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Iterative Schemes for Mixed Quasi Equilibrium-like Problems*

Muhammad Aslam Noor†
Mathematics Department, COMSATS Institute of Information Technology, Islamabad, Pakistan

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Abstract

In this paper, we introduce a new class of equilibrium problems, known as *mixed quasi invex equilibrium (equilibrium-like).* This class of equilibrium problems includes invex equilibrium problems, variational inequalities and variational-like inequalities as special cases. We use the auxiliary principle technique to suggest and analyze an implicit iterative scheme for solving invex equilibrium problems and study the convergence criteria of these methods under mild conditions. Our results represent significant and important refinements of the previously known results.

Keywords and Phrases: *Variational-like inequalities, Invex equilibrium problems, Auxiliary principle, Proximal methods, Convergence, Skew-symmetric functions.*

1. Introduction

Equilibrium problems introduced and investigated by Blum and Oettli [1] and Noor and Oettli [2] are being used to study a wide class of unrelated problems

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†E-mail: noormaslam@hotmail.com

arising in pure and applied sciences in a general and unified framework. It is well-known that equilibrium problems include variational inequalities and optimization problems as special cases, see [1-24]. Due to the nature of the equilibrium problems, it is not possible to extend the projection and its variant forms including the Wiener-Hopf equations, resolvent equations techniques for solving the equilibrium type problems. To overcome this drawback, one usually uses the auxiliary principle technique The main and basic idea in this technique is to consider an auxiliary invex equilibrium problem related to the original problem. This technique has been used to suggest and analyze a number of iterative methods for solving various classes of, equilibrium problems, variational-like inequalities and variational inequalities. It has been shown that a substantial number of numerical methods can be obtained as special cases from this technique, see [1-24].

Inspired and motivated by the research going on in this area, we introduce and consider a new class of equilibrium problems, which is called the mixed quasi equilibrium-like problems. It is shown that the mixed equilibrium-like problems include several classes of variational-like inequalities and invex equilibrium problems as special cases. Hence collectively the invex equilibrium problems cover a vast range of applications. We here use the auxiliary principle technique to suggest and consider an implicit iterative method for solving these equilibrium-like problems. We prove that the convergence of this method requires pseudomonotonicity, which is a weaker condition than monotonicity. In this respect, our results represent an improvement of the previously known results. Our results can be considered as a novel and important application of the auxiliary principle technique. Since the equilibrium-like problems include several classes of variational-like inequalities, variational inequalities, equilibrium and related optimization problems as special cases, results obtained in this paper continue to hold for these problems.

2. Preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed set in H. Let $f : K \rightarrow H$ and $\eta(.,.) : K \times K \rightarrow H$ be continuous functions. Let $\varphi : K \longrightarrow R$ be a continuous function.

First of all, we recall the following well know results and concepts.

Definitions 2.1[15]. *Let $u \in K$. Then the set K is said to be φ-invex at u with respect to $\eta(.,.)$ and $\varphi(.)$, if*

$$u + te^{i\varphi}\eta(v,u) \in K, \quad \forall u, v \in K, t \in [0,1].$$

K is said to be an φ-invex set with respect to η and φ, if K is φ-invex at each $u \in K$. The φ-invex set K is also called $\varphi\eta$-connected set. Note that the convex set with $\varphi = 0$ and $\eta(v,u) = v - u$ is an φ-invex set, but the converse is not true. For example, the set $K = R - (-\frac{1}{2}, \frac{1}{2})$ is an φ-invex set with respect to η and $\varphi = 0$, where

$$\eta(v,u) = \begin{cases} v - u, & \text{for} \quad v > 0, u > 0 \quad \text{or} \quad v < 0, u < 0 \\ u - v, & \text{for} \quad v < 0, u > 0 \quad \text{or} \quad v < 0, u < 0. \end{cases}$$

It is clear that K is not a convex set.

Remark 2.1.
(i). *If $\varphi = 0$, then the set K is called the invex (η-connected) set, see [2,4, 9-12].* **(ii).** *If If $\eta(v,u) = v - u$, then the set K is called the φ-convex set, see Noor [14].* **(iii).** *If $\varphi = 0$ and $\eta(v,u) = v - u$, then the set K is called the convex set*

From now onward K is a nonempty closed φ-invex set in H with respect to φ and $\eta(.,.)$, unless otherwise specified.

Definition 2.2[15]. *The function F on the φ-invex set K is said to be φ-preinvex with respect to η and φ, if*

$$F(u + te^{i\varphi}\eta(v,u)) \leq (1-t)F(u) + tF(v), \quad \forall u, v \in K, \quad t \in [0,1].$$

The function F is said to be φ-preconcave if and only if $-F$ is φ-preinvex. Note that every convex function is a φ-preinvex function, but the converse is not true. For example, the function $F(u) = -|u|$ is not a convex function, but it is a φ-preinvex function with respect to η and $\varphi = 0$, where

$$\eta(v,u) = \begin{cases} v - u, & \text{if} \quad v \leq 0, u \leq 0 \quad \text{and} \quad v \geq 0, u \geq 0 \\ u - v, & \text{otherwise} \end{cases}$$

Definition 2.3[15]. *The differentiable function F on the φ-invex set K is said to be an φ-invex function with respect to φ and $\eta(.,.)$, if*

$$F(v) - F(u) \geq \langle F'_\varphi(u), \eta(v,u)\rangle, \quad \forall u, v \in K,$$

where $F'_\varphi(u)$ is the differential of F at u in the direction of $v - u \in K$.

Note that for $\varphi = 0$, we obtain the original definition of invexity which is due to Hanson [10]. It is well known that the concepts of preinvex and invex functions play a significant role in the mathematical programming and optimization theory, see [8-24] and the references therein.

One can prove the following result, using the technique of Noor and Noor [15, 24].

Lemma 2.1. *Let f be a differentiable function on the φ-invex set K in H and let the Assumption 2.1 hold. Then the following are equivalent.*
(i). The function f is a φ-preinvex function.
(ii). The function f is an φ-invex function.
(iii). $f'_\varphi(u)$ is monotone, that is,

$$\langle f'_\varphi(u), \eta(v,u)\rangle + \langle f'_\varphi(v), \eta(u,v)\rangle \leq 0, \forall u, v \in K,$$

where $f'_\varphi(u)$ is the differential of the function f at $u \in K$.

Definition 2.4. *A function f is said to be strongly φ-preinvex function on K with respect to the function $\eta(.,.)$ and φ with modulus μ, if, for all $u, v \in K, t \in [0,1]$,*

$$f(u + te^{i\varphi}\eta(v,u)) \leq (1-t)f(u) + tf(v) - t(1-t)\mu\|\eta(v,u)\|^2.$$

Clearly the differentiable strongly φ-preinvex function F is a stronglyφ-y invex functions with module constant μ, that is,

$$f(v) - f(u) \geq \langle f'_\varphi(u), \eta(v,u)\rangle + \mu\|\eta(v,u)\|^2,$$

and the converse is also true under some conditions.

Definition 2.5. *The bifunction* $\phi(.,.) : H \times H \longrightarrow R \cup \{+\infty\}$ *is called skew-symmetric, if and only if,*

$$\phi(u,u) - \phi(u,v) - \phi(v,u) - \phi(v,v) \geq 0, \quad \forall u, v \in H.$$

Clearly if the skew-symmetric bifunction $\phi(.,.)$ is bilinear, then

$$\phi(u,u) - \phi(u,v) - \phi(v,u) + \phi(v,v) = \phi(u-v, u-v) \geq 0, \quad \forall u, v \in H.$$

For given continuous bifunction $F(.,.) : K \times K \longrightarrow R$ and continuous bifunction $\phi(.,.) : K \times K \longrightarrow R \cup \{\infty\}$, consider the problem of finding $u \in K$ such that

$$F(u, \eta(v,u)) + \phi(v,u) - \phi(u,u) \geq 0, \quad \forall v \in K, \tag{1}$$

which is called the *mixed quasi invex equilibrium (equilibrium-like) problem.*

If $F(u, \eta(v,u)) \equiv \langle Tu, \eta(v,u)$, where $T : H \longrightarrow H$, and $\eta(.,.) : K \times K \longrightarrow R \cup \infty$, then problem (1) is equivalent to finding $u \in K$ such that

$$\langle Tu, \eta(v,u)\rangle + \phi(v,u) - \phi(u,u) \geq 0, \quad \forall v \in K, \tag{2}$$

which is known as the mixed quasi variational-like inequality problem. Problem (2) and its variant forms have been studied extensively by many authors in the setting of convexity using the KKM mappings and fixed-point theory. It is worth mentioning the concept of variational-like inequalities in the convexity setting is not well-defined and consequently all the results so far obtained in the convexity (scalar and vector) are misleading and wrong.

If $\eta(v,u) = v - u$, then the invex set K becomes the convex set and problem (1) is called the mixed quasi equilibrium problem with bifucntion of finding $u \in K$ such that

$$F(u, v-u) + \phi(v,u) - \phi(u,u) \geq 0, \quad \forall v \in K, \tag{3}$$

which is called the mixed quasi equilibrium problem ans appears to be a new one.

Also the variational-like inequality (2) is equivalent to finding $u \in K$ such that

$$\langle Tu, v-u\rangle + \phi(v,u) - \phi(u,u) \geq 0, \quad \forall v \in K, \tag{4}$$

which is known as the mixed quasi variational inequality.

For the recent applications, numerical methods and formulations of variational inequalities, variational-like inequalities and invex equilibrium problems; see Refs. 1-38.

Definition 2.6. *The bifunction $F(.,.)$ is said to be:*
(i). *pseudomonotone with respect to the bifunction $\phi(.,.)$, if*

$$\begin{aligned} & F(u,\eta(v,u)) + \phi(v,u) - \phi(u,u) \geq 0 \\ \Longrightarrow \quad & -F(v,\eta(u,v)) + \phi(v,u) - \phi(u,u) \geq 0, \quad \forall u,v \in K. \end{aligned}$$

For $F(u,\eta(v,u)) = \langle Tu, \eta(v,u)\rangle$, and $F(u,\eta(v,u)) = \langle Tu, v-u\rangle$, Definition 2.5 reduces to the well known concepts in variational inequalities theory, see the references.

3. Main Results

In this section, we use the auxiliary principle technique to suggest and analyze an implicit iterative algorithm for solving equilibrium-like problem (1). For a given $u \in K$, consider the problem of finding $w \in K$ such that

$$\begin{aligned} & \rho F(w,\eta(v,w)) + \langle E'_{\varphi}(w) - E'_{\varphi}(u), \eta(v,w)\rangle \\ \geq \quad & \rho\{\phi(w,w) - \phi(v,w)\} \quad \forall v \in K, \end{aligned} \tag{5}$$

which is known as the auxiliary equilibrium-like problem. Here $E'_{\varphi}(u)$ is the differential of a strong φ-preinvex function $E(u$ at the point $u \in K$. Problem (5) has a unique solution, since the functions E is a strongly φ-preinvex function.

Remark 3.1 *The function $B(z,u) = E(z) - E(u) - \langle E'_{\varphi}(u), \eta(z,u)\rangle$ associated with the φ-preinvex function $E(u)$ is called the generalized Bregman function. We note that if $\eta(z,u) = z-u$, then $B(z,u) = E(z) - E(u) - \langle E'(u), z-u\rangle$ is the well known Bregman function. For the applications of the Bregman function in solving variational inequalities and complementarity problems, see [13,14,15, 16].*

We remark that if $w = u$, then w is a solution of (1). On the basis of this observation, we suggest and analyze the following iterative algorithm for solving (1) as long as (5) is easier to solve than (1).

Algorithm 3.1. *For a given* $u_0 \in H$, *calculate the approximate solution* u_{n+1} *by the iterative scheme*

$$\begin{aligned}&\rho F(u_{n+1}, \eta(v, u_{n+1})) + \langle E'_\varphi(u_{n+1}) - E'_\varphi(u_n), \eta(v, u_{n+1})\rangle \\ &\rho\{\phi(v, u_{n+1}) - \phi(u_{n+1}, u_{n+1})\} \geq 0, \quad \forall v \in K, \end{aligned} \tag{6}$$

Algorithm 3.1 is called the proximal point method for solving mixed quasi equilibrium-like problems with bifunction (1). Note that if $\eta(v, u) = v - u$, then Algorithm 3.1 reduces to the following method for solving mixed quasi equilibrium problem (3).

Algorithm 3.2. *For a given* $u_0 \in H$, *find the approximate solution* u_{n+1} *by the iterative scheme*

$$\begin{aligned}\rho F(u_{n+1}, v - u_{n+1}) \quad + \quad &\langle E'_\varphi(u_{n+1}) - E'_\varphi(u_n), v - u_{n+1}\rangle \\ &+\rho\{\phi(v, u_{n+1}) - \phi(u_{n+1}, u_{n+1})\} \geq 0, \quad \forall v \in K,\end{aligned}$$

which is called the proximal method. Note that $E'(u)$ *is the differential of a strongly convex function* $E(u)$ *at* $u \in K$, *a convex set in* H.

If $F(u, \eta(v, u)) = \langle Tu, \eta(v, u)\rangle$, then Algorithm 3.1 collapse to the following method for solving mixed quasi variational-like inequalities (2).

Algorithm 3.3. *For a given* $u_0 \in H$, *calculate the approximate solution* u_{n+1} *by the iterative scheme*

$$\begin{aligned}\langle \rho T u_{n+1} \quad + \quad &E'_\varphi(u_{n+1}) - E'_\varphi(u_n), \eta(v, u_{n+1})\rangle \\ &\rho\{\phi(v, u_{n+1}) - \phi(u_{n+1}, u_{n+1})\} \geq 0 \quad \forall v \in K.\end{aligned}$$

Here $E'_\varphi(u)$ *is the differential of a differentiable strongly* φ*- preinvex function* $E(u)$ *at a point* $u \in K$, *an* φ*-invex set in* H. *Algorithm 3.3 can be considered as a correct algorithm for solving variational-like inequalities (2).*

In brief, for suitable and appropriate choice of the bifunction $F(.,.), \phi(.,.)$ and the sets K, one can obtain a large number of iterative methods for solving various classes of equilibrium problems and variational inequalities. This shows that the iterative methods suggested in this paper are quite general and flexible.

We now study the convergence criteria of Algorithm 3.1, which is the main motivation of our next result.

Theorem 3.1. *Let the function $F(.,.)$ be pseudomonotone. If E is a differentiable strongly φ-preinvex function with modulus $\beta > 0$, and*

$$\eta(u,v) = \eta(u,z) + \eta(z,v), \quad \forall u,v,z \in H, \tag{7}$$

then the approximate solution u_{n+1} obtained from Algorithm 3.1 converges to a solution $u \in K$ satisfying (1).

Proof. Let $u \in K$ be a solution of (1). Then

$$-F(v,\eta(u,v)) + \phi(v,u) - \phi(u,u) \geq 0, \quad \forall v \in K \tag{8}$$

since the bifunction $F(.,.)$ is pseudomonotone with respect to the bifunction $\phi(.,.)$..

Taking $v = u_{n+1}$ in (8), we have

$$-F(u_{n+1},\eta(u_{n+1,u)}, u) + \phi(u_{n+1},u) - \phi(u,u) \geq 0. \tag{9}$$

Consider the function,

$$\begin{aligned} B(u,z) &= E(u) - E(z) - \langle E'_{\varphi}(z), \eta(u,z)\rangle \\ &\geq \beta\|\eta(u,z)\|^2, \quad \text{using the strongly } \varphi\text{-invexity of } E. \end{aligned} \tag{10}$$

Combining (6)- (10), we have

$$\begin{aligned} B(u,u_n) - B(u,u_{n+1}) &= E(u_{n+1}) - E(u_n) - \langle E'_{\varphi}(u_n), \eta(u,u_n)\rangle \\ &\quad + \langle E'_{\varphi}(u_{n+1}), \eta(u,u_{n+1})\rangle \\ &= E(u_{n+1}) - E(u_n) - \langle E'_{\varphi}(u_n) - E'_{\varphi}(u_{n+1}), \eta(u,u_{n+1})\rangle \\ &\quad - \langle E'(u_n), \eta(u_{n+1},u_n)\rangle \\ &\geq \beta\|\eta(u_{n+1},u_n)\|^2 + \langle E'_{\varphi}(u_{n+1}) - E'_{\varphi}(u_n), \eta(u,u_{n+1})\rangle \\ &\geq \beta\|\eta(u_{n+1},u_n)\|^2 - F(u_{n+1},Tu_{n+1},u) \\ &\quad + \rho\{\phi(u_{n+1},u_{n+1}) - \phi(u,u_{n+1})\}, \\ &\geq \beta\|\eta(u_{n+1},u_n)\|^2 + \rho\{\phi(u_{n+1},u_{n+1}) - \rho\phi(u,u_{n+1}) \\ &\quad - \phi(u_{n+1},u) + \phi(u,u)\} \\ &\geq \beta\|\eta(u_{n+1},u_n)\|^2, \end{aligned}$$

since the bifunction $\varphi(.,.)$ is skew-symmetric.

If $u_{n+1} = u_n$, then clearly u_n is a solution of equilibrium-like problem (1). Otherwise, the sequence $B(u, u_n) - B(u, u_{n+1})$ is nonnegative and we must have

$$\lim_{n\to\infty} (\|\eta(u_{n+1}, u_n)\|) = 0.$$

Now by using the technique of Zhu and Marcotte[20], it can be shown that the entire sequence $\{u_n\}$ converges to the cluster point $\overline{u}$ satisfying the equilibrium-like problem (1). □

Conclusion. In this paper, we have introduced and studied a new class of equilibrium problems, which is called the *Mixed Quasi Equilibrium-like Problems.* It is shown that this class includes variational-like inequalities, equilibrium problems and variational inequalities as special cases. We have used the auxiliary principle technique to suggest and analyze an implicit method for solving the equilibrium-like problems. The convergence of the proposed method is proved under some mild conditions. Results proved in this paper can be regarded as a refinement and improvement of the previously known results. It is an open problem to implement this algorithm numerically and compare its performance with other methods.

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