^{i\varpi T} I_5 + be^{-i\varpi T} I_6 \right], \tag{3.13}$$

where

$$I_5 = \int_0^T \tau^{-\frac{3}{2}} e^{-(1+i\varpi)\tau - \frac{\eta^2}{16\tau}} d\tau, \tag{3.14}$$

$$I_6 = \int_0^T \tau^{-\frac{3}{2}} e^{-(1-i\varpi)\tau - \frac{\eta^2}{16\tau}} d\tau. \tag{3.15}$$

Equation (3.13) after evaluating the integrals (3.14) and (3.15) becomes

$$g(\eta, T) = \left(ae^{i\varpi T} + be^{-i\varpi T}\right) + \frac{\eta e^{-\frac{\eta}{2} - T - \frac{\eta^2}{16T}}}{4\sqrt{\pi T}} \left[\frac{a}{(1+i\varpi)\, T - \frac{\eta^2}{16T}} + \frac{b}{(1-i\varpi)\, T - \frac{\eta^2}{16T}}\right]. \tag{3.16}$$

4. Conclusion

In this note we have furnished the results for unidimensional two directional incompressible Navier-Stokes equations when the boundary is porous. The flow is subject to pressure gradient in $x-$direction which is considered in general a function of time. Several limiting situations are discussed by letting the general free stream velocity of a special form.

References

[1] R. B. Bird, R. C. Armstrong, and O. Hassager. *Dynamics of Polymeric Liquids*, volume 1. Wiley, New York, 1977.

[2] H. Schlichting, *Boundary Layer Theory*, sixth ed., Mc Graw-Hill, New York, 1968.

[3] J. T. Stuart, A solution of the Navier-Stokes and energy equations illustrating the response of the skin friction and temperature of an infinite plate thermometer to fluctuations in the stream velocity. *Proc. Roy. Soc. London* **A231**, 116-130 (1955).

[4] J. Watson, A solution of the Navier-Stokes equations illustrating the response of a laminar layer to a given changes in the external stream velocity. *Quart. J. Mech. Appl. Math. II.* 302-325 (1958).