Tamsui Oxford Journal of Mathematical Sciences **23(4)** (2007) 517-534
Aletheia University

On the Hyers-Ulam-Rassias Stability for a Class of Quadratic Functional Equations*

Hark-Mahn Kim†

Department of Mathematics Education, Cheongju University,
NaeDok-Dong, SangDang-Gu, 360-764, Cheongju, Korea

Received January 15, 2006, Accepted February 22, 2006.

Abstract

In this paper, we obtain the general solution and the generalized Hyers-Ulam-Rassias stability theorem for an Euler-Lagrange type quadratic functional equation

$$f(ax+by)+f(ax-by)=b^2f(x+y)+b^2f(x-y)+2(a^2-b^2)f(x)$$

for any fixed integers a, b with $a \neq -1, 0, 1$, $b \neq 0$ and $a \pm b \neq 0$. The concept of Hyers-Ulam-Rassias stability originated from Th.M. Rassias' stability Theorem that was proved in his paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72(1978), 297-300.

Keywords and Phrases: *Hyers-Ulam-Rassias stability, Quadratic mapping.*

1. Introduction

In 1940, S. M. Ulam [34] gave the following question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function

*2000 *Mathematics subject classification:* 39B82, 39B52, 46H99.
†E-mail: hmkim@@math.cnu.ac.kr

$h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$? In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. If we turn our attention to the case of functional equations, we can ask the question: Under what conditions does there exist a true solution near an approximate function differing slightly from a functional equation? If the answer is affirmative, we say that the functional equation is stable.

D.H. Hyers [11] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X$ and for some $\varepsilon \ge 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \varepsilon$$

for all $x \in X$. Th.M. Rassias [26] succeeded in extending the result of Hyers' theorem by weakening the condition for the Cauchy difference to be unbounded. A number of mathematicians were attracted to this result of Th.M. Rassias and stimulated to investigate the stability problems of functional equations. The stability phenomenon that was introduced and proved by Th.M. Rassias in his 1978 paper is called the Hyers-Ulam-Rassias stability. And then, G.L. Forti [8] and P. Găvruta [10] have generalized the result of Th.M. Rassias' theorem, which permitted the Cauchy difference to become arbitrary unbounded. The terminology, generalized Hyers-Ulam-Rassias stability originates from these historical backgrounds. These terminologies are also applied to the case of other functional equations. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem. A large list of references can be found, for example, in the papers [2, 9, 10, 13, 14, 17, 19, 23].

Now, a square norm on an inner product space satisfies the important parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

for all vectors x, y. If $\triangle ABC$ is a triangle in a finite dimensional Euclidean space and I is the center of the side $\overline{BC}$, then the following identity

$$\|\overrightarrow{AB}\|^2 + \|\overrightarrow{AC}\|^2 = 2(\|\overrightarrow{AI}\|^2 + \|\overrightarrow{CI}\|^2)$$

holds for all vectors A, B and C. The following functional equation which was motivated by these equations

$$f(x+y) + f(x-y) = 2f(x) + 2f(y) \tag{1.1}$$

is called a quadratic functional equation, and every solution of the equation (1.1) is said to be a quadratic mapping. The quadratic functional equation and several other functional equations are useful to characterize inner product spaces ([7, 27]). It is well known that a mapping f is a solution of (1.1) if and only if there exists a unique symmetric biadditive mapping Q such that $f(x) = Q(x,x)$ for all x, where the mapping Q is given by $Q(x,y) = \frac{1}{4}(f(x+y) - f(x-y))$. See [1, 14] for the details. A stability problem for the quadratic functional equation (1.1) was solved by Skof [33] for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. The theorem of S. Czerwik [5] for the functional equation (1.1) states that if a function $f : G \to Y$, where G is a normed space and Y a Banach space, satisfies the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varepsilon(\|x\|^p + \|y\|^p)$$

for $p \neq 2$ and for all $x, y \in G$, then there exists a unique quadratic function q such that

$$\|f(x) - q(x)\| \le \frac{\varepsilon\|x\|^p}{|4 - 2^p|} + \frac{\|f(0)\|}{3}$$

for all $x \in G$ if $p > 0$, and for all $x \in G \setminus \{0\}$ if $p \le 0$, where $\|f(0)\| = 0$ if $p > 0$. A stability problem for the quadratic functional equation (1.1) was solved by a lot of authors [6, 15, 20, 29, 30, 31]. In particular, J. M. Rassias [24, 25] has solved the stability problem of Ulam for the Euler-Lagrange type quadratic functional equation.

Now, we are concerned with the following functional equations, which are related with each other to prove our main subject;

$$f(2x+y)+f(2x-y)=f(x+y)+f(x-y)+6f(x), \tag{1.2}$$

$$\begin{aligned} & f(2x+y)+f(2x-y)+4f(x)+f(y)+f(-y) \\ = \ & 2f(x+y)+2f(x-y)+2f(2x), \end{aligned} \tag{1.3}$$

$$\begin{aligned} & f(ax+y)+f(ax-y) \\ = \ & f(x+y)+f(x-y)+2(a^2-1)f(x) \end{aligned} \tag{1.4}$$

for any fixed integer a with $a \neq -1, 0, 1$. More generally, we consider the following Euler-Lagrange type quadratic functional equation

$$\begin{aligned} & f(ax+by)+f(ax-by) \\ = \ & b^2 f(x+y)+b^2 f(x-y)+2(a^2-b^2)f(x) \end{aligned} \tag{1.5}$$

for any fixed integers a, b with $a \neq -1, 0$ and 1, $b \neq 0$ and $a \pm b \neq 0$. Let both E_1 and E_2 be real vector spaces. The authors [4] proved that a mapping $f : E_1 \to E_2$ satisfies the functional equation (1.2) if and only if there exists a mapping $Q : E_1 \times E_1 \to E_2$ such that $f(x) = Q(x, x)$ for all $x \in E_1$, where Q is symmetric biadditive. Thus f is a quadratic mapping. They have also investigated the generalized Hyers-Ulam-Rassias stability problem for the equation (1.2). However it should be noted that (1.2) is a special case of the functional equation (1.4). The authors showed in [18] that a mapping $f : E_1 \to E_2$ satisfies the functional equation (1.3) if and only if there exist mappings $B : E_1 \times E_1 \times E_1 \to E_2$, $Q : E_1 \times E_1 \to E_2$ and $A : E_1 \to E_2$ such that $f(x) = B(x, x, x) + Q(x, x) + A(x)$ for all $x \in E_1$, where B is symmetric for each fixed one variable and additive for each fixed two variables, Q is symmetric biadditive and A is additive.

In this paper, we will establish the general solutions of (1.4) which are related with (1.2) and (1.3). Furthermore, we are going to solve the generalized Hyers-Ulam-Rassias stability problem for the equation (1.5) and to extend the results of the generalized Hyers-Ulam-Rassias stability problem for the equation (1.2).

2. Solution of (1.4)

Let $\mathbb{R}^+$ denote the set of all nonnegative real numbers and let both E_1 and E_2 be real vector spaces. We first present the general solution of the functional equation (1.4).

Theorem 2.1. *(A) A function $f : E_1 \to E_2$ satisfies the functional equation (1.1) if and only if (B) $f : E_1 \to E_2$ satisfies the functional equation (1.2) if and only if (C) $f : E_1 \to E_2$ satisfies the functional equation (1.4). Therefore, every solution of functional equations (1.2) and (1.4) is also a quadratic function.*

Proof. Let $f : E_1 \to E_2$ satisfy the functional equation (1.1). Then according to [4, Theorem 2.1] the assertion (A) is equivalent to (B).

Let $f : E_1 \to E_2$ satisfy the functional equation (1.2). Putting $x = 0 = y$ in (1.2), we get $f(0) = 0$. Set $y = 0$ in (1.2) to get $f(2x) = 4f(x)$. Letting $y = x$ and $y = 2x$ in (1.2) separately, we obtain that $f(3x) = 9f(x)$ and $f(x) = f(-x)$ for all $x \in E_1$. To use an induction argument we assume that (C) is true for all n with $1 < n \leq N$. Putting y by $x + y$ and y by $x - y$ in (1.4) separately, we obtain

$$\begin{aligned} & f((N+1)x+y) + f((N-1)x-y) & (2.1)\\ = \;& f(2x+y) + f(y) + 2(N^2-1)f(x), & \\ & f((N+1)x-y) + f((N-1)x+y) & (2.2)\\ = \;& f(2x-y) + f(y) + 2(N^2-1)f(x). & \end{aligned}$$

Adding (2.1) to (2.2) and using an inductive assumption for $N - 1$ together with (1.2), we lead to

$$\begin{aligned} & f((N+1)x+y) + f((N+1)x-y) & (2.3)\\ = \;& f(x+y) + f(x-y) + 2[(N+1)^2-1]f(x), & \end{aligned}$$

which proves the validity of (C) for $N + 1$.

For a negative integer $n < -1$, replacing n by $-n > 1$ and using the evenness of f, one can easily prove the validity of (C).

Therefore (1.2) implies (1.4) for all $a \in \mathbb{Z}$ with $a \neq 0, 1, -1$.

Now, let $f : E_1 \to E_2$ satisfy the functional equation (1.4). Putting $x = 0 = y$ in (1.4), we get $f(0) = 0$. Letting $y = 0$ in (1.4), we obtain that $f(ax) = a^2 f(x)$ for all $x \in E_1$. Replacing x and y by $2x$ and ay in (1.4), respectively, we have

$$\begin{aligned} & a^2 f(2x+y) + a^2 f(2x-y) & (2.4)\\ = \;& f(2x+ay) + f(2x-ay) + 2(a^2-1)f(2x) & \end{aligned}$$

for all $x, y \in E_1$. Putting y by $x + ay$ in (1.4), we obtain

$$\begin{aligned} & f(a(x+y)+x) + f(a(x-y)-x) \\ = \ & f(2x+ay) + f(-ay) + 2(a^2-1)f(x). \end{aligned} \tag{2.5}$$

Interchange y with $-y$ in (2.5) to get the relation

$$\begin{aligned} & f(a(x-y)+x) + f(a(x+y)-x) \\ = \ & f(2x-ay) + f(ay) + 2(a^2-1)f(x). \end{aligned} \tag{2.6}$$

Adding (2.5) to (2.6), by use of (1.4) we lead to

$$\begin{aligned} & f(2x+y) + f(y) + 2(a^2-1)f(x+y) \\ & +f(2x-y) + f(-y) + 2(a^2-1)f(x-y) \\ = \ & f(2x+ay) + f(2x-ay) + 4(a^2-1)f(x) + a^2 f(y) + a^2 f(-y) \end{aligned} \tag{2.7}$$

for all $x, \ y \in E_1$. Subtracting (2.7) from (2.4) side by side and that dividing by $a^2 - 1$, we obtain the result,

$$\begin{aligned} & f(2x+y) + f(2x-y) + 4f(x) + f(y) + f(-y) \\ = \ & 2f(x+y) + 2f(x-y) + 2f(2x), \end{aligned} \tag{2.8}$$

from which it follows that f is quadratic by [18, Theorem 2.1] since $f(ax) = a^2 f(x)$ for all $x \in E_1$. □

We note that (1.4) implies (1.5). In fact, if $b = \pm 1$ in (1.4), the equation (1.4) reduces (1.5) of itself. Let $b \neq \pm 1$ in (1.4). Then the equation (1.4) implies by Theorem 2.1

$$f(bx+y) + f(bx-y) = f(x+y) + f(x-y) + 2(b^2-1)f(x). \tag{2.9}$$

Setting $y = 0$ in (2.9), one gets $f(bx) = b^2 f(x)$, and thus $f(\frac{x}{b}) = \frac{1}{b^2} f(x)$. Replacing y by by in (2.9), we obtain

$$f(x+by) + f(x-by) + 2(b^2-1)f(x) = b^2 f(x+y) + b^2 f(x-y). \tag{2.10}$$

Thus we figure out by (2.10)

$$\begin{aligned}
& f(ax+by)+f(ax-by) \\
= \;& b^2\Big[f(a\cdot\frac{x}{b}+y)+f(a\cdot\frac{x}{b}-y)\Big] \\
= \;& b^2\Big[f(\frac{x}{b}+y)+f(\frac{x}{b}-y)+2(a^2-1)f(\frac{x}{b})\Big] \\
= \;& [f(x+by)+f(x-by)+2(a^2-1)f(x)] \\
= \;& [b^2f(x+y)+b^2f(x-y)-2(b^2-1)f(x)+2(a^2-1)f(x)] \\
= \;& b^2f(x+y)+b^2f(x-y)+2(a^2-b^2)f(x).
\end{aligned}$$

Therefore (1.4) implies (1.5) as desired.

3. Stability of (1.5)

From now on, let X be a real vector space and let Y be a Banach space unless we give any specific reference. We will investigate the generalized Hyers-Ulam-Rassias stability problem for the functional equation (1.5). Thus we find the condition that there exists a true quadratic function near a approximately quadratic function. For convenience, we use the following abbreviation: for any fixed integers a, b with $a \neq -1, 0, 1,\ b \neq 0$ and $a \pm b \neq 0$ and for all $x, y \in X$

$$\begin{aligned}
& D_{a,b}f(x,y) \\
:= \;& f(ax+by)+f(ax-by)-b^2f(x+y)-b^2f(x-y)-2(a^2-b^2)f(x),
\end{aligned}$$

which is called the approximate remainder of the functional equation (1.5) and acts as a perturbation of the equation.

Theorem 3.1 [15]. *Let $\phi : X^2 \to \mathbb{R}^+$ be a function such that*

$$\sum_{i=0}^{\infty}\frac{\phi(a^ix,0)}{|a|^{2i}} \qquad \Big(\sum_{i=1}^{\infty}|a|^{2i}\phi(\frac{x}{a^i},0),\ \textit{respectively}\Big) \tag{3.1}$$

converges and

$$\lim_{n\to\infty}\frac{\phi(a^nx,a^ny)}{|a|^{2n}}=0 \qquad \Big(\lim_{n\to\infty}|a|^{2n}\phi(\frac{x}{a^n},\frac{y}{a^n})=0\Big) \tag{3.2}$$

for all $x,\ y \in X$. Suppose that a function $f : X \to Y$ satisfies

$$\|D_{a,b}f(x,y)\| \le \phi(x,y) \tag{3.3}$$

for all $x, y \in X$. Then there exists a unique quadratic function $T : X \to Y$ which satisfies the equation (1.5) and the inequality

$$\begin{aligned} \|f(x) - T(x)\| &\le \frac{1}{2|a|^2}\sum_{i=0}^{\infty}\frac{\phi(a^i x, 0)}{|a|^{2i}} \\ \Big(\|f(x) - T(x)\| &\le \frac{1}{2|a|^2}\sum_{i=1}^{\infty}|a|^{2i}\phi(\frac{x}{a^i}, 0)\Big) \end{aligned} \tag{3.4}$$

for all $x \in X$. The function T is given by

$$T(x) = \lim_{n\to\infty}\frac{f(a^n x)}{a^{2n}} \qquad \Big(T(x) = \lim_{n\to\infty} a^{2n} f(\frac{x}{a^n})\Big) \tag{3.5}$$

for all $x \in X$. Further, if either f is measurable or the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous for each fixed $x \in X$, then $T(rx) = r^2 T(x)$ for all $r \in \mathbb{R}$.

Proof. Putting $y = 0$ in (3.3) and dividing by $2|a|^2$, we have

$$\left\|\frac{f(ax)}{a^2} - f(x)\right\| \le \frac{1}{2|a|^2}\phi(x,0) \tag{3.6}$$

for all $x \in X$. Replacing x by ax in (3.6) and dividing by $|a|^2$ and summing the resulting inequality with (3.6), we get

$$\left\|\frac{f(a^2 x)}{a^4} - f(x)\right\| \le \frac{1}{2|a|^2}\left[\phi(x,0) + \frac{\phi(ax,0)}{|a|^2}\right] \tag{3.7}$$

for all $x \in X$. Using the induction on a positive integer n, we obtain that

$$\begin{aligned} \left\|\frac{f(a^n x)}{a^{2n}} - f(x)\right\| &\le \frac{1}{2|a|^2}\sum_{i=0}^{n-1}\frac{\phi(a^i x, 0)}{|a|^{2i}} \\ &\le \frac{1}{2|a|^2}\sum_{i=0}^{\infty}\frac{\phi(a^i x, 0)}{|a|^{2i}} \end{aligned} \tag{3.8}$$

for all $x \in X$. In order to prove the convergence of the sequence $\{\frac{f(a^n x)}{a^{2n}}\}$, we divide inequality (3.8) by $|a|^{2m}$ and also replace x by $a^m x$ to find that for $n > m \geq 0$,

$$\begin{aligned} \left\| \frac{f(a^n a^m x)}{a^{2n+2m}} - \frac{f(a^m x)}{a^{2m}} \right\| &= \frac{1}{|a|^{2m}} \left\| \frac{f(a^n a^m x)}{a^{2n}} - f(a^m x) \right\| \\ &\leq \frac{1}{2|a|^2 \cdot a^{2m}} \sum_{i=0}^{n-1} \frac{\phi(a^i a^m x, 0)}{a^{2i}} \\ &\leq \frac{1}{2|a|^2} \sum_{i=0}^{\infty} \frac{\phi(a^i a^m x, 0)}{a^{2m+2i}}. \end{aligned} \tag{3.9}$$

Since the right hand side of the inequality tends to 0 as m tends to infinity, the sequence $\{\frac{f(a^n x)}{a^{2n}}\}$ is a Cauchy sequence in the Banach space Y. Therefore, we may define

$$T(x) = \lim_{n \to \infty} \frac{f(a^n x)}{a^{2n}}$$

for all $x \in X$. By letting $n \to \infty$ in (3.8), we arrive at the formula (3.4).

To show that T satisfies the equation (1.5), replace x, y by $a^n x, a^n y$, respectively, in (3.3) and divide by $|a|^{2n}$, then it follows that

$$\begin{aligned} & a^{-2n} \| f(a^n(ax + y)) + f(a^n(ax - y)) - f(a^n(x + y)) \\ & -f(a^n(x - y)) - 2(a^2 - 1) f(a^n x)) \| \\ \leq \ & a^{-2n} \phi(a^n x, a^n y). \end{aligned}$$

Taking the limit as $n \to \infty$, we find that T satisfies (1.5) for all $x, y \in X$.

To prove the uniqueness of the quadratic function T subject to (3.4), let us assume that there exists a quadratic function $S : X \to Y$ which satisfies (1.5) and the inequality (3.4). Obviously, we have $S(a^n x) = a^{2n} S(x)$ and $T(a^n x) = a^{2n} T(x)$ for all $x \in X$ and $n \in \mathbb{N}$. Hence it follows from (3.4) that

$$\begin{aligned} \|S(x) - T(x)\| &= |a|^{-2n} \|S(a^n x) - T(a^n x)\| \\ &\leq |a|^{-2n} (\|S(a^n x) - f(a^n x)\| + \|f(a^n x) - T(a^n x)\|) \\ &\leq \frac{1}{|a|^2} \sum_{i=0}^{\infty} \frac{\phi(a^i a^n x, 0)}{|a|^{2n+2i}} \end{aligned}$$

for all $x \in X$. By letting $n \to \infty$ in the preceding inequality, we find immediately the uniqueness of T.

The proof of assertion indicated by parentheses in the theorem is similarly proved by the following inequality originated from (3.6),

$$\left\| f(x) - a^{2n} f(\frac{x}{a^n}) \right\| \leq \frac{1}{2|a|^2} \sum_{i=1}^{n} |a|^{2i} \phi(\frac{x}{a^i}, 0).$$

In this case, $f(0) = 0$ since $\sum_{i=1}^{\infty} a^{2i}\phi(0,0) < \infty$ and so $\phi(0,0) = 0$ by assumption.

The last assertion of homogeneous of degree two of T in the theorem follows by the same reasoning as the proof of [6]. This completes the proof of the theorem. □

From the main Theorem 3.1,we obtain the following corollary concerning the stability of the equation (1.5). We note that p need not be equal to q.

Corollary 3.2. *Let X and Y be a real normed space and a Banach space, respectively, and let ε, p, q be real numbers such that $\varepsilon \geq 0$, $q > 0$ and either $p, q < 2$ or $p, q > 2$. Suppose that a function $f : X \to Y$ satisfies*

$$\|D_{a,b} f(x, y)\| \leq \varepsilon(\|x\|^p + \|y\|^q) \tag{3.10}$$

for all $x, y \in X$. Then there exists a unique quadratic function $T : X \to Y$ which satisfies the equation (1.5) and the inequality

$$\|f(x) - T(x)\| \leq \frac{\varepsilon \|x\|^p}{2|a^2 - |a|^p|}$$

for all $x \in X$ and for all $x \in X \setminus \{0\}$ if $p < 0$. The function T is given by

$$T(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{2n}} \quad \text{if } p, q < 2 \qquad \left(T(x) = \lim_{n\to\infty} a^{2n} f(\frac{x}{a^n}) \quad \text{if } p, q > 2 \right)$$

for all $x \in X$. If moreover either f is measurable or for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^2 T(x)$ for all $r \in \mathbb{R}$.

It is significant for us to decrease the possible estimator of the stability problem for the functional equations. This work is possible if we consider the stability problem in the sense of Hyers and Ulam for the functional equation

(1.5) with an appropriate large integer a. As a result, the following corollary is an immediate consequence of Theorem 3.1

Coeollery 3.3. *Let X and Y be a real normed space and a Banach space, respectively, and let $\varepsilon \geq 0$ be a real number. Suppose that a function $f : X \to Y$ satisfies*

$$\|D_{a,b}f(x,y)\| \leq \varepsilon \tag{3.11}$$

for all $x, y \in X$. Then there exists a unique quadratic function $T : X \to Y$ defined by $T(x) = \lim_{n\to\infty} \frac{f(a^n x)}{a^{2n}}$ which satisfies the equation (1.5) and the inequality

$$\|f(x) - T(x)\| \leq \frac{\varepsilon}{2(|a|^2 - 1)} \tag{3.12}$$

for all $x \in X$. Furthermore, if either f is measurable or for each fixed $x \in X$ the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to Y is continuous, then $T(rx) = r^2 T(x)$ for all $r \in \mathbb{R}$.

4. Stability in Banach modules over Banach $*$-algebras

In the last part of this paper, let A be a unital Banach $*$-algebra with norm $|\cdot|$ and let ${}_A\mathbb{B}_1$ and ${}_A\mathbb{B}_2$ be Banach left A-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. We denote $\hat{a} := aa^*, a^*a$, or $\frac{aa^*+a^*a}{2}$ for each $a \in A$. A mapping $Q : {}_A\mathbb{B}_1 \to {}_A\mathbb{B}_2$ is called A_{sa}-quadratic if $Q(ax) = \hat{a}Q(x)$ and $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ for all $a \in A$ and all $x \in {}_A\mathbb{B}_1$ [22]. If two Banach spaces E_1 and E_2 are considered as Banach modules over $A := \mathbb{C}$, then the A_{sa}-quadratic mapping $Q : E_1 \to E_2$ implies $Q(ax) = |a|^2 Q(x)$ for all $a \in \mathbb{C}$. We are going to prove the generalized Hyers-Ulam stability problem of the functional equation (1.5) in Banach modules over a unital Banach algebra. As an application of the above Theorem 3.1, we have the following.

Theorem 4.1. *Suppose that a mapping $f : {}_A\mathbb{B}_1 \to {}_A\mathbb{B}_2$ satisfies*

$$\begin{aligned} \|D_{a,b,u}f(x,y)\| &:= \|f(aux+buy) + f(aux-buy) \\ &\quad -b^2\hat{u}f(x+y) - b^2\hat{u}f(x-y) - 2(a^2-b^2)\hat{u}f(x)\| \\ &\leq \phi(x,y) \end{aligned} \tag{4.1}$$

for all $u \in A$ ($|u| = 1$) and for all $x, y \in {}_A\mathbb{B}_1$, and that the upper bound $\phi : {}_A\mathbb{B}_1 \times {}_A\mathbb{B}_1 \to \mathbb{R}^+$ for the approximate remainder $D_{a,b,u}f$ satisfies the assumptions of Theorem 3.1.

If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathbb{B}_1$, then there exists a unique A_{sa}-quadratic mapping $Q : {}_A\mathbb{B}_1 \to {}_A\mathbb{B}_2$, defined by

$$Q(x) = \lim_{i\to\infty} \frac{f(a^i x)}{a^{2i}} \qquad \Big(Q(x) = \lim_{i\to\infty} a^{2i} f(\frac{x}{a^i})\Big), \tag{4.2}$$

which satisfies the equation (1.5) and the inequality (3.4)

Proof. By Theorem 3.1, it follows from the inequality of the statement for $u = 1$ that there exists a unique quadratic mapping $Q : {}_A\mathbb{B}_1 \to {}_A\mathbb{B}_2$ defined by (4.2) which satisfies the equation (1.5) and the inequality (3.4).

Under the assumption that either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathbb{B}_1$, the quadratic mapping $Q : {}_A\mathbb{B}_1 \to {}_A\mathbb{B}_2$ satisfies the following equation by the same reasoning as the proof of [6]

$$Q(tx) = t^2 Q(x), \quad \forall x \in {}_A\mathbb{B}_1, \forall t \in \mathbb{R}.$$

That is, Q is $\mathbb{R}$-quadratic.

Replacing x, y by $a^{i-1}x, 0$ in (4.1) respectively, we obtain that for each $u \in A$ ($|u| = 1$)

$$2\|f(a^i ux) - a^2 \hat{u} f(a^{i-1}x)\| \le \phi(a^{i-1}x, 0) \tag{4.3}$$

for all $x \in {}_A\mathbb{B}_1$. Using the fact that there exists a positive constant K such that $\|uz\| \le K|u|\|z\|$ for all $u \in A$ and for each $z \in {}_A\mathbb{B}_2$ [3], one can show from (4.3) that

$$\begin{aligned} \|\hat{u} f(a^i x) - \hat{u} a^2 f(a^{i-1}x)\| &\le K|\hat{u}|\|f(a^i x) - a^2 f(a^{i-1}x)\| \\ &\le \frac{K\phi(a^{i-1}x, 0)}{2} \end{aligned}$$

for all $u \in A(|u| = 1)$ and all $x \in {}_A\mathbb{B}_1$. Thus we get by the last inequality

$$\begin{aligned} &\|f(a^i ux) - \hat{u} f(a^i x)\| \\ \le\ &\Big\|f(a^i ux) - a^2 \hat{u} f(a^{i-1}x)\Big\| + \Big\|a^2 \hat{u} f(a^{i-1}x) - \hat{u} f(a^i x)\Big\| \\ \le\ &\frac{\phi(a^{i-1}x, 0)}{2} + \frac{K\phi(a^{i-1}x, 0)}{2} \end{aligned}$$

for all $u \in A(|u| = 1)$ and all $x \in {}_A\mathbb{B}_1$. Dividing the last inequality by $|a|^{2i}$ and then taking the limit, we have

$$\begin{aligned} \|Q(ux) - \hat{u}Q(x)\| &= \lim_{i\to\infty} \left\| \frac{f(a^i ux) - \hat{u}f(a^i x)}{a^{2i}} \right\| \\ &\le \lim_{i\to\infty} \frac{\phi(a^{i-1}x, 0) + K\phi(a^{i-1}x, 0)}{2|a|^{2i}} \\ &= 0. \end{aligned}$$

Hence Q satisfies the equation $Q(ux) = \hat{u}Q(x)$ for all $u \in A(|u| = 1)$ and all $x \in {}_A\mathbb{B}_1$. The last equality is also true for $u = 0$. Since Q is $\mathbb{R}$-quadratic and $Q(ux) = \hat{u}Q(x)$ for each element $u \in A(|u| = 1)$, we figure out

$$\begin{aligned} Q(ax) &= Q(|a| \cdot \frac{a}{|a|}x) = |a|^2 \cdot Q(\frac{a}{|a|}x) = |a|^2 \cdot \frac{\hat{a}}{|a|^2} \cdot Q(x) \\ &= \hat{a}Q(x) \end{aligned}$$

for all $a \in A(a \neq 0)$ and all $x \in {}_A\mathbb{B}_1$. So the unique $\mathbb{R}$-quadratic mapping $Q : {}_A\mathbb{B}_1 \to {}_A\mathbb{B}_2$ is also A_{sa}-quadratic, as desired.

The proof of assertion indicated by parentheses in the theorem is similarly proved. This completes the proof of the theorem. □

Corcllary 4.2. *Let E_1 and E_2 be Banach spaces over the complex field $\mathbb{C}$. Suppose that a mapping $f : E_1 \to E_2$ and a mapping $\phi : E_1 \times E_1 \to \mathbb{R}_+$ satisfy (4.1) for all $u \in \mathbb{C}$ ($|u| = 1$) and for all $x, y \in E_1$. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$, then there exists a unique quadratic mapping $Q : E_1 \to E_2$ which satisfies the equation (1.5) and $Q(ax) = |a|^2 Q(x)$ for all $a \in \mathbb{C}$ and for all $x \in E_1$, and the inequality (3.4).*

Proof. Since $\mathbb{C}$ is a Banach algebra, the Banach spaces E_1 and E_2 are considered as Banach modules over $\mathbb{C}$. By Theorem 4.1, there exists a unique $\mathbb{C}_{sa}$-quadratic mapping $Q : E_1 \to E_2$ satisfying the inequality (3.4). This completes the proof. □

Now let M be a Banach left A-module. Let us call a mapping $Q : M \to A$ an A-quadratic mapping if both relations $Q(ax) = aQ(x)a^*$ and $Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y)$ are fulfilled [35]. A mapping $Q : M \to A$ is called a generalized A-quadratic mapping if $Q(ax) = aQ(x)a^*$ for all $x \in M$, and the following identity holds:

$$Q\left(\sum_{i=1}^n a_i x_i\right) + \sum_{1\le i<j\le n} a_i a_j Q(x_i - x_j) = \left(\sum_{i=1}^n a_i\right)\left[\sum_{i=1}^n a_i Q(x_i)\right]$$

for all $x_i \in M$, some fixed a_i in $\mathbb{R}$ $(i = 1, \cdots, n)$ and at least two of them are nonzero such that $\sum_{i=1}^{n} a_i \neq 0$, and a fixed $n \geq 2$ [21]. It was shown that the notion of A-quadratic mapping is equivalent to the notion of generalized A-quadratic mapping if all spaces are over the complex number field and a mapping $B : M \times M \to A$ is defined in terms of the mapping Q as

$$B(x, y) = \frac{1}{4}[Q(x + y) - Q(x - y) + iQ(x + iy) - iQ(x - iy)] \tag{4.4}$$

for all x, y in M [21]. It was indicated in [35] that if the relation (4.4) holds and Q is an A-quadratic form, then B is an A-sesquilinear form and $Q(x) = B(x, x)$, and vice versa. Now it follows easily from Theorem 2.1that a mapping Q is an A-quadratic mapping if and only if

$$\begin{aligned} &Q(ax) = aQ(x)a^*, \\ &Q(ax + by) + f(ax - by) = b^2 f(x + y) + b^2 f(x - y) + 2(a^2 - b^2)f(x) \end{aligned}$$

for all $x, y \in {}_A\mathbb{B}_1$. As an application of Theorem **??**, we are going to investigate the generalized Hyers-Ulam-Rassias stability problem for A-quadratic mappings of (1.5) in Banach modules over a Banach $*$-algebra. In the following theorem, let ${}_A\mathbb{B}_1$ and ${}_A\mathbb{B}_2$ be Banach A-bimodules.

Theorem 4.3. *Let $f : {}_A\mathbb{B}_1 \to {}_A\mathbb{B}_2$ be a mapping for which there exists a mapping $\phi : {}_A\mathbb{B}_1 \times {}_A\mathbb{B}_1 \to \mathbb{R}_+$ such that*

$$\begin{aligned} \|D_{a,b,u}f(x, y)\| &:= \|f(aux + buy) + f(aux - buy) \\ &\quad -b^2uf(x + y)u^* - b^2uf(x - y)u^* - 2(a^2 - b^2)uf(x)u^*\| \\ &\leq \phi(x, y) \end{aligned} \tag{4.5}$$

for all $u \in A$ $(|u| = 1)$ and for all $x, y \in {}_A\mathbb{B}_1$ and the upper bound ϕ for the approximate remainder $D_{a,b,u}f$ satisfies the assumptions of Theorem 3.1. If either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathbb{B}_1$, then there exists a unique A-quadratic mapping $Q : {}_A\mathbb{B}_1 \to {}_A\mathbb{B}_2$, defined by $Q(x) = \lim_{k\to\infty} \frac{f(a^k x)}{a^{2k}}$ for all $x \in {}_A\mathbb{B}_1$, which satisfies the equation (1.5) and the inequality (3.4) for all $x \in {}_A\mathbb{B}_1$.

Proof. By Theorem 3.1, it follows from the inequality of the statement for $u = 1$ that there exists a unique quadratic mapping $Q : {}_A\mathbb{B}_1 \to {}_A\mathbb{B}_2$, defined by $Q(x) = \lim_{k\to\infty} \frac{f(a^k x)}{a^{2k}}$, which satisfies the equation (1.5) and the inequality (3.4).

Under the assumption that either f is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_A\mathbb{B}_1$, the quadratic mapping Q satisfies $Q(tx) = t^2Q(x)$ for all $x \in {}_A\mathbb{B}_1$ and for all $t \in \mathbb{R}$ by the same reasoning as the proof of [6]. That is, Q is $\mathbb{R}$-quadratic. Replacing x and y by a^kx and a^ky in (4.5),respectively and dividing the resulting inequality by $|a|^{2k}$, and then taking $k \to \infty$

$$\begin{aligned}
& \|D_{a,b,u}Q(x,y)\| \\
= & \|Q(aux+buy)+Q(aux-buy) \\
& -b^2uQ(x+y)u^* - b^2uQ(x-y)u^* - 2(a^2-b^2)uQ(x)u^*\| \\
= & 0
\end{aligned}$$

for all $x \in {}_A\mathbb{B}_1$ and for each $u \in A(|u|=1)$. Setting $y := 0$ in the last equation, we obtain that $Q(ux) = uQ(x)u^*$ for all $x \in {}_A\mathbb{B}_1$ and for each $u \in A(|u|=1)$. The last relation is also true for $u = 0$. Since Q is $\mathbb{R}$-quadratic, for each element $a(a \neq 0) \in A$

$$\begin{aligned}
Q(ax) &= Q(|a|\frac{a}{|a|}x) = |a|^2Q(\frac{a}{|a|}x) = |a|^2\frac{a}{|a|}Q(x)\frac{a^*}{|a|} \\
&= aQ(x)a^*, \quad \forall a \in A(a \neq 0), \forall x \in {}_A\mathbb{B}_1.
\end{aligned}$$

So the unique $\mathbb{R}$-quadratic mapping Q is also A-quadratic, as desired. This completes the proof. □

References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Cambridge Univ. Press, 1989.

[2] J. Baker, The stability of the cosine equation, *Proc. Amer. Math. Soc.* **80**(1980), 411-416.

[3] F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York, Heidelberg and Berlin, 1973.

[4] I. Chang and H. Kim, On the Hyers-Ulam stability of quadratic functional equations, *J. Ineq. Pure and Appl. Math.* **3(3)**(2002).

[5] S. Czerwik, On the stability of the quadratic mapping in normed spaces, *Abh. Math. Sem. Univ. Hamburg,* **62**(1992), 59-64.

[6] S. Czerwik, The stability of the quadratic functional equation, in *Stability of Mappings of Hyers-Ulam Type* (edited by Th. M. Rassias and J. Tabor), Hadronic Press, Florida, 1994, pp 81-91.

[7] Dan Amir, *Characterizations of Inner Product Spaces*, Birkhäuser-Verlag, Basel, 1986.

[8] G. L. Forti, An existence and stability theorem for a class of functional equations, *Stochastica,* **4**(1980), 23-30.

[9] G. L. Forti, Hyers-Ulam stability of functional equations in several variables, *Aequationes Math.* **50**(1995), 143-190.

[10] P. Găvruta, A generalization of the Hyers-Ulam-Rassias Stability of approximately additive mappings, *J. Math. Anal. Appl.* **184**(1994), 431-436.

[11] D. H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci.* **27**(1941), 222-224.

[12] D. H. Hyers, G. Isac and Th.M. Rassias, *Stability of Functional Equations in Several Variables*, Birkhäuser, Basel, 1998.

[13] D. H. Hyers, G. Isac and Th.M. Rassias, On the asymptoticity aspect of Hyers-Ulam stability of mappings, *Proc. Amer. Math. Soc.* **126**(1998), 425-430.

[14] D. H. Hyers and Th.M. Rassias, Approximate homomorphisms, *Aequationes Math.* **44**(1992), 125-153.

[15] K. Jun and Y. Lee, On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality, *Math. Ineq. Appl.* **4(1)**(2001), 93-118.

[16] K. Jun and H. Kim, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation, *J. Math. Anal. Appl.* **274(2)**(2002), 867-878.

[17] K. Jun and H. Kim, On the Hyers-Ulam-Rassias stability of a general cubic functional equation, *Math. Ineq. Appl.* **6(2)**(2003), 289-302.

[18] K. Jun, H. Kim and I. Chang, On the Hyers-Ulam stability of an Euler-Lagrange type cubic functional equation, it J. Comput. Anal. Appl. to appear.

[19] K. Jun and H. Kim, On the Hyers-Ulam stability of a generalized quadratic and additive functional equations, *Bull. Korean Math. Soc.* **42**(2005),133-148.

[20] S. Jung, On the Hyers-Ulam stability of the functional equations that have the quadratic property, *J. Math. Anal. Appl.* **222**(1998), 126-137.

[21] C. S. Lin, Sesquilinear and quadratic forms on modules over $*$-algebra, *Publ. Inst. Math.* **51**(1992), 81-86.

[22] C. Park, On the stability of the quadratic mapping in Banach modules, *J. Math. Anal. Appl.* **276**(2002), 135-144.

[23] A. Prastaro and Th. M. Rassias, Ulam stability in geometry of PDE's, *Nonlinear Funct. Anal. Appl.* **8(2)**(2003), 259-278.

[24] J. M. Rassias, On the stability of the Euler-Lagrange functional equation, *Chinese J. Math.* **20**(1992), 185-190.

[25] J. M. Rassias, Solution of the Ulam stability problem for Euler-Lagrange quadratic mappings, *J. Math. Anal. Appl.* **220**(1998), 613-639.

[26] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.* **72**(1978), 297-300.

[27] Th. M. Rassias, *Inner Product Spaces and Applications*, Longman, 1997.

[28] Th. M. Rassias, On the stability of the quadratic functional equation and its applications, *Studia, Univ. Babes-Bolyai,* bf XLlll(3)(1998), 89-124.

[29] Th. M. Rassias, On the stability of functional equations and a problem of Ulam, *Acta Applicandae Mathematicae* **62**(2000), 23-130.

[30] Th. M. Rassias, The problem of S.M. Ulam for approximately multiplicative mappings, *J. Math. Anal. Appl.* **246**(2000), 352-378.

[31] Th. M. Rassias, On the stability of functional equations in Banach spaces, *J. Math. Anal. Appl.* **251**(2000), 264-284.

[32] Th. M. Rassias, *Functional Equations and Inequalities*, Kluwer Academic Publishers, Dorderecht, 2000.

[33] F. Skof, Local properties and approximations of operators(Italian), *Rend. Sem. Mat. Fis. Milano* **53**(1983), 113-129.

[34] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publ. New York, 1960.

[35] J. Vukman, Some functional equations in Banach algebras and an application, *Proc. Amer. Math. Soc.* **100**(1987), 133-136.