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Explicit Estimators of the Location and Scale Parameters of a Symmetric Hyperbolic Distribution

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Abstract

Explicit modified maximum likelihood estimators (MMLE) of the location and scale parameters of a symmetric hyperbolic distribution are obtained as linear functions of the order statistics of a random sample. These new estimators are unbiased and have some other nice properties. Numerical comparisons with respect to bias and mean squared error of MMLEs, moment estimators, MLEs, least square estimators and asymptotically most efficient estimators have been carried out.

Keywords and phrases: *Order statistics, Modified maximum likelihood estimators, Moment estimators, Least square estimators, Asymptotically most efficient estimators.*

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1. Introduction

Barndorff-Nielsen (1977) introduced the hyperbolic distribution as a model for the log size of a sand sample found in Aeolian sand deposits. In his subsequent studies he and his coworkers (See e.g.Barndorff-Nielsen et. al. (1983)) described modeling applications of this distribution in a variety of fields. The pdf of symmetric hyperbolic distribution is given by

$$g(x) = \frac{1}{2\tau k_1(\nu)} \exp\left[-\nu \sqrt{1+\left(\frac{x-\mu}{\tau}\right)^2}\right], \qquad -\infty < x < \infty \tag{1}$$

where $\tau, \nu > 0, -\infty < \mu < \infty$ are the parameters and $k_1(\nu)$ is the modified Bessel function of the third kind with index 1. The estimation of the location and scale parameters μ and τ have received special attention. The uniqueness and existence of maximum likelihood estimators is established in Barndorff-Nielsen et. al.(1983). The equations giving Bayes estimators for these parameters are obtained by Howlader (1989). The maximum likelihood and Bayes estimators are then obtained numerically. The explicit expressions for estimators of μ and τ are not available. The importance of explicit estimators is well known. The main objective of this paper is to provide explicit modified likelihood estimators and prove some of its properties. We will also like to see how do they compare numerically with other estimators.

In section 2, we derive some explicit estimators for μ and τ for known ν and discuss some of their properties. Modified maximum likelihood estimators and least square estimators are weighted sum of order statistics of a sample from density (1). The asymptotically most efficient estimators are also given. Section 3 provides a numerical comparison based on simulated results of the estimators obtained in section 2 with respect to their bias and mean squared error.

2. Estimators

Let $X_1, X_2, ..., X_n$ be a random sample of size n on random variable X having pdf (1). Set $Y = (X - \mu)/\tau$. Then pdf of Y is given by

$$h(y) = \frac{1}{2k_1(\nu)} \exp\left(-\nu\sqrt{1+y^2}\right), \quad -\infty < y < \infty. \tag{2}$$

Define

$$J_i(\nu) = \int_{-\infty}^{\infty} \phi_i(y) \exp(-\nu\sqrt{1+y^2})\, dy, \qquad i = 0,1,2, \tag{3}$$

where $\phi_0(y) = 1$, $\phi_1(y) = y^2$, and $\phi_2(y) = y^2 / (1+y^2)$. Clearly,

$$E(Y) = 0, \qquad Var(Y) = J_1(\nu) / J_0(\nu). \tag{4}$$

This yields moments estimators of μ and τ as follows.

$$\hat{\mu}_1 = \bar{X}, \quad \hat{\tau}_1 = S_X \Big/ \sqrt{Var(Y)}, \tag{5}$$

where $\bar{X}$ and S_X^2 are the mean and variance of the random sample. Note that $\hat{\mu}_1$ is unbiased for μ.

MODIFIED MAXIMUM LIKELIHOOD ESTIMATORS (MMLE):

The log-likelihood function is given by

$$L(\mu, \tau / \underline{X}) = -n \ln \tau - \nu \sum_{i=1}^{n} \sqrt{1 + \left(\frac{X_i - \mu}{\tau}\right)^2} - n \ln 2 - n \ln k_1(\nu) \tag{6}$$

which yields the likelihood equations

$$\sum_{i=1}^{n} Y_i / \sqrt{1+Y_i^2} = 0, \qquad \sum_{i=1}^{n} Y_i^2 / \sqrt{1+Y_i^2} = \frac{n}{\nu} \tag{7}$$

where $Y_i = (X_i - \mu)/\tau$. Let $X_{(1)}, X_{(2)}, \ldots X_{(n)}$ be the order statistics of the random sample and $Y_{(i)} = (X_{(i)} - \mu)/\tau$. Then the equation (7) can be written as

$$\sum_{i=1}^{n} Y_{(i)} \Big/ \sqrt{1+Y_{(i)}^2} = 0, \qquad \sum_{i=1}^{n} Y_{(i)}^2 \Big/ \sqrt{1+Y_{(i)}^2} = \frac{n}{\nu} \tag{8}$$

Following the approach of Tiku et. al. (1986), we use the Taylor's approximation to functions

$Y_{(i)} \Big/ \sqrt{1+Y_{(i)}^2}$ and $Y_{(i)}^2 \Big/ \sqrt{1+Y_{(i)}^2}$ about $\mu_{(i)} = E(Y_{(i)})$ in (8) to obtain the modified log- likelihood equations

$$\sum_{i=1}^{n} \left[\frac{\mu_{(i)}}{\sqrt{1+\mu_{(i)}^2}} + \frac{Y_{(i)} - \mu_{(i)}}{(1+\mu_{(i)}^2)^{3/2}} \right] = 0 \tag{9}$$

and

$$\sum_{i=1}^{n} \left[\frac{\mu_{(i)}^2}{\sqrt{1+\mu_{(i)}^2}} + \frac{(Y_{(i)} - \mu_{(i)})\,(2\mu_{(i)} + \mu_{(i)}^3)}{(1+\mu_{(i)}^3)^{3/2}} \right] = \frac{n}{\nu} \tag{10}$$

Using the relation $\mu_{(i)} = -\mu_{(n-i+1)}$ for symmetric distributions (see David (1981) and solving equations (9) and (10), we easily obtain the MMLE of μ and τ as follows.

$$\hat{\mu}_2 = \sum_{i=1}^{n} \left[X_{(i)} \Big/ (1+\mu_{(i)}^2)^{3/2} \right] \Big/ \sum_{i=1}^{n} (1+\mu_{(i)}^2)^{-3/2} \tag{11}$$

and

$$\tilde{\tau}_2 = \frac{\sum_{i=1}^{n}\left[[X_{(i)}\ \mu_{(i)}(2+\mu_{(i)}^2)] \Big/ (1+\mu_{(i)}^2)^{3/2} \right]}{\frac{n}{\nu} + \sum_{i=1}^{n} [\mu_{(i)}^2 \Big/ (1+\mu_{(i)}^2)^{3/2}]} \qquad (12)$$

Both the estimators are linear combinations of the order statistics and while $\hat{\mu}_2$ is unbiased for μ, but the estimator $\tilde{\tau}_2$ is biased for τ. However, as $\mathrm{E}(\tilde{\tau}_2) = \mathrm{C}\,\tau$, where C is a constant depending on $\mu_{(i)}$'s, the estimator $\hat{\tau}_2 = \tilde{\tau}_2 / \mathrm{C}$ as given below is unbiased for τ.

$$\hat{\tau}_2 = \frac{\sum_{i=1}^{n}\left[[X_{(i)}\ \mu_{(i)}(2+\mu_{(i)}^2)] \Big/ (1+\mu_{(i)}^2)^{3/2} \right]}{\sum_{i=1}^{n}\left[[\mu_{(i)}^2\ (2+\mu_{(i)}^2)] \Big/ (1+\mu_{(i)}^2)^{3/2} \right]} \qquad (13)$$

Next we discuss some properties of the estimators $\hat{\mu}_2, \tilde{\tau}_2$ and $\hat{\tau}_2$ as given by (11), (12) and (13) respectively.

(I) *The estimator* $(\hat{\mu}_2, \tilde{\tau}_2)$ *for* (μ, τ) *is equivariant under location-scale group of transformations.*

Proof. The pdf (1) belongs to location-scale family and therefore is invariant under location-scale transformation group. The numerator of $\tilde{\tau}_2$ in (12) by using the symmetry of pdf (2) about zero and the relation that $\mu_{(i)} = -\mu_{(n-i+1)}$ can be rewritten as

$$\sum_{i=1}^{\left[\frac{n}{2}\right]} \frac{\mu_{(n-i+1)}\ (2+\mu_{(n-i+1)}^2)}{(1+\mu_{(n-i+1)}^2)^{3/2}} (X_{(n-i+1)} - X_{(i)}) \qquad (14)$$

where [.] is the greatest integer function. Using the definition of equivariance, the result immediately follows from (11) and (14).

Remark $(\hat{\mu}_2, \hat{\tau}_2)$ is also equivariant for (μ, τ).

(II) *The estimators* $\hat{\tau}_2$ *and* $\tilde{\tau}_2$ *of* τ *take non-negative values.*

Proof. The pdf (2) being symmetric about zero, $\mu_{(n-i+1)}$ *is* > 0 *for* $i = 1, 2, \cdots, \left[\frac{n}{2}\right]$ and hence the numerators of both $\hat{\tau}_2$ and $\tilde{\tau}_2$ which are rewritten as (14) are non-negative. Since their denominators as in (13) and (12) are positive, the result follows.

(III) *The lower bounds for the variance of unbiased estimators of* μ *and* τ *are respectively* $1/I_{11}$ *and* $1/I_{22}$*, where*

$$I_{11} = \frac{\nu^2 \ J_2(\nu)}{\tau^2 \ J_0(\nu)}, \qquad I_{22} = \frac{1}{\tau^2}\left[\frac{\nu^2 (J_1(\nu) - J_2(\nu))}{J_0(\nu)} - 1\right] \tag{15}$$

Proof. The pdf (1) is a member of location-scale family and the pdf of
Y = $(X - \mu)/\tau$ given by (2) is symmetric about zero. Then by using well known results, we note that the information matrix has the form $\begin{bmatrix} I_{11} & 0 \\ 0 & I_{22} \end{bmatrix}$, where

$$I_{11} = \frac{1}{\tau^2}\int_{-\infty}^{\infty}\left(\frac{d}{dy}\ln h(y)\right)^2 h(y)\ dy \quad \text{and} \quad I_{22} = \frac{1}{\tau^2}\int_{-\infty}^{\infty}\left(y\ \frac{d}{dy}\ln h(y) + 1\right)^2 h(y)\ dy$$

which on simplification give (15).

(IV) *The estimator* $\hat{\mu}_2$ *of* μ *is a L-estimator. This estimator is consistent asymptotically normal and asymptotically the most efficient. More precisely*

$$\sqrt{n}\ (\hat{\mu}_2 - \mu) \xrightarrow{\ d\ } N(o, \sigma^2), \text{ where } \sigma^2 = 1/I_{11}.$$

Proof. We know that for large n, $\mu_{(i)} \approx H^{-1}(\frac{i}{n+1})$, where H is the df corresponding to pdf h in (2). The estimator given by (11), viz.

$$\hat{\mu}_2 = \sum_{i=1}^{n} w_{in} X_{(i)}, \qquad \text{where} \qquad w_{in} = \frac{\left(1+\mu_{(i)}^2\right)^{-3/2}}{\sum_{i=1}^{n}\left(1+\mu_{(i)}^2\right)^{-3/2}}$$

is a L-estimator. For large n, $w_{in} \propto \lambda_0\left(\frac{i}{n+1}\right)$, where

$$\lambda_0(t) = \frac{J_0(\nu)}{\nu\, J_2(\nu)\left[1+\left(H^{-1}(t)\right)^2\right]^{3/2}}, \quad 0 < t < 1. \tag{16}$$

Next we note that $\lambda_0(t)$ is a pdf and is symmetrical about ½, since h(y) is symmetrical about zero. Then a direct application of Cor. 5.1 and Cor. 5.2 of Lehmann (1983), pages 371-372 gives

$$\sqrt{n}\,(\hat{\mu}_2 - \mu) \xrightarrow{d} N(o, \sigma_0^2), \text{ for some } \sigma_0^2.$$

Now an application of Theorem 5.2 of Lehmann (1983), page 373 will prove the result if we can show that

$$\lambda_0(t) = \gamma'\left(G^{-1}(t)\right) \Big/ \int \gamma^2(x)\, g(x)\, dx$$, where $\gamma(x) = -g'(x)/g(x)$ and G is the df corresponding to the pdf g given by (1). A straight forward simplification of the expressions involved shows that this is indeed true and the proof is complete.

Though unique MLEs of μ and τ exist, explicit expressions could be obtained only for MMLEs which are asymptotically efficient. For comparison purposes we obtain the asymptotic efficient estimators of these parameters by using the standard methodology (see Lehmann (1983) ,pages 435-436) of linear approximations on log-likelihood about $\sqrt{n}$ - consistent estimators.

ASYMPTOTICALLY EFFICIENT ESTIMATORS (AEE):

As (1) is a symmetric density about the location parameter μ and $\tau \sqrt{\mathrm{Var(Y)}}$ is its standard deviation, the moment estimators $\hat{\mu}_1$ and $\hat{\tau}_1$ as given in (5) are respectively $\sqrt{n}$ - consistent for μ and τ. Thus AEEs $(\hat{\mu}_3 , \hat{\tau}_3)$ are the solution of equations

$$n\left[\hat{\mu}_3 - \hat{\mu}_1\right] \; I_{11}(\hat{\mu}_1, \hat{\tau}_1) = \frac{\partial L\,(\hat{\mu}_1, \hat{\tau}_1)}{\partial \mu}$$

$$\mathrm{n}\left[\hat{\tau}_3 - \hat{\tau}_1\right] \; I_{22}(\hat{\mu}_1, \hat{\tau}_1) = \frac{\partial L\,(\hat{\mu}_1, \hat{\tau}_1)}{\partial \tau}$$

where L is the log-likelihood function given by (6). Thus

$$\hat{\mu}_3 = \overline{X} + \frac{\mathrm{S_X}\,(\dot{J}_0(\upsilon))^{3/2}}{\upsilon\, J_2(\upsilon)\sqrt{J_1(\upsilon)}} \; \frac{1}{n}\sum_{i=1}^{n} Y_i' / \sqrt{1+Y_i'^2} \tag{17}$$

$$\hat{\tau}_3 = \frac{S_X\sqrt{\dfrac{\dot{J}_0(\upsilon)}{J_1(\upsilon)}}\left[\dfrac{\upsilon^2(J_1(\upsilon)-J_2(\upsilon))}{\dot{J}_0(\upsilon)} + \dfrac{\upsilon}{\mathrm{n}}\sum_{i=1}^{\mathrm{n}} Y_i'^2 / \sqrt{1+Y_i'^2} - 2\right]}{\left(\dfrac{\upsilon^2(J_1(\upsilon)-J_2(\upsilon))}{\dot{J}_0(\upsilon)} - 1\right)} \tag{18}$$

where $Y_i' = \dfrac{\mathrm{X}_i - \overline{\mathrm{X}}}{S_X}\sqrt{\dfrac{J_1(\upsilon)}{J_0(\upsilon)}}$. (19)

Lastly, we give below the least square estimators of μ and τ, that are linear function of order statistics.

LEAST SQUARE ESTIMATORS (LSE):

The LSEs of μ and τ (see Arnold at. el (1992), page 180) are respectively given

$$\hat{\mu}_4 = \frac{\sum_{i=1}^{n}\left[\sum_{j=1}^{n} \mu_{(j)}^2 - (\sum_{j=1}^{n} \mu_{(j)})\ \mu_{(i)}\right] X_{(i)}}{n\sum_{i=1}^{n} \mu_{(i)}^2 - (\sum_{i=1}^{n} \mu_{(i)})^2}$$

and

$$\hat{\tau}_4 = \frac{\sum_{i=1}^{n}\left[n\mu_{(i)} - \sum_{j=1}^{n} \mu_{(j)}\right] X_{(i)}}{n\sum_{i=1}^{n} \mu_{(i)}^2 - (\sum_{i=1}^{n} \mu_{(i)})^2} .$$

As $\sum_{i=1}^{n} \mu_{(i)} = 0$ for a symmetric probability distribution, these estimators simplify to

$$\hat{\mu}_4 = \overline{X} \quad \text{and} \quad \hat{\tau}_4 = \sum_{i=1}^{n} \mu_{(i)}\ X_{(i)} \Big/ \sum_{i=1}^{n} \mu_{(i)}^2 \qquad (20)$$

3. Simulation Study

In order to compare MMLEs, MLEs, AMEs, moment estimators, Bayes estimators and LSEs of μ and τ, means and mean squared errors (mse)of all estimators except MLEs and Bayes estimators were simulated. For this purpose, we generated 1000 samples each of size n = 10, 20, 30, 40, 50 60,from pdf (1) with several parameter combinations for μ, τ and ν. In this paper we report the results for $\mu = 10, \tau = 1$ and $\nu = 3$ as well as for $\mu = 20, \tau = 5$ and $\nu = 3$ as Howlader (1989) carried out the simulation with these parameter combinations for Bayes estimators and MLEs. We denote Bayes estimators and MLEs by $(\hat{\mu}_5, \hat{\tau}_5)$ and $(\hat{\mu}_6, \hat{\tau}_6)$ respectively.

The computation of MMLEs given in (11), (13) and LSEs given in (20) require the values of $\mu_{(i)}$'s which are given by

$$\mu_{(i)} = \frac{n|}{(i-1)|\ (n-i)|} \int_{-\infty}^{\infty} y\ (H(y))^{i-1}\ (1-H(y))^{n-i}\ h(y)\ dy,$$

where h(y) is the pdf defined by (2) and H(y) is its cdf. Numerical integration is required for computation of $\mu_{(i)}$ from the above expression. We used the Gauss- Laguerre scheme on MATLAB for this purpose. We also estimated $\mu_{(i)}$ from the order statistics of samples simulated from pdf (2) for some values of ν and these matched the values obtained through numerical integration up to 5 decimal places.

The simulated values of means and mse's of $(\hat{\mu}_i, \hat{\tau}_i)$, .i =1, 2, …,6 are presented in table 1 and table 2. In these tables LBV(mu^) and LBV(tau^) respectively denote the lower bound of the variance of unbiased estimators of μ and τ given by (14). Also E(mui^) and M(mui^) stand for mean and mse respectively of $\hat{\mu}_i$,whereas E(taui^) and M(taui^) stand for these values of $\hat{\tau}_i$ .The simulated values for Bayes estimators and MLEs appearing in last four columns in the tables were taken from Howlader(1989). We make the following observations from the tables1 and 2.

1. For estimating μ, the estimators $\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_4$ are unbiased by definition. Whereas $\hat{\mu}_3$ shows negligible bias, the estimators $\hat{\mu}_5, \hat{\mu}_6$ show systematic positive but negligible bias. Thus all the estimators considered are highly competitive bias wise.

2. With respect to mse criterion, $\hat{\mu}_2$ has the least mse in most cases in comparison to that of other estimators considered. The next best is the MLE $\hat{\mu}_6$. The mse of $\hat{\mu}_1$ or $\hat{\mu}_4$ (the two being equal) is largest. The difference in mse among the estimators narrows down as n increases.

3. For estimating τ, the estimator $\hat{\tau}_2$ is unbiased by definition. Both $\hat{\tau}_1$ and $\hat{\tau}_3$ underestimate τ, as is case with the MLE $\hat{\tau}_6$. Bias in $\hat{\tau}_3$ is the largest though it differs only marginally from that of $\hat{\tau}_4$ and $\hat{\tau}_6$. Bias in $\hat{\tau}_4$ is least for larger value of τ.

4. The mse-wise $\hat{\tau}_3$ is closest to MLE $\hat{\tau}_6$, even better sometimes. Their ranking from the worst to the best with respect to mse criterion is roughly $\hat{\tau}_1$, $\hat{\tau}_4$, $\hat{\tau}_2$, $\hat{\tau}_5$, $\hat{\tau}_3$, $\hat{\tau}_6$.

On the basis simulated results we may conclude that for medium sample sizes:

- For estimating μ, since the estimators do not differ much bias wise, the main criterion for comparison is the Mean Squared Error. From this point of view MMLE $\hat{\mu}_2$ is generally seen to perform better than other estimators considered here and hence is recommended. Further, since MMLE has an explicit expression, it has a definite computational advantage over MLE. The moment estimators and the least square estimators are not recommended for μ, because their MSE is highest among estimators under comparison.

- For estimating τ, if small bias is the main consideration then MMLE $\hat{\tau}_2$ being unbiased may be preferred, where as, MLE $\hat{\tau}_6$ is recommended if small MSE is the main criterion.

To conclude we remark that MMLEs of μ and τ proposed in this paper being unbiased, asymptotically the most efficient and being equivariant, apart from having explicit expressions, are a good alternative to their MLEs.

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