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A Generalization on a Certain Class of Salagean-Type Harmonic Univalent Functions and Distortion Theorems for Fractional Calculus *

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Abstract

We define and investigate a new class of Salagean-type harmonic univalent functions. We obtain some properties of this subclass. Furthermore, we give the Hadamard product of several functions and some distortion theorems for fractional calculus of this generalized class.

Keywords and Phrases: *Univalent functions, Harmonic functions, Salagean operator, Fractional calculus*

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1. Introduction

A continuous complex-valued function $f = u+iv$ defined in a simply connected complex domain $\mathbb{C}$ is said to be harmonic in $\mathbb{C}$ if both u and v are real harmonic in $\mathbb{C}$. In any simple connected domain, we can write $f = h + \bar{g}$, where h and g are analytic in $\mathbb{C}$. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathbb{C}$ is that $|h'(z)| > |g'(z)|, \ z \in \mathbb{C}$.

Denote by S_H the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense preserving in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in \S_H$ we may express the analytic functions h and g as

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad , \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, \qquad |b_1| < 1. \tag{1}$$

Clunie and Sheil-Small [1] studied the class S_H together with some geometric subclasses of S_H.

The differential operator D^n was introduced by Salagean [2]. For $f = h+\bar{g}$ given by (1), Jahangiri *et al.* [3] defined the modified Salagean operator of f as follows:

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)} \tag{2}$$

where

$$D^n h(z) = z + \textstyle\sum_{k=2}^{\infty} k^n a_k z^k \text{and} D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

For $0 \leq \alpha < 1 \quad , n \in \mathbb{N}_0 \quad , \lambda > \frac{1}{2}$ and $z \in \mathbb{U}$, we let $HP(n, \lambda, \alpha)$ denote the family of harmonic functions f of the form (1) such that

$$\mathrm{Re}\{\frac{\mathrm{D^n f(z) + \lambda(D^{n+1} f(z) - D^n f(z))}}{\mathrm{z}}\} \geq \alpha \tag{3}$$

where $D^n f$ is defined by (2).

If the co-analytic part of $f = h + \bar{g}$ is identically zero and $n = 0$, then the family $HP(n, \lambda, \alpha)$ turns out to be the class $F_\lambda(\alpha)$ introduced by Bhoosmurmath and Swamy [4] for the analytic case.

We let the subclass $HP^*(n,\lambda,\alpha)$ consist of harmonic functions $f_n = h + \bar{g_n}$ in $HP(n,\lambda,\alpha)$ so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \qquad a_k \geq 0 \quad, b_k \geq 0. \quad (4)$$

The class $HP(n,\lambda,\alpha)$ includes a variety of well-known subclasses of S_H. For example, $HP^*(0,1,\alpha) \equiv HP^*(\alpha)$ which denote the subclass of S_H satisfying $\mathrm{Re}\{h'(z) + g'(z)\} > \alpha \quad ; 0 \leq \alpha < 1$ in [5].

In this paper, the coefficient bounds given in [5] for the class $HP^*(\alpha)$ are extended to the class $HP^*(n,\lambda,\alpha)$ of the forms (3) above. Furthermore, we determine extreme points, distortion theorems for fractional calculus, convolution conditions, and convex combinations for the functions in $HP^*(n,\lambda,\alpha)$.

2. Main Results

Firstly, we introduce a sufficient coefficient condition for functions in $HP^*(n,\lambda,\alpha)$.

Theorem 1. *Let $f = h + \bar{g}$ be so that h and g are given by [1]. If*

$$\sum_{k=1}^{\infty} [(\lambda k^{n+1} + (1-\lambda)k^n)|a_k| + (\lambda k^{n+1} - (1-\lambda)k^n)|b_k|] \leq 2 - \alpha \qquad (5)$$

$a_1 = 1, 0 \leq \alpha < 1, n \in \mathbb{N}_0$ and $\lambda > \frac{1}{2}$ then f is harmonic univalent, sense-preserving in $\mathbb{U}$ and $f \in HP(n,\lambda,\alpha)$.

Proof. If $z_1 \neq z_2$, then

$$
\begin{aligned}
& |\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}| \\
\geq\ & 1-|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}| \\
=\ & 1-|\frac{\sum_{k=1}^{\infty} b_k(z_1^k-z_2^k)}{(z_1-z_2)\sum_{k=2}^{\infty} a_k(z_1^k-z_2^k)}| \\
>\ & 1-\frac{\sum_{k=1}^{\infty} k|b_k|}{1-\sum_{k=2}^{\infty} k|a_k|} \\
\geq\ & 1-\frac{\sum_{k=1}^{\infty} \lambda k^{n+1}-(1-\lambda)k^n)|b_k|}{\sum_{k=2}^{\infty} \lambda k^{n+1}+(1-\lambda)k^n)|a_k|} \\
\geq\ & 0
\end{aligned}
$$

which proves univalence of f. Note that f is sense- preserving in $\mathbb{U}$. This is because

$$
\begin{aligned}
|h'(z)| &\geq 1-\sum_{k=2}^{\infty} k|a_k||z|^{k-1} \\
&> 1-\sum_{k=2}^{\infty}(\lambda k^{n+1}+(1-\lambda)k^n)|a_k| \\
&\geq \sum_{k=1}^{\infty}(\lambda k^{n+1}-(1-\lambda)k^n)|b_k| \\
&> \sum_{k=1}^{\infty}(\lambda k^{n+1}-(1-\lambda)k^n)|b_k||z|^{k-1} \\
&\geq \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \\
&\geq |g'(z)|.
\end{aligned}
$$

Using the fact that $Re\{w\} \geq \alpha$ iff $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$
\begin{aligned}
& |(1-\alpha)z+D^n f(z)+\lambda[D^{n+1}f(z)-D^n f(z)]| \\
& -|(1+\alpha)z-D^n f(z)-\lambda[D^{n+1}f(z)-D^n f(z)]| \\
= \; & |(2-\alpha)z+\sum_{k=2}^{\infty}(\lambda k^{n+1}+(1-\lambda)k^n)a_k z^k-(-1)^n\sum_{k=1}^{\infty}(\lambda k^{n+1}-(1-\lambda)k^n)\overline{b_k z^k}| \\
& -|\alpha z-\sum_{k=2}^{\infty}(\lambda k^{n+1}+(1-\lambda)k^n)a_k z^k+(-1)^n\sum_{k=1}^{\infty}(\lambda k^{n+1}-(1-\lambda)k^n)\overline{b_k z^k}| \\
\geq \; & 2(1-\alpha)|z|-\sum_{k=2}^{\infty}2(\lambda k^{n+1}+(1-\lambda)k^n)|a_k||z|^k-\sum_{k=1}^{\infty}2(\lambda k^{n+1}-(1-\lambda)k^n)|b_k||z|^k \\
= \; & 2(1-\alpha)|z|\{1-\sum_{k=2}^{\infty}(\lambda k^{n+1}+(1-\lambda)k^n)|a_k||z|^{k-1}-\sum_{k=1}^{\infty}(\lambda k^{n+1}-(1-\lambda)k^n)|b_k||z|^{k-1}\} \\
> \; & 2\{(1-\alpha)-(\sum_{k=2}^{\infty}(\lambda k^{n+1}+(1-\lambda)k^n)|a_k|+\sum_{k=1}^{\infty}(\lambda k^{n+1}-(1-\lambda)k^n)|b_k|)\} \\
> \; & 0.
\end{aligned}
$$

So the proof of theorem is complete.

The harmonic mappings

$$f(z)=z+\sum_{k=2}^{\infty}\frac{1-\alpha}{(\lambda k^{n+1}+(1-\lambda)k^n)}x_k z^k+\sum_{k=1}^{\infty}\frac{1-\alpha}{(\lambda k^{n+1}-(1-\lambda)k^n)}\overline{y_k z^k} \quad (6)$$

$0 \leq \alpha < 1, n \in \mathbb{N}_0, \lambda > \frac{1}{2}$ and $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, show that the coefficient bound given by (5) is sharp. The functions of the form (6) are in $HP(n,\lambda,\alpha)$ because

$$
\begin{aligned}
& \sum_{k=1}^{\infty}[(\lambda k^{n+1}+(1-\lambda)k^n)|a_k|+(\lambda k^{n+1}-(1-\lambda)k^n)|b_k|] \\
= \; & 1+(1-\alpha)(\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|) \\
= \; & 2-\alpha.
\end{aligned}
$$

In the following theorem, it is shown that the condition(5) is also necessary for functions $f_n = h + \bar{g_n}$ where h and g_n are of the form (4).

Theorem 2. *Let* $f_n = h + \bar{g_n}$ *be given by (4). Then* $f_n \in HP^*(n, \lambda, \alpha)$ *if and only if*

$$\sum_{k=1}^{\infty}((\lambda k^{n+1} + (1-\lambda)k^n)a_k + (\lambda k^{n+1} - (1-\lambda)k^n)b_k) \leq 2 - \alpha. \tag{7}$$

Proof. Since $HP^*(n, \lambda, \alpha) \subset HP(n, \lambda, \alpha)$, we need to prove the "only if" part of the theorem. To this end, for functions f_n of the form (4), we notice that the condition

$$\mathrm{Re}\{\frac{\mathrm{D^nf(z) + \lambda(D^{n+1}f(z) - D^nf(z))}}{\mathrm{z}}\} \geq \alpha$$

is equivalent to

$$\mathrm{Re}\{\frac{\mathrm{(1-\alpha)z - \sum_{k=2}^{\infty}(\lambda k^{n+1} + (1-\lambda)k^n)a_kz^k - (-1)^n \sum_{k=1}^{\infty}(\lambda k^{n+1} - (1-\lambda)k^n)\overline{b_kz^k}}}{\mathrm{z}}\} \geq 0. \tag{8}$$

The above required condition (8) must hold for all values of z in $\mathbb{U}$. Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, we must have

$$(1-\alpha) - \sum_{k=2}^{\infty}(\lambda k^{n+1} + (1-\lambda)k^n)a_kr^{k-1} - (-1)^n \sum_{k=1}^{\infty}(\lambda k^{n+1} - (1-\lambda)k^n)b_kr^{k-1} \geq 0. \tag{9}$$

If the condition (7) doesn't hold, then the numerator in (9) is negative for r sufficiently close 1. Hence there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (9) is negative. This contradicts the required condition for $f_n \in HP^*(n, \lambda, \alpha)$ and so the proof is complete.

Next, we determine the extreme points of closed convex hulls of $HP^*(n, \lambda, \alpha)$ denoted by $clcoHP^*(n, \lambda, \alpha)$.

Theorem 3. *Let $f_n = h + \bar{g_n}$ be given by (4). Then $f_n \in HP^*(n, \lambda, \alpha)$ if and only if*

$$f_n(z) = \sum_{k=1}^{\infty} X_k h_k(z) + Y_k g_{n_k}(z) \tag{10}$$

where

$$h_1(z) = z \quad , h_k(z) = z - \frac{1-\alpha}{\lambda k^{n+1} + (1-\lambda)k^n} z^k \quad (k = 2, 3, \cdots), n \in \mathbb{N}_0,$$

$$g_{n_k}(z) = z + (-1)^n \frac{1-\alpha}{\lambda k^{n+1} - (1-\lambda)k^n} \bar{z}^k \quad (k = 1, 2, \cdots), n \in \mathbb{N}_0,$$

and

$$\sum_{k=1}^{\infty} X_k + Y_k = 1 \qquad X_k \geq 0, Y_k \geq 0.$$

In particular, the extreme points of $HP^*(n, \lambda, \alpha)$ are $\{h_k\}$ and $\{g_{n_k}\}$.

Proof. For functions f_n of the form (10), we have

$$\begin{aligned} f_n(z) &= \sum_{k=1}^{\infty} X_k h_k(z) + Y_k g_{n_k}(z) \\ &= \sum_{k=1}^{\infty} (X_k + Y_k) z - \sum_{k=2}^{\infty} \frac{1-\alpha}{\lambda k^{n+1} + (1-\lambda)k^n} X_k z^k \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{1-\alpha}{\lambda k^{n+1} - (1-\lambda)k^n} Y_k \bar{z}_k. \end{aligned}$$

Then

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{\lambda k^{n+1} + (1-\lambda)k^n}{1-\alpha} a_k + \sum_{k=1}^{\infty} \frac{\lambda k^{n+1} - (1-\lambda)k^n}{1-\alpha} b_k \\ = &\sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k \\ = &\ 1 - X_1 \leq 1 \end{aligned}$$

and so $f_n \in clcoHP^*(n, \lambda, \alpha)$.

Conversely, suppose that $f_n \in clcoHP^*(n, \lambda, \alpha)$. Setting

$$X_k = \frac{\lambda k^{n+1} + (1-\lambda)k^n}{1-\alpha} a_k \quad (k = 2, 3, \cdots), n \in \mathbb{N}_0,$$

$$Y_k = \frac{\lambda k^{n+1} - (1-\lambda)k^n}{1-\alpha} b_k \quad (k = 1, 2, \cdots), n \in \mathbb{N}_0,$$

where $\sum_{k=1}^{\infty} X_k + Y_k = 1$, we obtain $f_n(z) = \sum_{k=1}^{\infty} X_k h_k(z) + Y_k g_{n_k}(z)$ as required.

For the following theorem, we must define the fractional integral of order μ for a function in $HP^*(n, \lambda, \alpha)$. Denote the fractional integral of order μ, for a function $f_n = h + \bar{g_n}$, by

$$D_z^{-\mu} f_n(z) = \frac{1}{\Gamma(\mu)} \{ \int_0^z \frac{h(\xi)}{(z-\xi)^{1-\mu}} d\xi + (-1)^n \int_0^z \frac{\overline{g_n}(\xi)}{(z-\xi)^{1-\mu}} d\xi \} \qquad ; \mu > 0$$

where f_n is analytic harmonic function by h and g_n are analytic in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-\xi)^{\mu-1}$ is removed by requiring log(z − ξ) to be real when $z - \xi > 0$.

The following theorem gives the distortion bounds for fractional calculus for function in $HP^*(n, \lambda, \alpha)$ which yields a covering result for this class.

Theorem 4. *Let* $f_n = h + \bar{g_n} \in HP^*(n, \lambda, \alpha)$ *where*

$$h(z) = z - \sum_{k=2}^{\infty} a_k z^k, \qquad g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \qquad a_k \geq 0 \quad , b_k \geq 0.$$

Then

$$\begin{aligned} & \frac{|z|^{1+\mu}}{\Gamma(\mu+2)} \{ (1 - |b_1|) - \frac{1 - (2\lambda - 1)|b_1|}{2^{n-1}(\mu+2)(\lambda+1)} |z| \} \\ \leq & |D_z^{-\mu} f_n(z)| \\ \leq & \frac{|z|^{1+\mu}}{\Gamma(\mu+2)} \{ (1 + |b_1|) + \frac{1 - (2\lambda - 1)|b_1|}{2^{n-1}(\mu+2)(\lambda+1)} |z| \} \end{aligned} \tag{11}$$

for $\mu > 0, \lambda > \frac{1}{2}$ *and* $z \in \mathbb{U}$.

Proof. We note that

$$\begin{aligned}
& \Gamma(\mu+2)z^{-\mu}D_z^{-\mu}f_n(z) \\
= & \Gamma(\mu+2)z^{-\mu}\{D_z^{-\mu}h(z)+\overline{(-1)^nD_z^{-\mu}g_n(z)}\} \\
= & z-\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(\mu+2)}{\Gamma(k+\mu+1)}a_kz^k+(-1)^n\sum_{k=1}^{\infty}\frac{\Gamma(k+1)\Gamma(\mu+2)}{\Gamma(k+\mu+1)}b_k\bar{z}^k \\
= & z+(-1)^nb_1\bar{z}-\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(\mu+2)}{\Gamma(k+\mu+1)}(a_kz^k+(-1)^nb_k\bar{z}^k) \\
= & z+(-1)^nb_1\bar{z}-\sum_{k=2}^{\infty}\varphi(k)(a_kz^k+(-1)^nb_k\bar{z}^k)
\end{aligned}$$

where

$$\varphi(k)=\frac{\Gamma(k+1)\Gamma(\mu+2)}{\Gamma(k+\mu+1)} \quad ;k\geq 2.$$

Noting that $\varphi(k)$ is decreasing function of k, we have

$$0<\varphi(k)\leq\varphi(2)=\frac{2}{\mu+2}.$$

Therefore, we obtain

$$\begin{aligned}
& |\Gamma(\mu+2)z^{-\mu}D_z^{-\mu}f_n(z)| \\
\leq & (1+|b_1|)|z|+\sum_{k=2}^{\infty}\varphi(k)(a_k+b_k)|z|^k \\
\leq & (1+|b_1|)|z|+\varphi(2)|z|^2\sum_{k=2}^{\infty}(a_k+b_k) \\
\leq & (1+|b_1|)|z|+\frac{\varphi(2)}{2^{n+1}\lambda+2^n(1-\lambda)}|z|^2\sum_{k=2}^{\infty}[2^{n+1}\lambda+2^n(1-\lambda)](a_k+b_k) \\
\leq & (1+|b_1|)|z|+\frac{1}{2^{n-1}(\lambda+1)(\mu+2)}|z|^2\sum_{k=2}^{\infty}[k^{n+1}\lambda+k^n(1-\lambda)]a_k \\
& +[k^{n+1}\lambda-k^n(1-\lambda)]b_k \\
\leq & (1+|b_1|)r+\frac{1}{2^{n-1}(\lambda+1)(\mu+2)}\{1-(2\lambda-1)|b_1|\}r^2
\end{aligned}$$

which is equivalent (11).

The following covering result follows from the left hand inequality in Theorem 4.

Corollary. *Let* $f_n = h + \bar{g_n}$ *of the form (4) be so that* $f_n = h + \bar{g_n} \in HP^*(n, \lambda, \alpha)$. *Then*

$$\{w : |w| < \frac{2^{n-1}(\lambda+1)(\mu+2)-1}{2^{n-1}(\lambda+1)(\mu+2)\Gamma(\mu+2)} + \frac{(2\lambda-1)-2^{n-1}(\lambda+1)(\mu+2)}{2^{n-1}(\lambda+1)(\mu+2)\Gamma(\mu+2)}|b_1|\}$$
$$\subset f_n(\mathbb{U}).$$

Remark. *Letting* $n = 0, \lambda = 1$ *and* $\mu \to 0$ *in Theorem 4 and Corollary , we obtain the results similar to that were given in [5,Theorem 2.3] and [5,Corollary 2.4], respectively.*

For our next theorem, we need to define the convolution of two harmonic functions. For harmonic functions of the form$f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \bar{z}^k$ and $F_n(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^n \sum_{k=1}^{\infty} B_k \bar{z}^k$

we define the convolution of f_n and F_n as

$$(f_n * F_n)(z) = f_n(z) * F_n(z) = z - \sum_{k=2}^{\infty} a_k A_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k B_k \bar{z}^k. \qquad (12)$$

Theorem 5. *For* $0 \le \beta \le \alpha < 1$, *let* $f_n \in HP^*(n, \lambda, \alpha)$ *and* $F_n \in HP^*(n, \lambda, \alpha)$. *Then* $f_n * F_n \in HP^*(n, \lambda, \alpha) \subset HP^*(n, \lambda, \beta)$.

Proof. For f_n and F_n as in Theorem 5,write$f_n(z) = z - \sum_{k=2}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} b_k \bar{z}^k$ and $F_n(z) = z - \sum_{k=2}^{\infty} A_k z^k + (-1)^n \sum_{k=1}^{\infty} B_k \bar{z}^k$.

Then the convolution $f_n * F_n$ is given by (12). We wish to show that the coefficients of $f_n * F_n$ satisfy the required condition given by Theorem 2. For

$F_n \in HP^*(n,\lambda,\beta)$, we note that $|A_k| \le 1$ and $|B_k| \le 1$. Now, for the convolution function $f_n * F_n$, we obtain

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{\lambda k^{n+1} + (1-\lambda)k^n}{2-\beta} a_k A_k + \sum_{k=1}^{\infty} \frac{\lambda k^{n+1} - (1-\lambda)k^n}{2-\beta} b_k B_k \\ \le\ &\sum_{k=2}^{\infty} \frac{\lambda k^{n+1} + (1-\lambda)k^n}{2-\beta} a_k + \sum_{k=1}^{\infty} \frac{\lambda k^{n+1} - (1-\lambda)k^n}{2-\beta} b_k \\ \le\ &\sum_{k=2}^{\infty} \frac{\lambda k^{n+1} + (1-\lambda)k^n}{2-\alpha} a_k + \sum_{k=1}^{\infty} \frac{\lambda k^{n+1} - (1-\lambda)k^n}{2-\alpha} b_k \le 1 \end{aligned}$$

since $0 \le \beta \le \alpha < 1$ and $f_n \in HP^*(n,\lambda,\alpha)$. Therefore $f_n * F_n \in HP^*(n,\lambda,\alpha) \subset HP^*(n,\lambda,\beta)$.

Now, we show that $HP^*(n,\lambda,\alpha)$ is closed under convex combination.

Theorem 6. *The family $HP^*(n,\lambda,\alpha)$ is closed under convex combination.*

Proof. For $i = 1, 2, \cdots$ suppose that $f_{n_i} HP^*(n,\lambda,\alpha)$ where

$f_{n_i}(z) = z - \sum_{k=2}^{\infty} a_{k_i} z^k + (-1)^n \sum_{k=1}^{\infty} b_{k_i} \bar{z}^k.$

Then by Theorem 2,

$$\sum_{k=2}^{\infty} \frac{\lambda k^{n+1} + (1-\lambda)k^n}{2-\alpha} a_{k_i} + \sum_{k=1}^{\infty} \frac{\lambda k^{n+1} - (1-\lambda)k^n}{2-\alpha} b_{k_i} \le 1. \tag{13}$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \le t_i \le 1$, the convex combination of f_{n_i} may be written as

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) = z - \sum_{k=2}^{\infty} (\sum_{i=1}^{\infty} t_i a_{k_i}) z^k + (-1)^n \sum_{k=1}^{\infty} (\sum_{i=1}^{\infty} t_i b_{k_i}) \bar{z}^k.$$

Then, by (13),

$$\begin{aligned}
&\sum_{k=2}^{\infty}\frac{\lambda k^{n+1}+(1-\lambda)k^n}{2-\alpha}(\sum_{i=1}^{\infty}t_i a_{k_i})+\sum_{k=1}^{\infty}\frac{\lambda k^{n+1}-(1-\lambda)k^n}{2-\alpha}(\sum_{i=1}^{\infty}t_i b_{k_i})\\
=\;&\sum_{i=1}^{\infty}t_i(\sum_{k=2}^{\infty}\frac{\lambda k^{n+1}+(1-\lambda)k^n}{2-\alpha}a_{k_i}+\sum_{k=1}^{\infty}\frac{\lambda k^{n+1}-(1-\lambda)k^n}{2-\alpha}b_{k_i})\\
\leq\;&\sum_{i=1}^{\infty}t_i=1
\end{aligned}$$

and therefore $\sum_{i=1}^{\infty} t_i f_{n_i} \in HP^*(n,\lambda,\alpha)$.

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