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Multilinear Trif Mappings in Banach Modules over a C^*-Algebra*

Chun-Gil Park†

Department of Mathematics, Chungnam National University,
DaeJeon 305-764, South Korea

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Abstract

We prove the Hyers–Ulam–Rassias stability of multilinear Trif functional equations in Banach modules over a unital C^*-algebra.

Keywords and Phrases: *Banach module over C^*-algebra, Stability, Unitary group, Multilinear Trif functional equation.*

1. Introduction

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in R$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th. M. Rassias [6] showed that there exists a unique R-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p}||x||^p$$

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†E-mail: cgparkmath.cnu.ac.kr

for all $x \in X$.

Recently, T. Trif [11, Theorem 2.1] proved that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation

$$n\ {}_{n-2}C_{k-2} f(\frac{x_1 + \cdots + x_n}{n}) + {}_{n-2}C_{k-1} \sum_{l=1}^{n} f(x_l) = k \sum_{1 \le l_1 < \cdots < l_k \le n} f(\frac{x_{l_1} + \cdots + x_{l_k}}{k}) \qquad (A)$$

for all $x_1, \cdots, x_n \in X$ if and only if the mapping $f : X \to Y$ satisfies the additive Cauchy equation $f(x + y) = f(x) + f(y)$ for all $x, y \in X$. And he proved the stability of the functional equation (A). Several authors have investigated functional equations (see [2]–[4], [7]–[10]).

Throughout this paper, let A be a unital C^*-algebra with norm $|\cdot|$, $U(A)$ the unitary group of A, $A_1 = \{a \in A \mid |a| = 1\}$, and A_1^+ the set of positive elements in A_1. Let ${}_AB_l$ be left A-modules for $l = 1, \cdots, d$. Let ${}_AD$ be a left Banach A-module with norm $\|\cdot\|$. Let n_l and k_l be integers such that $2 \le k_l \le n_l - 1$ for all $l = 1, \cdots, d$.

The main purpose of this paper is to prove the Hyers–Ulam–Rassias stability of multilinear Trif functional equations in Banach modules over a unital C^*-algebra.

2. Stability of multilinear Trif functional equations in Banach modules over a C^*-algebra

For a given mapping $f : \prod_{l=1}^{d} {}_AB_l \to {}_AD$ and given $a_1, \cdots, a_d \in A$, we set

$$\begin{aligned} D_{a_1,\cdots,a_d} f(x_{11}, \cdots, x_{1n_1}, \cdots, x_{d1}, \cdots, x_{dn_d}) :&= \sum_{l=1}^{d} n_l\ {}_{n_l-2}C_{k_l-2} f(x_{11}, \cdots, x_{l-1\ 1}, \frac{a_l x_{l1} + \cdots + a_l x_{ln_l}}{n_l}, x_{l+1\ 1}, \cdots, x_{d1}) \\ &+ \sum_{l=1}^{d} {}_{n_l-2}C_{k_l-1} \sum_{j=1}^{n_l} f(x_{11}, \cdots, x_{l-1\ 1}, a_l x_{lj}, x_{l+1\ 1}, \cdots, x_{d1}) \\ &- \sum_{l=1}^{d} k_l \sum_{1 \le j_1 < \cdots < j_{k_l} \le n_l} a_l f(x_{11}, \cdots, x_{l-1\ 1}, \frac{x_{lj_1} + \cdots + x_{lj_{k_l}}}{k_l}, x_{l+1\ 1}, \cdots, x_{d1}) \end{aligned}$$

for all $x_{l1}, \cdots, x_{ln_l} \in {}_AB_l$, $l = 1, \cdots, d$.

Theorem 1. *Let* $q_l = \frac{k_l(n_l-1)}{n_l-k_l}$ *and* $r_l = -\frac{k_l}{n_l-k_l}$ *for* $l = 1, \cdots, d$. *Let* $f : \prod_{l=1}^d {}_AB_l \to {}_AD$ *be a mapping for which there exists a function* $\varphi : \prod_{l=1}^d {}_AB_l^{n_l} \to [0, \infty)$ *such that*

$$\begin{aligned}\widetilde{\varphi}(x_1, \cdots, x_d) := \sum_{j=0}^{\infty}\sum_{l=1}^{d}\frac{q_1^{-1}\cdots q_{l-1}^{-1}}{k_l\ {}_{n_l-1}C_{k_l-1}}q_1^{-j}\cdots q_d^{-j}\varphi(&\underbrace{q_1^{j+1}x_1, \cdots, q_1^{j+1}x_1}_{n_1 \text{ times}}, \cdots, \\ &\underbrace{q_{l-1}^{j+1}x_{l-1}, \cdots, q_{l-1}^{j+1}x_{l-1}}_{n_{l-1} \text{ times}}, q_l^{j+1}x_l, \underbrace{r_lq_l^jx_l, \cdots, r_lq_l^jx_l}_{n_l - 1 \text{ times}}, \\ &\underbrace{q_{l+1}^jx_{l+1}, \cdots, q_{l+1}^jx_{l+1}}_{n_{l+1} \text{ times}}, \cdots, \underbrace{q_d^jx_d, \cdots, q_d^jx_d}_{n_d \text{ times}}) < \infty\end{aligned} \quad (i)$$

$$\begin{aligned}&\|D_{u_1,\cdots,u_d}f(x_{11}, \cdots, x_{1n_1}, \cdots, x_{d1}, \cdots, x_{dn_d})\| \\ &\qquad \le \varphi(x_{11}, \cdots, x_{1n_1}, \cdots, x_{d1}, \cdots, x_{dn_d})\end{aligned} \quad (ii)$$

for all $u_1, \cdots, u_d \in U(A)$, *all* $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$, *and all* $x_{l1}, \cdots, x_{ln_l} \in {}_AB_l$, $l = 1, \cdots, d$. *Assume that* $f(x_1, \cdots, x_d) = 0$ *if* $x_l = 0$ *for any* $l = 1, \cdots, d$. *Then there exists a unique* A*-multilinear mapping* $M : \prod_{l=1}^d {}_AB_l \to {}_AD$ *such that*

$$\|f(x_1, \cdots, x_d) - M(x_1, \cdots, x_d)\| \le \widetilde{\varphi}(x_1, \cdots, x_d) \quad (iii)$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$.

Proof. Put $u_1 = \cdots = u_d = 1 \in U(A)$. For each fixed l, let $x_{11} = \cdots = x_{1n_1} = x_1, \cdots, x_{l-1\ 1} = \cdots = x_{l-1\ n_{l-1}} = x_{l-1}, x_{l+1\ 1} = \cdots = x_{l+1\ n_{l+1}} = x_{l+1}, \cdots, x_{d1} = \cdots = x_{dn_d} = x_d$ and $x_{l1} = q_lx_l$, $x_{l2} = \cdots = x_{ln_l} = r_lx_l$ in (ii). Then we get

$$\begin{aligned}&\|{}_{n_l-2}C_{k_l-1}f(x_1, \cdots, x_{l-1}, q_lx_l, x_{l+1}, \cdots, x_d) \\ &\qquad - k_l\ {}_{n_l-1}C_{k_l-1}f(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d)\| \\ &\qquad \le \varphi(\underbrace{x_1, \cdots, x_1}_{n_1 \text{ times}}, \cdots, \underbrace{x_{l-1}, \cdots, x_{l-1}}_{n_{l-1} \text{ times}}, q_lx_l, \underbrace{r_lx_l, \cdots, r_lx_l}_{n_l - 1 \text{ times}}, \\ &\qquad\qquad \underbrace{x_{l+1}, \cdots, x_{l+1}}_{n_{l+1} \text{ times}}, \cdots, \underbrace{x_d, \cdots, x_d}_{n_d \text{ times}}),\end{aligned}$$

hence

$$\begin{aligned}\|f(x_1,\cdots,&x_{l-1},x_l,x_{l+1},\cdots,x_d)-q_l^{-1}f(x_1,\cdots,x_{l-1},q_lx_l,x_{l+1},\cdots,x_d)\|\\ &\le\frac{1}{k_l\ {}_{n_l-1}C_{k_l-1}}\varphi(\underbrace{x_1,\cdots,x_1}_{n_1\text{ times}},\cdots,\underbrace{x_{l-1},\cdots,x_{l-1}}_{n_{l-1}\text{ times}},q_lx_l,\underbrace{r_lx_l,\cdots,r_lx_l}_{n_l-1\text{ times}},\\ &\qquad\underbrace{x_{l+1},\cdots,x_{l+1}}_{n_{l+1}\text{ times}},\cdots,\underbrace{x_d,\cdots,x_d}_{n_d\text{ times}})\end{aligned}$$

for all $(x_1,\cdots,x_d)\in\prod_{l=1}^d {}_AB_l$. So one can obtain that

$$\begin{aligned}\|q_1^{-1}\cdots q_{l-1}^{-1}&f(q_1x_1,\cdots q_{l-1}x_{l-1},x_l,\cdots,x_d)\\ &-q_1^{-1}\cdots q_{l-1}^{-1}q_l^{-1}f(q_1x_1,\cdots,q_lx_l,x_{l+1},\cdots,x_d)\|\\ &\le\frac{q_1^{-1}\cdots q_{l-1}^{-1}}{k_l\ {}_{n_l-1}C_{k_l-1}}\varphi(\underbrace{q_1x_1,\cdots,q_1x_1}_{n_1\text{ times}},\cdots,\underbrace{q_{l-1}x_{l-1},\cdots,q_{l-1}x_{l-1}}_{n_{l-1}\text{ times}},\\ &\qquad q_lx_l,\underbrace{r_lx_l,\cdots,r_lx_l}_{n_l-1\text{ times}},\underbrace{x_{l+1},\cdots,x_{l+1}}_{n_{l+1}\text{ times}},\cdots,\underbrace{x_d,\cdots,x_d}_{n_d\text{ times}})\end{aligned}$$

for all $(x_1,\cdots,x_d)\in\prod_{l=1}^d {}_AB_l$. Thus

$$\begin{aligned}\|f(x_1,\cdots,x_d)&-q_1^{-1}\cdots q_d^{-1}f(q_1x_1,\cdots,q_dx_d)\|\\ &\le\sum_{l=1}^d\frac{q_1^{-1}\cdots q_{l-1}^{-1}}{k_l\ {}_{n_l-1}C_{k_l-1}}\varphi(\underbrace{q_1x_1,\cdots,q_1x_1}_{n_1\text{ times}},\cdots,\underbrace{q_{l-1}x_{l-1},\cdots,q_{l-1}x_{l-1}}_{n_{l-1}\text{ times}},\\ &\qquad q_lx_l,\underbrace{r_lx_l,\cdots,r_lx_l}_{n_l-1\text{ times}},\underbrace{x_{l+1},\cdots,x_{l+1}}_{n_{l+1}\text{ times}},\cdots,\underbrace{x_d,\cdots,x_d}_{n_d\text{ times}})\end{aligned}$$

for all $(x_1,\cdots,x_d)\in\prod_{l=1}^d {}_AB_l$. Hence we get

$$\begin{aligned}\|q_1^{-j}\cdots q_d^{-j}&f(q_1^jx_1,\cdots,q_d^jx_d)-q_1^{-j-1}\cdots q_d^{-j-1}f(q_1^{j+1}x_1,\cdots,q_d^{j+1}x_d)\|\\ &\le\sum_{l=1}^d\frac{q_1^{-1}\cdots q_{l-1}^{-1}}{k_l\ {}_{n_l-1}C_{k_l-1}}q_1^{-j}\cdots q_d^{-j}\varphi(\underbrace{q_1^{j+1}x_1,\cdots,q_1^{j+1}x_1}_{n_1\text{ times}},\cdots,\\ &\qquad\underbrace{q_{l-1}^{j+1}x_{l-1},\cdots,q_{l-1}^{j+1}x_{l-1}}_{n_{l-1}\text{ times}},q_l^{j+1}x_l,\underbrace{r_lq_l^jx_l,\cdots,r_lq_l^jx_l}_{n_l-1\text{ times}},\\ &\qquad\underbrace{q_{l+1}^jx_{l+1},\cdots,q_{l+1}^jx_{l+1}}_{n_{l+1}\text{ times}},\cdots,\underbrace{q_d^jx_d,\cdots,q_d^jx_d}_{n_d\text{ times}})\end{aligned}$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$. So

$$\begin{aligned}\|f(x_1, \cdots, x_d) - q_1^{-n} \cdots q_d^{-n} f(q_1^n x_1, \cdots, q_d^n x_d)\| \\ \le \sum_{j=0}^{n-1} \sum_{l=1}^{d} \frac{q_1^{-1} \cdots q_{l-1}^{-1}}{k_l \ {}_{n_l-1}C_{k_l-1}} q_1^{-j} \cdots q_d^{-j} \varphi(\underbrace{q_1^{j+1}x_1, \cdots, q_1^{j+1}x_1}_{n_1 \text{ times}}, \cdots, \\ \underbrace{q_{l-1}^{j+1}x_{l-1}, \cdots, q_{l-1}^{j+1}x_{l-1}}_{n_{l-1} \text{ times}}, q_l^{j+1}x_l, \underbrace{r_l q_l^j x_l, \cdots, r_l q_l^j x_l}_{n_l - 1 \text{ times}}, \\ \underbrace{q_{l+1}^j x_{l+1}, \cdots, q_{l+1}^j x_{l+1}}_{n_{l+1} \text{ times}}, \cdots, \underbrace{q_d^j x_d, \cdots, q_d^j x_d}_{n_d \text{ times}})\end{aligned} \tag{1}$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$.

For each $l = 1, \cdots, d$, let x_l be an element in ${}_AB_l$. For positive integers n and m with $n > m$,

$$\begin{aligned}\|q_1^{-m} \cdots q_d^{-m} f(q_1^m x_1, \cdots, q_d^m x_d) - q_1^{-n} \cdots q_d^{-n} f(q_1^n x_1, \cdots, q_d^n x_d)\| \\ \le \sum_{j=m}^{n-1} \sum_{l=1}^{d} \frac{q_1^{-1} \cdots q_{l-1}^{-1}}{k_l \ {}_{n_l-1}C_{k_l-1}} q_1^{-j} \cdots q_d^{-j} \varphi(\underbrace{q_1^{j+1}x_1, \cdots, q_1^{j+1}x_1}_{n_1 \text{ times}}, \cdots, \\ \underbrace{q_{l-1}^{j+1}x_{l-1}, \cdots, q_{l-1}^{j+1}x_{l-1}}_{n_{l-1} \text{ times}}, q_l^{j+1}x_l, \underbrace{r_l q_l^j x_l, \cdots, r_l q_l^j x_l}_{n_l - 1 \text{ times}}, \\ \underbrace{q_{l+1}^j x_{l+1}, \cdots, q_{l+1}^j x_{l+1}}_{n_{l+1} \text{ times}}, \cdots, \underbrace{q_d^j x_d, \cdots, q_d^j x_d}_{n_{l+1} \text{ times}}),\end{aligned}$$

which tends to zero as $m \to \infty$ by (i). So $\{q_1^{-n} \cdots q_d^{-n} f(q_1^n x_1, \cdots, q_d^n x_d)\}$ is a Cauchy sequence for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$. Since ${}_AD$ is complete, the sequence $\{q_1^{-n} \cdots q_d^{-n} f(q_1^n x_1, \cdots, q_d^n x_d)\}$ converges for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$. We can define a mapping $M : \prod_{l=1}^d {}_AB_l \to {}_AD$ by

$$M(x_1, \cdots, x_d) = \lim_{j \to \infty} q_1^{-j} \cdots q_d^{-j} f(q_1^j x_1, \cdots, q_d^j x_d) \tag{2}$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$.

By (i) and (2), we get

$$\begin{aligned}
&\|D_{1,\cdots,1}M(\underbrace{x_1,\cdots,x_1}_{n_1 \text{ times}},\cdots,\underbrace{x_{l-1},\cdots,x_{l-1}}_{n_{l-1} \text{ times}},x_{l1},\cdots,x_{ln_l},\\
&\qquad\underbrace{x_{l+1},\cdots,x_{l+1}}_{n_{l+1} \text{ times}},\cdots,\underbrace{x_d,\cdots,x_d}_{n_d \text{ times}})\|\\
&= \lim_{j\to\infty} q_1^{-j}\cdots q_d^{-j}\|D_{1,\cdots,1}f(\underbrace{q_1^jx_1,\cdots,q_1^jx_1}_{n_1 \text{ times}},\cdots,\underbrace{q_{l-1}^jx_{l-1},\cdots,q_{l-1}^jx_{l-1}}_{n_{l-1} \text{ times}},\\
&\qquad q_l^jx_{l1},\cdots,q_l^jx_{ln_l},\underbrace{q_{l+1}^jx_{l+1},\cdots,q_{l+1}^jx_{l+1}}_{n_{l+1} \text{ times}},\cdots,\underbrace{q_d^jx_d,\cdots,q_d^jx_d}_{n_d \text{ times}})\|\\
&\le \lim_{j\to\infty} q_1^{-j}\cdots q_d^{-j}\varphi(\underbrace{q_1^jx_1,\cdots,q_1^jx_1}_{n_1 \text{ times}},\cdots,\underbrace{q_{l-1}^jx_{l-1},\cdots,q_{l-1}^jx_{l-1}}_{n_{l-1} \text{ times}},\\
&\qquad q_l^jx_{l1},\cdots,q_l^jx_{ln_l},\underbrace{q_{l+1}^jx_{l+1},\cdots,q_{l+1}^jx_{l+1}}_{n_{l+1} \text{ times}},\cdots,\underbrace{q_d^jx_d,\cdots,q_d^jx_d}_{n_d \text{ times}}) = 0,
\end{aligned}$$

hence

$$\begin{aligned}
&D_{1,\cdots,1}M(\underbrace{x_1,\cdots,x_1}_{n_1 \text{ times}},\cdots,\underbrace{x_{l-1},\cdots,x_{l-1}}_{n_{l-1} \text{ times}},x_{l1},\cdots,x_{ln_l},\\
&\qquad\underbrace{x_{l+1},\cdots,x_{l+1}}_{n_{l+1} \text{ times}},\cdots,\underbrace{x_d,\cdots,x_d}_{n_d \text{ times}}) = 0
\end{aligned}$$

for all $(x_1,\cdots,x_{l-1},x_{l1},x_{l+1},\cdots,x_d) \in \prod_{l=1}^d {}_AB_l$ and all $x_{l2},\cdots,x_{ln_l} \in {}_AB_l$, which implies that the mapping $M : \prod_{l=1}^d {}_AB_l \to {}_AD$ satisfies the functional equation (A) for each $l = 1,\cdots,d$. So M is additive for each $l = 1,\cdots,d$. Moreover, by passing to the limit in (1) as $n \to \infty$, we get the inequality (iii).

Now let $L : \prod_{l=1}^d {}_AB_l \to {}_AD$ be another multi-additive mapping satisfying

$$\|f(x_1,\cdots,x_d) - L(x_1,\cdots,x_d)\| \le \widetilde{\varphi}(x_1,\cdots,x_d)$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$.

$$\begin{aligned}\|M(x_1, \cdots, x_d) - L(x_1, \cdots, x_d)\| &= q_1^{-j} \cdots q_d^{-j} \|M(q_1^j x_1, \cdots, q_d^j x_d) - L(q_1^j, \cdots, q_d^j x_d)\| \\ &\le q_1^{-j} \cdots q_d^{-j} \|M(q_1^j x_1, \cdots, q_d^j x_d) - f(q_1^j x_1, \cdots, q_d^j x_d)\| \\ &\quad + q_1^{-j} \cdots q_d^{-j} \|f(q_1^j x_1, \cdots, q_d^j x_d) - L(q_1^j x_1, \cdots, q_d^j x_d)\| \\ &\le 2 \cdot q_1^{-j} \cdots q_d^{-j} \widetilde{\varphi}(q_1^j x_1, \cdots, q_d^j x_d),\end{aligned}$$

which tends to zero as $j \to \infty$ by (i). Thus $M(x_1, \cdots, x_d) = L(x_1, \cdots, x_d)$ for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$. This proves the uniqueness of M.

By the assumption, for each $u_l \in U(A)$,

$$\begin{aligned}&q_1^{-j} \cdots q_d^{-j} \|D_{1,\cdots,1,u_l,1,\cdots,1} f(\underbrace{q_1^j x_1, \cdots, q_1^j x_1}_{n_1 \text{ times}}, \cdots, \\ &\qquad \underbrace{q_l^j x_l, \cdots, q_l^j x_l}_{n_l \text{ times}}, \cdots, \underbrace{q_d^j x_d, \cdots, q_d^j x_d}_{n_d \text{ times}})\| \\ &\le q_1^{-j} \cdots q_d^{-j} \varphi(\underbrace{q_1^j x_1, \cdots, q_1^j x_1}_{n_1 \text{ times}}, \cdots, \underbrace{q_l^j x_l, \cdots, q_l^j x_l}_{n_l \text{ times}}, \cdots, \underbrace{q_d^j x_d, \cdots, q_d^j x_d}_{n_d \text{ times}})\end{aligned}$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$, and so

$$\begin{aligned}&q_1^{-j} \cdots q_d^{-j} \|D_{1,\cdots,1,u_l,1,\cdots,1} f(\underbrace{q_1^j x_1, \cdots, q_1^j x_1}_{n_1 \text{ times}}, \cdots, \\ &\qquad \underbrace{q_l^j x_l, \cdots, q_l^j x_l}_{n_l \text{ times}}, \cdots, \underbrace{q_d^j x_d, \cdots, q_d^j x_d}_{n_d \text{ times}})\| \to 0\end{aligned}$$

as $j \to \infty$ for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$. So

$$\begin{aligned}&D_{1,\cdots,1,u_l,1,\cdots,1} M(\underbrace{x_1, \cdots, x_1}_{n_1 \text{ times}}, \cdots, \underbrace{x_l, \cdots, x_l}_{n_l \text{ times}}, \cdots, \underbrace{x_d, \cdots, x_d}_{n_d \text{ times}}) \\ &= \lim_{j \to \infty} q_1^{-j} \cdots q_d^{-j} D_{1,\cdots,1,u_l,1,\cdots,1} f(\underbrace{q_1^j x_1, \cdots, q_1^j x_1}_{n_1 \text{ times}}, \cdots, \\ &\qquad \underbrace{q_l^j x_l, \cdots, q_l^j x_l}_{n_l \text{ times}}, \cdots, \underbrace{q_d^j x_d, \cdots, q_d^j x_d}_{n_d \text{ times}}) = 0\end{aligned}$$

for all $u_l \in U(A)$ and all $(x_1, \cdots, x_d) \in \prod_{l=1}^{d} {}_A B_l$. So we get

$$M(x_1, \cdots, x_{l-1}, u_l x_l, x_{l+1}, \cdots, x_d) = u_l M(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d)$$

for all $u_l \in U(A)$ and all $(x_1, \cdots, x_d) \in \prod_{l=1}^{d} {}_A B_l$.

Now let $a \in A$ ($a \neq 0$) and K an integer greater than $4|a|$. Then

$$|\frac{a}{K}| = \frac{1}{K}|a| < \frac{|a|}{4|a|} = \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}.$$

By [5, Theorem 1], there exist three elements $u_1, u_2, u_3 \in U(A)$ such that $3\frac{a}{K} = u_1 + u_2 + u_3$. And

$$\begin{aligned} M(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d) =& M(x_1, \cdots, x_{l-1}, 3 \cdot \frac{1}{3} x_l, x_{l+1}, \cdots, x_d) \\ =& 3M(x_1, \cdots, x_{l-1}, \frac{1}{3} x_l, x_{l+1}, \cdots, x_d) \end{aligned}$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^{d} {}_A B_l$. So

$$M(x_1, \cdots, x_{l-1}, \frac{1}{3} x_l, x_{l+1}, \cdots, x_d) = \frac{1}{3} M(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d)$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^{d} {}_A B_l$. Thus

$$\begin{aligned} M(x_1, \cdots, a x_l, \cdots, x_d) &= M(x_1, \cdots, \frac{K}{3} \cdot 3\frac{a}{K} x_l, \cdots, x_d) \\ &= \frac{K}{3} M(x_1, \cdots, 3\frac{a}{K} x_l, \cdots, x_d) \\ &= \frac{K}{3} M(x_1, \cdots, u_1 x_l + u_2 x_l + u_3 x_l, \cdots, x_d) \\ &= \frac{K}{3} (u_1 + u_2 + u_3) M(x_1, \cdots, x_l, \cdots, x_d) \\ &= \frac{K}{3} \cdot 3\frac{a}{K} M(x_1, \cdots, x_l, \cdots, x_d) \\ &= a M(x_1, \cdots, x_l, \cdots, x_d) \end{aligned}$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^{d} {}_A B_l$. Obviously,

$$M(x_1, \cdots, 0 x_l, \cdots, x_d) = 0 M(x_1, \cdots, x_l, \cdots, x_d)$$

for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$. Hence

$$\begin{aligned} &M(x_1, \cdots, ax_l + by_l, \cdots, x_d) \\ &\quad = M(x_1, \cdots, x_{l-1}, ax_l, x_{l+1}, \cdots, x_d) + M(x_1, \cdots, x_{l-1}, by_l, x_{l+1}, \cdots, x_d) \\ &\quad = aM(x_1, \cdots, x_{l-1}, x_l, x_{l+1}, \cdots, x_d) + bM(x_1, \cdots, x_{l-1}, y_l, x_{l+1}, \cdots, x_d) \end{aligned}$$

for all $a, b \in A$ and all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$ and all $y_l \in {}_AB_l$. So the unique multi-additive mapping $M : \prod_{l=1}^d {}_AB_l \to {}_AD$ is an A-multilinear mapping, as desired.

Theorem 2. *Let $q_l = \frac{k_l(n_l-1)}{n_l-k_l}$ and $r_l = -\frac{k_l}{n_l-k_l}$ for $l = 1, \cdots, d$. Let $f : \prod_{l=1}^d {}_AB_l \to {}_AD$ be a mapping for which there exists a function $\varphi : \prod_{l=1}^d {}_AB_l^{n_l} \to [0, \infty)$ satisfying* (i) *such that*

$$\begin{aligned} &\|D_{a_1,\cdots,a_d} f(x_{11}, \cdots, x_{1n_1}, \cdots, x_{d1}, \cdots, x_{dn_d})\| \\ &\qquad \le \varphi(x_{11}, \cdots, x_{1n_1}, \cdots, x_{d1}, \cdots, x_{dn_d}) \end{aligned}$$

for all $a_1, \cdots, a_d \in A_1^+ \cup \{i\}$ and all $x_{l1}, \cdots, x_{ln_l} \in {}_AB_l$, $l = 1, \cdots, d$. Assume that $f(x_1, \cdots, x_d) = 0$ if $x_l = 0$ for any $l = 1, \cdots, d$. Assume that for each $l = 1, \cdots, d$, $f(x_1, \cdots, x_{l-1}, \lambda x_l, x_{l+1}, \cdots, x_d)$ is continuous in $\lambda \in R$ for each fixed $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$, and that $\{q_1^{-j} \cdots q_d^{-j} f(q_1^j x_1, \cdots, q_d^j x_d)\}$ converges uniformly for all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$. Then there exists a unique A-multilinear mapping $M : \prod_{l=1}^d {}_AB_l \to {}_AD$ satisfying (iii).

Proof. Put $a_1 = \cdots = a_d = 1 \in A_1^+$. By the same reasoning as in the proof of Theorem 1, there exists a unique multi-additive mapping $M : \prod_{l=1}^d {}_AB_l \to {}_AD$ satisfying (iii).

For each fixed $l = 1, \cdots, d$, since $f(x_1, \cdots, \lambda x_l, \cdots, x_d)$ is continuous in $\lambda \in R$ for each fixed $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$, the mapping $M(x_1, \cdots, \lambda x_l, \cdots, x_d)$ is continuous in $\lambda \in R$ for each fixed $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_i$ by the uniform convergence. By the same reasoning as in the proof of [6, Theorem], the multi-additive mapping $M : \prod_{l=1}^d {}_AB_l \to {}_AD$ is R-linear in the l-th variable. So the multi-additive mapping $M : \prod_{l=1}^d {}_AB_l \to {}_AD$ is R-multilinear.

By the same reasoning as in the proof of Theorem 1,

$$M(x_1, \cdots, x_{j-1}, ax_j, x_{j+1}, \cdots, x_d) = aM(x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_d) \quad (4)$$

for all $a \in A_1^+ \cup \{i\}$ and $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$.

For any element $a \in A$, $a = \frac{a+a^*}{2} + i\frac{a-a^*}{2i}$, and $\frac{a+a^*}{2}$ and $\frac{a-a^*}{2i}$ are self-adjoint elements, furthermore, $a = (\frac{a+a^*}{2})^+ - (\frac{a+a^*}{2})^- + i(\frac{a-a^*}{2i})^+ - i(\frac{a-a^*}{2i})^-$, where $(\frac{a+a^*}{2})^+$, $(\frac{a+a^*}{2})^-$, $(\frac{a-a^*}{2i})^+$, and $(\frac{a-a^*}{2i})^-$ are positive elements (see [?, ?]). Using the R-multilinearity and (4), one can easily show that

$$M(x_1, \cdots, x_{j-1}, ax_j, x_{j+1}, \cdots, x_d) = aM(x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_d)$$

for all $a \in A$ and all $(x_1, \cdots, x_d) \in \prod_{l=1}^d {}_AB_l$. Hence

$$\begin{aligned} &M(x_1, \cdots, x_{j-1}, ax_j + by_j, x_{j+1}, \cdots, x_d) \\ &\quad = M(x_1, \cdots, x_{j-1}, ax_j, x_{j+1}, \cdots, x_d) + M(x_1, \cdots, x_{j-1}, by_j, x_{j+1}, \cdots, x_d) \\ &\quad = aM(x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_d) + bM(x_1, \cdots, x_{j-1}, y_j, x_{j+1}, \cdots, x_d) \end{aligned}$$

for all $a, b \in A$, all $(x_1, \cdots, x_d) \in \prod_{j=1}^d {}_AB_j$ and $y_j \in {}_AB_j$. So the unique multi-additive mapping $M : \prod_{l=1}^d {}_AB_l \to {}_AD$ is an A-multilinear mapping, as desired.

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