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Strong Convergence Theorems of an Implicit Iteration Process for Generalized Hemi-contractive Mappings*

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Abstract

In this paper, we introduce the concept of generalized pseudo-contractive and generalized hemi-contractive mappings, and study the strong convergence theorems for approximation of common fixed points of generalized hemi-contractive mappings in p-uniformly convex Banach spaces by using an implicit iteration process recently introduced by Xu and Ori [17].

Keywords and Phrases: *Implicit iteration process; generalized pseudo-contractive mapping; generalized hemi-contractive mapping; p-uniformly convex Banach space.*

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1. Introduction and Preliminaries

Let $F(T)$ and I denote the set of all fixed points of the mapping T and the identity mapping, respectively.

Definition 1.1. Let C be a nonempty subset of a normed linear space E and let $T : C \to C$ be a mapping. Suppose that $p > 0$, $\theta \in [0, \infty)$ are two constants.

(i) T is called *generalized pseudo-contractive with constants* p *and* θ if

$$\|Tx - Ty\|^p \leq \|x - y\|^p + \theta\|(I - T)x - (I - T)y\|^p, \tag{1.1}$$

for all $x, y \in C$.

(ii) T is called *generalized hemi-contractive with constants* p *and* θ if $F(T) \neq \varnothing$ and

$$\|Tx - u\|^p \leq \|x - u\|^p + \theta\|x - Tx\|^p, \tag{1.2}$$

for all $x \in C$ and $u \in F(T)$.

(iii) T is called *Lipschitzian* if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

for all $x, y \in C$. If $L = 1$, then T is called *nonexpansive.*

(iv) T is called *semi-compact* if C is closed and for any bounded sequence $\{x_n\}_{n=1}^{\infty}$ in C with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\lim_{j\to\infty} x_{n_j} = u \in C$.

Remark 1.1. *(1) If* $p = 1$ *and* $\theta = 0$*, the mappings in (1.1) and (1.2) are said to be nonexpansive and quasi-nonexpansive, respectively.*
(2) If E *is a Hilbert space and if* $p = 2$ *and* $\theta \in [0, 1)$*, the mappings in (1.1) and (1.2) are said to be strictly pseudo-contractive in the terminology of Browder and Petryshyn [11] and demi-contractive [8], respectively.*

(3) If E is a Hilbert space and if $p = 2$ and $\theta = 1$, the mappings in (1.1) and (1.2) are said to be pseudo-contractive [1, 2] and hemi-contractive [12, 13], respectively.

It is obvious that the above classes of mappings with fixed points in the setting of Hilbert spaces, we have the following implications:

nonexpansive	$\Longrightarrow$	quasi-nonexpansive
$\Downarrow$		$\Downarrow$
strictly pseudo-contractive	$\Longrightarrow$	demi-contractive
$\Downarrow$		$\Downarrow$
pseudo-contractive	$\Longrightarrow$	hemi-contractive
$\Downarrow$		$\Downarrow$
generalized pseudo-contractive	$\Longrightarrow$	generalized hemi-contractive

Definition 1.2. Let C be a nonempty closed convex subset of a real Banach space E and let $\{\alpha_n\}_{n=1}^{\infty}$ be a sequence in [0, 1]. Then the sequence $\{x_n\} \subset C$ defined by

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N, \\ x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})T_1 x_{N+1}, \\ &\vdots \end{aligned}$$

The scheme is expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \quad n \geq 1, \tag{1.3}$$

where $T_n = T_{n(\operatorname{mod} N)}$, is called *the implicit iteration process for a finite family of nonexpansive mappings* $\{T_i\}_{i=1}^{N}$.

In 2001, Xu and Ori [17] introduced the above implicit iteration process for a finite family of nonexpansive mappings and proved the weak convergence of this process to a common fixed point of the finite family defined in a Hilbert space. They further remarked that *it is yet unclear what assumptions on the mappings and/or the parameters* $\{\alpha_n\}$ *are sufficient to guarantee the strong convergence of the sequence* $\{x_n\}$. Since then, the convergence problems of an implicit iteration process to a common fixed point for a finite family of nonexpansive, strictly pseudo-contractive, and pseudo-contractive mappings on Banach spaces have been studied by several authors (see, for example, Osilike [9, 10], Chidume and Shahzad [5], Su and Li [15], Chen, Song and Zhou [6], Chen, Lin and Song [7] and Zeng and Yao [19] and the references therein). More recently, Rhoades and Soltuz [14] noted that the existence of $(I - tT_i)^{-1}$ for all $t \in (0, 1)$ and $i = 1, 2, \ldots, N$ should be assumed in order to have the iteration (1.3) well-defined.

Let C be a nonempty convex subset of a real Banach space E with dual E^*. Observe that if $T : C \to C$ is a L-Lipschitzian map, then for every fixed $u \in C$ and $t \in (L/(1+L), 1)$, the mapping $S_t : C \to C$ defined by $S_t x = tu+(1-t)Tx$ satisfies

$$\begin{aligned} & < S_t x - S_t y, \quad j(x-y) > \\ = \; & (1-t) < Tx - Ty, \quad j(x-y) > \\ \leq \; & (1-t)L\|x-y\|^2 \end{aligned}$$

for all $x, y \in C$ and some $j(x-y) \in J(x-y)$ where $J : E \to 2^{E^*}$ denotes the normalized duality map on E (see, e.g. [3]). Since $(1-t)L \in (0, 1)$, it follows that S_t is a strongly pseudo-contractive and hence has a unique fixed point x_t in C (see [4]) such that

$$x_t = tu + (1-t)Tx_t.$$

Thus the implicit iteration process (1.3) is well defined in C for the family $\{T_i\}_{i=1}^N$ of N L_i-Lipschitzian self-maps of a nonempty convex subset C of a Banach space E provided that $(I - tT_i)^{-1}$ exists for all $t \in (0,1)$, $i = 1, 2, \ldots, N$, and $\alpha_n \in (\alpha, 1)$ for all $n \geq 1$, where $\alpha := L/(1+L)$ and $L := \max_{1 \leq i \leq N}\{L_i\}$.

In this paper, we introduce the concept of generalized pseudo-contractive and generalized hemi-contractive mappings and then study the strong convergence theorems of implicit iteration process for a finite family of Lipschitzian and generalized hemi-contractive mappings in p-uniformly convex Banach spaces with $p > 1$. Thus we provide a positive answer to Xu and Ori's guestion for the general class of mappings which contains nonexpansive, quasi-nonexpensive, strictly pseudo-contractive, demi-contractive, pseudo-contractive, and hemi-con-
tractive mappings. In particular, our theorems hold in L_p, l_p, $W^{1,p}$ spaces, for $1 < p < \infty$.

In what follows, we shall make use of the following lemma.

Lemma 1.1.(see [18, Theorem 1]) *Let $p > 1$ be a given real number. Let E be a p-uniformly convex Banach space. Then, there exists a constant $d > 0$ such that*

$$\|\lambda x + (1-\lambda)y\|^p \leq \lambda\|x\|^p + (1-\lambda)\|y\|^p - dW_p(\lambda)\|x-y\|^p, \tag{1.4}$$

for all $\lambda \in [0,1]$ and $x, y \in E$, where $W_p(\lambda) := \lambda^p(1-\lambda) + \lambda(1-\lambda)^p$.

The following proposition was obtained by Chidume and Shahzad (see [5, p.1151] for the below contractive definitions).

Proposition 1.1. (see [5, Proposition 3.4]) *Let E be a uniformly convex Banach space and C be a nonempty closed bounded convex subset of E. Suppose $T : C \to C$. Then T is semi-compact if T satisfies any of the following conditions:*

(1) *T is either set-condensing or ball-condensing (or compact);*

(2) T *is a generalized contraction;*

(3) T *is uniformly strictly contractive;*

(4) T *is strictly semi-contractive;*

(5) T *is of strictly semi-contractive type;*

(6) T *is of strongly semi-contractive type.*

2. Main Results

Now, we state and prove the following theorems.

Theorem 2.1. *Let C be a nonempty closed convex subset of a real p-uniformly convex Banach space E with $p > 1$. Let $\{T_i : C \to C\}_{i=1}^N$ be N L_i-Lipschitzian and generalized hemi-contractive mapping with constants p and $\theta_i \in [0, \infty)$ such that $F := \cap_{i=1}^N F(T_i) \neq \varnothing$ and $(I - tT_i)^{-1}$ exists $\forall t \in (0,1)$ and $i = 1, 2, \ldots, N$. Let $L := \max\limits_{1\le i\le N}\{L_i\}$ and let $\{\alpha_n\}_{n=1}^\infty$ be a sequence in $(0,1)$ such that*

$$0 < \alpha \le \alpha_n \le \beta < \left[\left(\frac{\theta}{d} - 1\right)^{\frac{1}{p-1}} + 1\right]^{-1} \le 1, \tag{2.1}$$

for all $n \ge 1$, where $\alpha := L/(1+L)$, $\theta := \max\limits_{1\le i\le N}\{\theta_i, d\} > 0$, and d denotes the constant appearing in inequality (1.4). Then, the implicit iteration sequence $\{x_n\}_{n=1}^\infty$ generated by (1.3) exists in C and

(i) $\lim\limits_{n\to\infty} \|x_n - u\|$ *exists for all $u \in F$,*

(ii) $\lim\limits_{n\to\infty} d(x_n, F)$ *exists where $d(x_n, F) = \inf\limits_{u\in F} \|x_n - u\|$,*

(iii) $\lim\limits_{n\to\infty} \|x_n - T_n x_n\| = 0.$

Proof. Let $u \in F$. By the definition of $\{x_n\}$ and Lemma 1.1, we have

$$\begin{aligned}\|x_n - u\|^p &= \|\alpha_n(x_{n-1} - u) + (1-\alpha_n)(T_n x_n - u)\|^p \\ &\le \alpha_n\|x_{n-1} - u\|^p + (1-\alpha_n)\|T_n x_n - u\|^p \\ &\quad -dW_p(\alpha_n)\|x_{n-1} - T_n x_n\|^p. \qquad (2.2)\end{aligned}$$

Since each T_i is generalized hemi-contractive mapping with constant θ_i, it follows that for $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}\|T_i x_n - u\|^p &\le \|x_n - u\|^p + \theta_i\|x_n - T_i x_n\|^p \\ &\le \|x_n - u\|^p + \theta\|x_n - T_i x_n\|^p.\end{aligned}$$

Thus we obtain from (2.2) and $x_n - T_n x_n = \alpha_n(x_{n-1} - T_n x_n)$ that

$$\begin{aligned}\|x_n - u\|^p &\le \alpha_n\|x_{n-1} - u\|^p + (1-\alpha_n)\Big[\|x_n - u\|^p + \theta\|x_n - T_n x_n\|^p\Big] \\ &\quad -dW_p(\alpha_n)\|x_{n-1} - T_n x_n\|^p \\ &\le \|x_{n-1} - u\|^p - \frac{1}{\alpha_n}\Big[dW_p(\alpha_n) - (1-\alpha_n)\alpha_n^p\theta\Big]\|x_{n-1} - T_n x_n\|^p \\ &= \|x_{n-1} - u\|^P - (1-\alpha_n)\alpha_n^{p-1}\left[d\Big(\frac{1}{\alpha_n} - 1\Big)^{p-1} + d - \theta\right]\|x_{n-1} - T_n x_n\|^p, \qquad (2.3)\end{aligned}$$

for all $n \ge 1$. Using inequality (2.1), we obtain that

$$\begin{aligned}& d\Big(\frac{1}{\alpha_n} - 1\Big)^{p-1} + d - \theta \\ \ge\ & d\Big(\frac{1}{\beta} - 1\Big)^{p-1} + d - \theta \\ >\ & d\Big(\frac{\theta}{d} - 1\Big) + d - \theta \\ =\ & 0.\end{aligned}$$

Put $t := d(\frac{1}{\beta} - 1)^{p-1} + d - \theta > 0$. Hence, (2.3) can be written as

$$\|x_n - u\|^p \le \|x_{n-1} - u\|^p - t(1-\beta)\alpha^{p-1}\|x_{n-1} - T_n x_n\|^p, \qquad (2.4)$$

for all $n \geq 1$. It now follows from (2.4) that

$$\|x_n - u\| \leq \|x_{n-1} - u\| \qquad (\forall\, n \geq 1). \tag{2.5}$$

Taking infimum over all $u \in F$, we have

$$d(x_n, F) \leq d(x_{n-1}, F),$$

hence $\lim_{n\to\infty} \|x_n - u\|$ exists and $\lim_{n\to\infty} d(x_n, F)$ exists. Furthermore, (2.4) implies that

$$t(1-\beta)\alpha^{p-1}\|x_{n-1} - T_n x_n\|^p \leq \|x_{n-1} - u\|^p - \|x_n - u\|^p \qquad (\forall\, n \geq 1).$$

Thus $\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0$, so that

$$\|x_n - T_n x_n\| = \alpha_n \|x_{n-1} - T_n x_n\| \to 0 \text{ as } n \to \infty.$$

The proof is complete.

Theorem 2.2. *Let $C, E, \{T_i\}_{i=1}^N, F, \{\alpha_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be as in Theorem 2.1. Then $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of the family of $\{T_i\}_{i=1}^N$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.*

Proof. Since $0 \leq d(x_n, F) \leq \|x_n - u\| \quad (\forall\, u \in F)$. Thus, the necessity is obvious and so we show the sufficiency. Suppose $\liminf_{n\to\infty} d(x_n, F) = 0$. From Theorem 2.1, we know that $\lim_{n\to\infty} d(x_n, F) = 0$. Moreover, from (2.5), we have

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - u\| + \|x_n - u\| \leq 2\|x_n - u\|, \tag{2.6}$$

for all $u \in F$ and $m, n \geq 1$. Taking infimum over all $u \in F$, from (2.6) we obtain

$$\|x_{n+m} - x_n\| \leq 2d(x_n, F) \to 0 \text{ as } n \to \infty,$$

so that $\{x_n\}$ is Cauchy sequence and hence $\lim_{n\to\infty} x_n = x^*$ for some $x^* \in C$. Since each T_i is continuous mapping, it is easy to see that $F(T_i)$ is closed, so that F is closed. Therefore, $d(x^*, F) = \lim_{n\to\infty} d(x_n, F) = 0$ implies that $x^* \in F$. The proof is complete.

Theorem 2.3. *Let $C, E, \{T_i\}_{i=1}^N, F, \{\alpha_n\}_{n=1}^\infty$ and $\{x_n\}_{n=1}^\infty$ be as in Theorem 2.1. Suppose that there exists one map $T \in \{T_i\}_{i=1}^N$ to be semi-compact. Then $\{x_n\}_{n=1}^\infty$ converges strongly to a common fixed point of the family of $\{T_i\}_{i=1}^N$.*

Proof. It follows from Theorem 2.1 that $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$ and hence $\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0$. Then, we have

$$\|x_n - x_{n-1}\| = (1 - \alpha_n)\|x_{n-1} - T_n x_n\| \to 0 \text{ as } n \to \infty.$$

Thus $\lim_{n\to\infty} \|x_n - x_{n+i}\| = 0$ for each $i = 1, 2, \ldots, N$. Since every T_i is L_i-Lipschitzian and $L := \max_{1\le i\le N}\{L_i\}$, then

$$\|T_i x - T_i y\| \le L\|x - y\|$$

for all $x, y \in C$ and each $i = 1, 2, \ldots, N$.

Therefore,

$$\begin{aligned} \|x_n - T_{n+i}x_n\| &\le \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\le (1 + L)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \to 0 \text{ as } n \to \infty, \end{aligned}$$

i.e.,

$$\lim_{n\to\infty} \|x_n - T_{n+i}x_n\| = 0 \tag{2.7}$$

for each $i = 1, 2, \ldots, N$. It follows from (2.7) (see also [16], [15]) that

$$\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \tag{2.8}$$

for each $i = 1, 2, \ldots, N$.

It follows from Theorem 2.1 that $\lim_{n\to\infty} \|x_n - u\|$ exists for all $u \in F$. Thus $\{x_n\}$ is bounded. By hypothesis that there exists one map $T \in \{T_i\}_{i=1}^N$ to be semi-compact, we may assume that T_1 is semi-compact without loss of generality. Thus, from (2.8), we have $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$. Hence, by the definition of semi-compact, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ such that $\{x_{n_j}\}_{j=1}^{\infty}$ converges strongly to some $x^* \in C$. Therefore, for each $i = 1, 2, \ldots, N$, we have

$$\|x^* - T_i x^*\| = \lim_{j\to\infty} \|x_{n_j} - T_i x_{n_j}\| = 0,$$

i.e., $x^* \in F$. It follows that $\liminf_{n\to\infty} d(x_n, F) = 0$, so that by Theorem 2.2 we conclude that $\lim_{n\to\infty} x_n = x^* \in F$. The proof is complete.

Remark 2.1. *Theorem 2.3 provide a positive answer to the question raised in Xu and Ori [17]. Moreover, it is possible to replace the semi-conpactness assumption in Theorem 2.3 by any one of the contractive assumptions (1)-(6) of Porposition 1.1.*

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