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A New Version of The Stirling Formula*

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Abstract

A new version of the Stirling formula is given as

$$n! = (\frac{n}{e})^n \sqrt{2\pi n} exp \int_n^\infty \frac{\frac{1}{2} - \{x\}}{x} \, dx,$$

and it is applied to provide a new and more natural proof of a recent version due to L. C. Hsu.

Keywords and Phrases: *Stirling formula, Wallis' product formula, Infinite integral.*

1. Introduction

Stirling formula and its different versions have a fascinating history. The classical form containing Bernoulli numbers which has been studied deeply and thoroughly in [1] and [2] where an infinite numbers of recurrence relations for the Bernoulli numbers are obtained.

It is very interesting that in the last decade Hsu in [3] has given a new version without using Bernoulli numbers as the following identity

$$n! = (\frac{n}{e})^n \sqrt{2\pi n} exp(\sum_{k=n}^\infty \sum_{j=2}^\infty \frac{j-1}{2j(j+1)} (\frac{-1}{k})^j), \tag{1}$$

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whose proof is elementary and simple in nature, and it is applied in [4] to get a more accurate asymptotic relation.

Instead of the double summation in the right hand side of (1), in this short note, we will derive a new version of the Stirling formula via an infinite integral as

$$n! = (\frac{n}{e})^n\sqrt{2\pi n}exp\int_n^\infty \frac{\frac{1}{2}-\{x\}}{x}\,dx, \tag{2}$$

and it is applied to give a new and more natural proof of (1).

We will need the following well known Dirichlet test for convergence of infinite integral (see e.g. [5]).

Lemma. If $F(A) = \int_a^A f(x)\,dx$ is bounded on $[a,\infty)$, $g(x)$ is monotonic on $[a,\infty)$ and $lim_{x\to\infty}g(x) = 0$, then the infinite integral $\int_a^\infty f(x)g(x)\,dx$ is convergence.

2. Proof of (2)

From [6] and [7] we may find that

$$log\,n! = (n+\frac{1}{2})log\,n - n + 1 - \int_1^n \frac{\frac{1}{2}-\{x\}}{x}\,dx. \tag{3}$$

where $\{x\} = x - [x]$ and $[x]$ denotes the integral part of x.

Put $\delta_n := 1 - \int_1^n \frac{\frac{1}{2}-\{x\}}{x}\,dx$. By Lemma, it is not difficult to find that the infinite integral

$$\int_1^\infty \frac{\frac{1}{2}-\{x\}}{x}\,dx$$

is convergence, since

$$F(A) = \int_1^A (\frac{1}{2}-\{x\})\,dx = \int_{[A]}^A (\frac{1}{2}-\{x\})\,dx = \int_{[A]}^A ([A]+\frac{1}{2}-x)\,dx = \frac{1}{2}\{A\}(1-\{A\})$$

is bounded on $[1,\infty)$ and $g(x) = \frac{1}{x}$ is strictly decreasing on $[1,\infty)$ with $\lim_{x\to\infty}g(x) = 0$. So, we have

$$\lim_{n\to\infty} \delta_n = 1 - \int_1^\infty \frac{\frac{1}{2} - \{x\}}{x}\, dx := \delta. \tag{4}$$

Then from (3) we get

$$n! = (\frac{n}{e})^n \sqrt{n} e^{\delta_n}. \tag{5}$$

Similarly we have

$$(2n)! = (\frac{2n}{e})^{2n} \sqrt{2n} e^{\delta_{2n}}. \tag{6}$$

By (4) we get

$$\lim_{n\to\infty}\delta_{2n} = \lim_{n\to\infty}\delta_n = \delta. \tag{7}$$

Substituting (5), (6) and (7) into the Wallis' product formula

$$\lim_{n\to\infty}\frac{(n!)^2 2^{2n}}{(2n)!\sqrt{n}} = \sqrt{\pi},$$

we get

$$e^\delta = \sqrt{2\pi}.$$

Thus (5) may be rewritten in the form

$$n! = (\frac{n}{e})^n \sqrt{2\pi n} exp(\delta_n - \delta). \tag{8}$$

Finally, notice that

$$\delta_n - \delta = \int_n^\infty \frac{\frac{1}{2} - \{x\}}{x}\, dx,$$

and substitute it into (8), the formula (2) is obtained.

3. A New Proof of (1)

It is immediate to prove (1) by applying (2), since

$$
\begin{aligned}
& \int_n^\infty \frac{\frac{1}{2}-\{x\}}{x}\,dx \\
= \;& \Sigma_{k=n}^\infty \int_k^{k+1} \frac{\frac{1}{2}-\{x\}}{x}\,dx \\
= \;& \Sigma_{k=n}^\infty \int_k^{k+1} \frac{k+\frac{1}{2}-x}{x}\,dx \\
= \;& \Sigma_{k=n}^\infty[(k+\tfrac{1}{2})log\tfrac{k+1}{k}-1] \\
= \;& \Sigma_{k=n}^\infty[(1-\tfrac{1}{2k}+\tfrac{1}{3k^2}-\tfrac{1}{4k^3}+\cdots+(-1)^{j-1}\tfrac{1}{jk^{j-1}}+\dots) \\
& +\tfrac{1}{2}(\tfrac{1}{k}-\tfrac{1}{2k^2}+\tfrac{1}{3k^3}-\cdots+(-1)^{j-1}\tfrac{1}{jk^j}+\dots)-1] \\
= \;& \Sigma_{k=n}^\infty\Sigma_{j=2}^\infty(\tfrac{1}{j+1}-\tfrac{1}{2j})(\tfrac{-1}{k})^j \\
= \;& \Sigma_{k=n}^\infty\Sigma_{j=2}^\infty\tfrac{j-1}{2j(j+1)}(\tfrac{-1}{k})^j.
\end{aligned}
$$

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