Tamsui Oxford Journal of Mathematical Sciences **23(4)** (2007) 449-459
Aletheia University

On the Hyers-Ulam-Rassias Stability of the Cauchy Linear Functional Equation*

Bouikhalene Belaid†
University of Ibn Tofail, Faculty of Sciences,
Department of Mathematics, Kenitra, Morocco

and

Elqorachi Elhoucien‡
Department of Mathematics Faculty of Sciences,
University Ibn Zohr Agadir, Morocco

Received February 10, 2006, Accepted March 22, 2006.

Abstract

In this paper, we obtain the general solution and we provide a proof of the functional stability in the spirit of Hyers-Ulam, Th. M. Rassias and P. Găvruta for the Cauchy linear functional equation

$$f(x+y+a) = f(x) + f(y),$$

where $f : E_1 \longrightarrow E_2$ and a is an arbitrary element in E_1.

Keywords and Phrases: *Quadratic equation, Hyers-Ulam-Rassias stability.*

---

*2000 *Mathematics Subject Classification.* 39B52, 39B72.
†E-mail: bbouikhalene@yahoo.fr
‡E-mail: elqorachi@hotmail.com

1. Introduction

The main purpose of this paper is to generalize the results obtained in [4], [8] and [2] for the Cauchy linear functional equation

$$f(x+y+a) = f(x) + f(y),\ x,y \in E_1. \tag{1}$$

A particular case of this linear functional equation is:

$$f(x+y) = f(x) + f(y),\ x,y \in E_1. \tag{2}$$

If f is a solution of (2) it is said to be additive or satisfies the Cauchy equation. In [9] S. M. Ulam posed the question of the stability of Cauchy equation: If a function f approximately satisfies Cauchy's functional equation (2) when does there exists an exact solution of (2) which f approximates. The problem has been considered for many different types of spaces by a number of writers including D. H. Hyers [4,5,6], Th. M. Rassias [6,8], Z. Gajda [1] and P. Găvruţă [3]. The interested reader should refer to the book by D. H. Hyers, G. Isac and Th. M. Rassias [7].

This paper is organized as follows: In the first section after this introduction we establish the general solution of (1). In the other section we prove the stability problem in the sense Hyers-Ulam, Th. M. Rassias and P. Găvruţă for the Cauchy linear functional equation (1).

2. Solution of (1)

In this section we present the general solution of (1).

Theorem 2.1 *A function $f : E_1 \longrightarrow E_2$ satisfies the functional equation (1) if and only if $f(x) = A(x+a)$ for all $x \in G$, where A is an additive function.*

Proof. Let $f : E_1 \longrightarrow E_2$ satisfy the functional equation (1). Putting $x = y = 0$ in (1), we get $f(a) = 2f(0)$. Set $x = -a$, $y = 0$ in (1) to get $f(-a) = 0$. Letting $x = -2a$, $y = 0$ in (1), we obtain that $f(-2a) + f(0) =$

$f(-2a+a) = f(-a) = 0$, so $f(-2a) = -f(0)$.
Replacing y by $-2a$ in (1) yields

$$f(x-a) = f(x-2a+a) = f(x) + f(-2a) = f(x) - f(0),\ x \in E_1.$$

Now let $A : E_1 \longrightarrow E_2$ given by $A =: f - f(0)$. For all $x, y \in E_1$, A satisfies the Cauchy equation (2). In fact

$$\begin{aligned} A(x+y) &= f(x+y) - f(0) \\ &= f(x-a+y+a) - f(0) \\ &= f(x-a) + f(y) - f(0) \\ &= f(x) - f(0) + f(y) - f(0) \\ &= A(x) + A(y). \end{aligned}$$

Since $f(a) = 2f(0)$ it follows that $A(a) = f(a) - f(0) = 2f(0) - f(0) = f(0)$. Hence $f(x) = A(x+a)$ for all $x \in G$.

3. Hyers-Ulam Stability of (1)

In this section, we give the Hyers-Ulam stability for the linear Cauchy functional equation (1). The results obtained here extend the ones obtained by D. H. Hyers in [4].

Theorem 3.1 *Let G be an abelian group and let E be a Banach space. If a function $f : G \longrightarrow E$ satisfies the functional inequality*

$$\|f(x+y+a) - f(x) - f(y)\| \leq \delta, \quad x, y \in G \tag{3}$$

for some $\delta > 0$, then the limit

$$T(x) = \lim_{n \longrightarrow \infty} \frac{1}{2^n} f(2^n x + (2^n - 1)a) \tag{4}$$

exists for all $x \in G$ and $T : G \longrightarrow E$ is the unique function such that

$$T(x+y+a) = T(x) + T(y) \tag{5}$$

and

$$\|f(x) - T(x)\| \leq \delta$$

for any $x, y \in G$.

Proof. By letting $x = y$ in (3), one obtain the inequality

$$\|f(2x+a) - 2f(x)\| \leq \delta \tag{6}$$

from which we get by replacing x by $2x + a$ that

$$\|f(4x+3a) - 2f(2x+a)\| \leq \delta. \tag{7}$$

Now, make the inductive assumption

$$\|f(x) - \frac{1}{2^n}f(2^n x + (2^n - 1)a)\| \leq \delta(1 - \frac{1}{2^n}) \tag{8}$$

for some positive integer n. Clearly the inductive assumption is true for the case $n = 1$, since replacing n by 1 in (8) would give (6). for $n + 1$ we get by using (8) and (6) that

$$\begin{aligned} & \|f(x) - \frac{1}{2^{n+1}}f(2^{n+1}x + (2^{n+1} - 1)a)\| \\ \leq \ & \frac{1}{2^{n+1}}\|f(2^{n+1}x + (2^{n+1} - 1)a) - 2f(2^n x + (2^n - 1)a)\| \\ & + \frac{1}{2^n}\|f(2^n x + (2^n - 1)a) - 2^n f(x)\| \\ \leq \ & \frac{\delta}{2^{n+1}} + \delta(1 - \frac{1}{2^n}) = \delta(1 - \frac{1}{2^{n+1}}). \end{aligned}$$

We claim that the sequence $T_n(x) = \frac{1}{2^n}\{f(2^n x + (2^n - 1)a)\}$ is a Cauchy sequence. Indeed for all positive integer n we have

$$\begin{aligned} & \| \ T_{n+1}(x) - T_n(x) \ \| \\ = \ & \|\frac{1}{2^{n+1}}f(2^{n+1}x + (2^{n+1} - 1)a) - \frac{1}{2^n}f(2^n x + (2^n - 1)a) \ \| \\ = \ & \frac{1}{2^{n+1}}\|2f(2^n x + (2^n - 1)a) - f(2^{n+1}x + (2^{n+1} - 1)a)\| \\ \leq \ & \frac{\delta}{2^{n+1}}. \end{aligned}$$

Since $\frac{1}{2} < 1$, then a small computation shows that $(T_n(x))_n$ is a Cauchy sequence. Now, E is a Banach space, consequently the limit of this sequence exists and is in E. Define $T : G \longrightarrow E$ by

$$T(x) = \lim_{n \longrightarrow \infty} \frac{1}{2^n}f(2^n x + (2^n - 1)a).$$

First, inequality (8) implies that $\| f - T \| \leq \delta$.
For all $x, y \in G$ and for all positive integer n we have

$$\begin{aligned} & \|T_n(x+y+a) - T_n(x) - T_n(y)\| \\ = & \|\frac{1}{2^n} f(2^n x + 2^n y + (2^{n+1}-1)a) - \frac{1}{2^n} f(2^n x + (2^n - 1)a) - \frac{1}{2^n} f(2^n y + (2^n - 1)a)\| \\ \leq & \frac{\delta}{2^n} \end{aligned}$$

As $n \longrightarrow +\infty$ if follows that $T(x+y+a) = T(x) + T(y)$ for all $x, y \in G$ which establishes (5).
Suppose now that there exists an other mapping $T' : G \longrightarrow E$ solution of (5) which satisfies the inequality $\| f - T' \| \leq \delta$. Then

$$\begin{aligned} \|T(x) - T'(x)\| &= \|\frac{1}{2^n} T(2^n x + (2^n - 1)a) - \frac{1}{2^n} T'(2^n x + (2^n - 1)a)\| \\ &\leq \|\frac{1}{2^n} T(2^n x + (2^n - 1)a) - \frac{1}{2^n} f(2^n x + (2^n - 1)a)\| \\ &\quad \|\frac{1}{2^n} f(2^n x + (2^n - 1)a) - \frac{1}{2^n} T'(2^n x + (2^n - 1)a)\| \\ &\leq \frac{\delta}{2^n} + \frac{\delta}{2^n} = \frac{\delta}{2^{n-1}}. \end{aligned}$$

Thus, $\| T(x) - T'(x) \| = 0$ for any $x \in G$, which implies $T(x) = T'(x)$ for all $x \in G$. This completes the proof of theorem.

4. Rassias Stability

In this section, we give the Rassias stability for the linear Cauchy functional equation (1). The results obtained here extend the ones obtained by Th. M. Rassias in [8].

Theorem 4.1 *Let G be a normed space and let E be a Banach space. If a function $f : G \longrightarrow E$ satisfies the functional inequality*

$$\|f(x+y+a) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p), \quad x, y \in G \tag{9}$$

for some $\theta > 0$ and $p \in [0, 1[$ for all $x, y \in G$, then there exists a unique function $T : G \longrightarrow E$ such that

$$T(x+y+a) = T(x) + T(y) \tag{10}$$

and

$$\|f(x)-T(x)\| \leq \theta \sum_{k=0}^{+\infty} 2^{k(p-1)}\|x+(1-\frac{1}{2^k})a\|^p \tag{11}$$

for any $x, y \in G$.

Proof. By letting $x=y$ in (9), one obtain the inequality

$$\|f(2x+a)-2f(x)\| \leq 2\theta\|x\|^p \tag{12}$$

from which we get by replacing x by $2x+a$ that

$$\|f(4x+3a)-2f(2x+a)\| \leq 2\theta\|2x+a\|^p. \tag{13}$$

By induction we prove

$$\|f(x)-\frac{1}{2^n}f(2^n x+(2^n-1)a)\| \leq \theta \sum_{k=0}^{n-1} \frac{1}{2^k}\|2^k x+(2^k-1))a\|^p \tag{14}$$

for any positive integer n. Clearly the inductive assumption is true for the case $n=1$. For $n+1$ we get by using (14) and (12) that

$$\begin{aligned}
& \|f(x)-\frac{1}{2^{n+1}}f(2^{n+1}x+(2^{n+1}-1)a)\| \\
\leq\ & \frac{1}{2^{n+1}}\|f(2^{n+1}x+(2^{n+1}-1)a)-2f(2^n x+(2^n-1)a)\| \\
& |\frac{1}{2^n}\|f(2^n x+(2^n-1)a)-2^n f(x)\| \\
\leq\ & \theta\frac{1}{2^n}\|2^n x+(2^n-1)a\|^p+\theta\sum_{k=0}^{n-1}\frac{1}{2^k}\|2^k x+(2^k-1)a\|^p \\
=\ & \theta\sum_{k=0}^{n}\frac{1}{2^k}\|2^k x+(2^k-1)a\|^p.
\end{aligned}$$

Therefore, the inequality (14) is true for any $x \in G$ and for any positive integer n.

Put $T_n(x)=\frac{1}{2^n}\{f(2^n x+(2^n-1)a)\}$, where n is a positive integer and $x \in G$.

According to (14) we have

$$\begin{aligned}
\| T_{n+1}(x) - T_n(x) \| &= \|\frac{1}{2^{n+1}} f(2^{n+1}x + (2^{n+1}-1)a) - \frac{1}{2^n} f(2^n x + (2^n - 1)a)\| \\
&= \frac{1}{2^{n+1}} \|2f(2^n x + (2^n-1)a) - f(2^{n+1}x + (2^{n+1}-1)a)\| \\
&\quad \frac{\theta}{2^n}\|2^n x + (2^n - 1)a)\|^p = 2^{n(p-1)}\theta\|x + (1 - \frac{1}{2^n})a)\|^p. \\
&\leq 2^{n(p-1)}\theta(\| x \| + \| a \|)^p.
\end{aligned}$$

Since $p < 1$ and $\frac{1}{2} < 1$, $(T_n(x))_n$ is a Cauchy sequence for each $x \in G$. As E is complete we can define $T : G \longrightarrow E$ by

$$T(x) = \lim_{n \longrightarrow \infty} \frac{1}{2^n} f(2^n x + (2^n - 1)a).$$

For all $x, y \in G$ and for all positive integer n we have

$$\begin{aligned}
& \|T_n(x + y + a) - T_n(x) - T_n(y)\| \\
= & \|\frac{1}{2^n} f(2^n x + 2^n y + (2^{n+1} - 1)a) - \frac{1}{2^n} f(2^n x + (2^n - 1)a) - \frac{1}{2^n} f(2^n y + (2^n - 1)a)\| \\
\leq & \frac{\theta}{2^n}(\|2^n x + (2^n - 1)a)\|^p + \|2^n y + (2^n - 1)a)\|^p) \\
= & 2^{n(p-1)}\theta(\|x + (1 - \frac{1}{2^n})a)\|^p + \|y + (1 - \frac{1}{2^n})a)\|^p).
\end{aligned}$$

As $n \longrightarrow +\infty$ if follows that $T(x+y+a) = T(x)+T(y)$ for all $x, y \in G$ which establishes (10).
The inequality (11) immediately follows from (14).
The uniqueness of the mapping T can be obtained, by using the similar proof used in the precedent paragraph. This completes the proof of theorem.

5. Găvruta Stability

In this section, we give the Găvruta stability (see [2,3]), for the linear Cauchy functional equation (1). This result generalizes and modifies the Hyers-Ulam-Rassias stability.

Theorem 5.1 *Let G be an abelian group and let E be a Banach space and let $\varphi : G \times G \longrightarrow [0, +\infty[$ be a function satisfying*

$$\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \varphi(2^k x + (2^k - 1)a, 2^k y + (2^k - 1)a) < +\infty \tag{15}$$

for all $x, y \in G$. If a function $f : G \longrightarrow E$ satisfies the functional inequality

$$\|f(x+y+a) - f(x) - f(y)\| \leq \varphi(x, y), \quad x, y \in G, \tag{16}$$

then there exists a unique function $T : G \longrightarrow E$ solution of the functional equation

$$T(x+y+a) = T(x) + T(y) \tag{17}$$

and

$$\|f(x) - T(x)\| \leq \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \varphi(2^k x + (2^k - 1)a, 2^k x + (2^k - 1)a) \tag{18}$$

for any $x, y \in G$.

Proof. By letting $x = y$ in (16) yields

$$\|f(2x+a) - 2f(x)\| \leq \varphi(x, x) \tag{19}$$

from which we get by replacing x by $2x + a$ that

$$\|f(4x+3a) - 2f(2x+a)\| \leq \varphi(2x+a, 2x+a). \tag{20}$$

Now, make the induction assumption

$$\|f(x) - \frac{1}{2^n} f(2^n x + (2^n - 1)a)\| \leq \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} \varphi(2^k x + (2^k - 1)a, 2^k x + (2^k - 1)a) \tag{21}$$

for any positive integer n. By considering (19) it follows that the inductive assumption is true for the case $n = 1$. For $n + 1$ we have by using (21) and

(19) that

$$\begin{aligned}
& \|f(x) - \frac{1}{2^{n+1}} f(2^{n+1}x + (2^{n+1}-1)a)\| \\
\leq\ & \frac{1}{2^{n+1}} \|f(2^{n+1}x + (2^{n+1}-1)a) - 2f(2^n x + (2^n - 1)a)\| \\
& + \frac{1}{2^n} \|f(2^n x + (2^n - 1)a) - 2^n f(x)\| \\
\leq\ & \frac{1}{2^{n+1}} \varphi(2^n x + (2^n - 1)a, 2^n x + (2^n - 1)a) \\
& + \sum_{k=0}^{n-1} \frac{1}{2^{k+1}} \varphi(2^k x + (2^k - 1)a, 2^k x + (2^k - 1)a) \\
=\ & \sum_{k=0}^{n} \frac{1}{2^{k+1}} \varphi(2^k x + (2^k - 1)a, 2^k x + (2^k - 1)a)
\end{aligned}$$

which ends the proof of (21). For any positive integer p and $x \in G$ we have

$$\begin{aligned}
& |\frac{1}{2^{p+1}} f(2^{p+1}x + (2^{p+1}-1)a) - \frac{1}{2^p} f(2^p x + (2^p - 1)a)\| \\
=\ & \frac{1}{2^p} \|f(2^p x + (2^p - 1)a) - \frac{1}{2} f(2^{p+1}x + (2^{p+1}-1)a)\| \\
\leq\ & \frac{1}{2^{p+1}} \varphi(2^p x + (2^p - 1)a, 2^p x + (2^p - 1)a).
\end{aligned}$$

Hence for $n > m$ we get by using the triangular inequality

$$\begin{aligned}
& \|\frac{1}{2^n} f(2^n x + (2^n - 1)a) - \frac{1}{2^m} f(2^m x + (2^m - 1)a)\| \\
\leq\ & \sum_{k=m}^{n} \frac{1}{2^{k+1}} \varphi(2^k x + (2^k - 1)a, 2^k x + (2^k - 1)a).
\end{aligned}$$

Taking the limit as $m \longrightarrow +\infty$ and considering (15) it follows that the sequence $T_n(x) = \frac{1}{2^n}\{f(2^n x + (2^n - 1)a)\}$ is a Cauchy sequence for each $x \in G$. As E is complete we can define $T : G \longrightarrow E$ by

$$T(x) = \lim_{n \longrightarrow \infty} \frac{1}{2^n} f(2^n x + (2^n - 1)a). \tag{22}$$

For all $x, y \in E$ and for all positive integer n we have

$$\begin{aligned}
& \|T_n(x+y+a) - T_n(x) - T_n(y)\| \\
= & \|\frac{1}{2^n} f(2^n x + 2^n y + (2^{n+1}-1)a) - \frac{1}{2^n} f(2^n x + (2^n-1)a) - \frac{1}{2^n} f(2^n y + (2^n-1)a)\| \\
\leq & \frac{1}{2^n}\varphi(2^n x + (2^n-1)a, 2^n y + (2^n-1)a).
\end{aligned}$$

From (15) and (21) it follows that $T(x+y+a) = T(x)+T(y)$ for all $x, y \in G$.
Taking the limit in (20) we obtain the inequality (18).
It remains to show that T is uniquely defined. Let $T' : G \longrightarrow E$ be an other solution of (17) and (18). Then for all $x \in G$ we have

$$\begin{aligned}
\|T(x) - T'(x)\| &= \|\frac{1}{2^n} T(2^n x + (2^n-1)a) - \frac{1}{2^n} T'(2^n x + (2^n-1)a)\| \\
&\leq \frac{1}{2^n}\|T(2^n x + (2^n-1)a) - f(2^n x + (2^n-1)a)\| \\
&\quad + \frac{1}{2^n}\|T'(2^n x + (2^n-1)a) - f(2^n x + (2^n-1)a)\| \\
&\leq \frac{1}{2^n}\sum_{k=0}^{\infty}\frac{1}{2^k}\varphi(2^{k+n}x + (2^{k+n}-1)a, 2^{k+n}x + (2^{k+n}-1)a) \\
&= 2\sum_{k=n}^{\infty}\frac{1}{2^{k+1}}\varphi(2^k x + (2^k-1)a, 2^k x + (2^k-1)a).
\end{aligned}$$

Taking the limit as $n \longrightarrow +\infty$ we get $T(x) = T'(x)$ for all $x \in G$. This ends the proof.

References

[1] Z. Gajda, On stability of additive mappings, *Internat. J. Math. Sci.* **14** (1991), 431-434.

[2] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, *J. Math. Anal. Appl.* **184** (1994), 431-436.

[3] P. Găvruta, An answer to a question of John M. Rassias concerning the stability of Cauchy equation, in: *Advences in Equations and Inequalities*, in: *Hadronic Math. Ser.* (1999), 67-71.

[4] D. H. Hyers, On the stability of the linear functional equation, *Proc. Nat. Acad. Sci. U. S. A.* **27** (1941), 222-224.

[5] D. H. Hyers, The stability of homomorphisms and related topics, *in Global Analysis-Analysis on Manifolds*, in: *Teubner-Texte Math.* **75** (1983), pp 140-153.

[6] D. H. Hyers and Th. M. Rassias, Approximate homomorphisms, *Aequationes Math.* **44** (1992), 125-153.

[7] D. H. Hyers, G. I. Isac and Th. M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhauser, Basel. 1998.

[8] Th. M. Rassias, On the stability of linear mappings in Banach spaces, *Proc. Amer. Math. Soc.* **72** (1978), 297-300.

[9] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publ. New York, 1961. Problems in Modern Mathematics, Wiley, New York 1964.