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Generating Functions Involving Arbitrary Products*

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Abstract

We show that the use of operational methods and of multi-index Bessel functions allow the derivation of generating functions, involving the product of an arbitrary number of Laguerre polynomials.

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1. Introduction

In this paper we will discuss a method allowing the derivation of generating functions involving the product of an arbitrary order of Laguerre polynomials. The technique, we will exploit, is based on the use of negative derivative operators [1], on their algebraic manipulation and on the properties of multi-index Bessel functions [2]. We remind therefore that the symbol ${}_a\widehat{\mathcal{D}}_x^{-1}$ denotes an operator whose action on a function $f(x)$ is such that

$$ {}_a\widehat{\mathcal{D}}_x^{-1} f(x) = \int_a^x f(\xi) d\xi. \tag{1} $$

The subscript on the left hand side will be omitted, if the lower integration limit is $a = 0$. The repeated action of the above operator can be denoted as ${}_a\widehat{\mathcal{D}}_x^{-n}$ and its action on a given function can be written in terms of repeated integrals, thus getting e. g.

$$ \widehat{\mathcal{D}}_x^{-n} f(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t) dt. \tag{2} $$

In the particular case of $f(x) = 1$ we get

$$ \widehat{\mathcal{D}}_x^{-n}(1) = \frac{x^n}{n!}. \tag{3} $$

From now on, when the unit function is involved, 1 will be omitted from the right side of the operator.

The exponential operator $\exp(\alpha \widehat{\mathcal{D}}_x^{-1})$ will play a central role in the theory we are going to develop. We remind therefore that [2]

$$ \exp(-\alpha \widehat{\mathcal{D}}_x^{-1}) = \sum_{n=0}^{\infty} \frac{(-\alpha)^n \widehat{\mathcal{D}}_x^{-n}}{n!} = \sum_{n=0}^{\infty} \frac{(-\alpha)^n x^n}{(n!)^2} = C_0(\alpha x). \tag{4} $$

Where $C_0(x)$ is the 0^{th} order Tricomi function defined as

$$ C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}, \tag{5} $$

and linked to the cylindrical Bessel functions by

$$C_n(x) = x^{-\frac{n}{2}} J_n(2\sqrt{x}). \tag{6}$$

The Tricomi functions satisfy the generating functions

$$\begin{aligned} \sum_{n=-\infty}^{\infty} t^n C_n(x) &= \exp\left(t - \tfrac{x}{t}\right) \\ \sum_{n=\ell}^{\infty} \frac{t^n}{n!}\, C_n(x) &= C_\ell(x-t). \end{aligned} \tag{7}$$

Along with the function one index-one variable Tricomi functions we will introduce its three variable two index extension, defined as

$$C_{m,n}(x,y,z) = \sum_{s=0}^{\infty} \frac{C_{m+s}(x) C_{n+s}(y) z^s}{s!}. \tag{8}$$

The relevant generating functions can easily be derived

$$\sum_{m,n=-\infty}^{\infty} u^m v^n\, C_{m,n}(x,y,z) = \exp\left(u + v - \frac{x}{u} - \frac{y}{v} + \frac{z}{uv}\right). \tag{9}$$

and it is also easily understood that

$$\begin{aligned} \sum_{s=0}^{\infty} \frac{u^s}{s!}\, C_{m+s,n}(x,y,z) &= C_{m,n}(x-u,y,z) \\ \sum_{s=0}^{\infty} \frac{v^s}{s!}\, C_{m,n+s}(x,y,z) &= C_{m,n}(x,y-v,z) \\ \sum_{s=0}^{\infty} \frac{\xi^s}{s!}\, C_{m+s,n+s}(x,y,z) &= C_{m,n}(x,y,z+\xi) \end{aligned} \tag{10}$$

It is also evident that

$$C_{m,n}(0,0,z) = \sum_{s=0}^{\infty} \frac{z^s}{s!(m+s)!(n+s)!}. \tag{11}$$

In the following sections we will see how the above results can be extended to the theory of Laguerre polynomials.

2. Generating Functions of Laguerre Polynomials

The Laguerre polynomials can be written, in terms of negative derivative operators, as [2]

$$L_n(x) = (1 - \widehat{\mathcal{D}}_x^{-1})^n = n! \sum_{r=0}^{n} \frac{(-1)^r x^r}{(r!)^2 (n-r)!}. \tag{12}$$

According to the above relations we note that

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \exp(t(1 - \widehat{\mathcal{D}}_x^{-1})) = \exp(t) C_0(xt). \tag{13}$$

Let us now come to the specific problem of this paper, by considering the infinite sum

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) L_n(y) &= \exp\left[t(1 - \widehat{\mathcal{D}}_x^{-1})(1 - \widehat{\mathcal{D}}_y^{-1})\right] = \\ &= \exp(t) \exp(-t\widehat{\mathcal{D}}_x^{-1}) \exp(-t\widehat{\mathcal{D}}_y^{-1}) \exp(t\widehat{\mathcal{D}}_x^{-1}\widehat{\mathcal{D}}_y^{-1}). \end{aligned} \tag{14}$$

We can now explicitly evaluate the above expression by noting that

$$\exp(t\widehat{\mathcal{D}}_x^{-1}\widehat{\mathcal{D}}_y^{-1}) = \sum_{r=0}^{\infty} \frac{(txy)^r}{r!^3} = C_{0,0}(0, 0, txy). \tag{15}$$

Furthermore, we also find

$$\begin{aligned} \exp(-t\widehat{\mathcal{D}}_x^{-1}) C_{0,0}(0, 0, txy) &= \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} x^r C_{r,0}(0, 0, txy) = \\ &= C_{0,0}(xt, 0, txy) \end{aligned} \tag{16}$$

and

$$\exp(-t\widehat{\mathcal{D}}_y^{-1}) C_{0,0}(xt, 0, txy) = C_{0,0}(xt, yt, xyt). \tag{17}$$

Thus getting in conclusion

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) L_n(y) = \exp(t)\, C_{0,0}(xt, yt, xyt). \tag{18}$$

This last expression generalizes the result given in equation (13) and can be recognized as a different way of formulating a particular case of the Hille-Hardy formula [4]-[5]. The use of the two index Tricomi function is useful because it allows the derivation of generating functions involving an arbitrary product of Laguerre polynomials, as we will show in the forthcoming section.

3. Concluding Remarks

Before discussing the problem of the generalization of the previous results, we consider the following generating function yielding a three index and seven variable Tricomi function

$$\begin{aligned} G(x_\alpha, y_\alpha, z\,; u_\alpha) &= \sum_{m,n,p=-\infty}^{\infty} u_1^m u_2^n u_3^p \; C_{m,n,p}(x_\alpha, y_\alpha, z) = \\ &= \exp\left[\sum_{\alpha=1}^{3}\left(u_\alpha - \frac{x_\alpha}{u_\alpha} + \frac{y_\alpha}{u_\alpha u_{\alpha+1}}\right) - \frac{z}{u_1 u_2 u_3}\right], \\ & \quad (\alpha = 1,2,3), \quad u_4 = u_1. \end{aligned} \tag{19}$$

Even though the properties of this function can be derived from the above generating function, through its integral representation

$$C_{m,n,p}(x_\alpha, y_\alpha, z) =$$

$$= \frac{1}{(2\pi)^3}\int_0^{2\pi} d\phi_1 \int_0^{2\pi} d\phi_2 \int_0^{2\pi} d\phi_3 G(x_\alpha, y_\alpha, z\,; \exp(i\phi_\alpha)) \exp\left[-i\left(\sum_{\alpha=1}^{3}\phi_\alpha\right)\right]$$

we give here, for completeness, the relevant series expansion

$$C_{m,n,p}(x_\alpha, y_\alpha, z) = \sum_{r=0}^{\infty} \frac{(-z)^r B_{m+r,n+r,p+r}\,(x_\alpha, y_\alpha)}{r!}\,, \tag{20}$$

where

$$\begin{aligned} B_{m,n,p} &= \sum_{r=0}^{\infty} \frac{y_3^r \; A_{m+r,n+r,p+r}\,(x_\alpha, y_1, y_2)}{r!}\,, \\ A_{m,n,p} &= \sum_{r=0}^{\infty} \frac{y_2^r \; C_{m+r,n+r}\,(x_1, x_2, y_1) \; C_{p+r}(x_3)}{r!}\,. \end{aligned} \tag{21}$$

The properties of this function too can be easily derived and we note that they are just an extension of those given in equation (10) with the further relation

$$\sum_{s=0}^{\infty} \frac{u^s}{s!}\; C_{m+s,n+s,p+s}(x_\alpha, y_\alpha, z) \;=\; C_{m,n,p}(x_\alpha, y_\alpha, z+u)\,. \tag{22}$$

Let us consider the following extension of equation (14),

$$\sum_{n=0}^{\infty} \frac{t^n}{n!}\; L_n(x)L_n(y)L_n(z) = \exp\left[t(1-\widehat{\mathcal{D}}_x^{-1})(1-\widehat{\mathcal{D}}_y^{-1})(1-\widehat{\mathcal{D}}_z^{-1})\right], \tag{23}$$

according to the previous considerations we end up with

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!}\; L_n(x)L_n(y)L_n(z) = \\ &= \exp(t)\; C_{0,0,0}(xt, yt, zt, xyt, xzt, yzt, xyzt). \end{aligned} \tag{24}$$

It is now evident that the extension of the above results to the case of the generating function

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \prod_{s=1}^{m} L_n(x_s) \tag{25}$$

requires the introduction of a Tricomi function with m-index and $2^m - 1$ variables, and in general we find

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!} \prod_{s=1}^{m} L_n(x_s) = \\ &= \exp(t)\; C_{\{0\}}\left(\{x_s\}\,t, \{x_s x_j\}_{s<j}\,t, \{x_s x_j x_k\}_{s<j<k}\,t, ..., \textstyle\prod_{s=1}^{m} x_s t\right), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \{0\} &= 0, ..., 0 \quad m\text{−}times \\ \{x_s\} &= x_1, ..., x_m, \\ \{x_s x_j\}_{s<j} &= \binom{m}{2} \text{ variables } \quad (x_1x_2,\, x_2x_3, ...), \\ \{x_s x_j x_k\}_{s<j<k} &= \binom{m}{3} \text{ variables } \quad (x_1x_2x_3,\, x_2x_3x_4, ...). \end{aligned} \tag{27}$$

Remark 1. We have already remarked that equation (18) can be viewed as an alternative formulation of the Hille-Hardy formula ([4]-[5]) which is extended to a product of two associated Laguerre polynomials $L_n^{(\alpha)}$ with arbitrary α, it has been shown in ref. [5] that the use of operational methods can usefully be exploited to derive the Hille-Hardy result in a fairly direct way and it would be interesting to obtain its generalization to the case of the arbitrary product. To achieve this result we note that, according to ref. [2], the associated Laguerre polynomials can be written by means of the operational formula

$$L_n^{(\alpha)}(x) = \left(1 - \frac{\partial}{\partial x}\right)^{\alpha} \left(1 - \hat{D}_x^{-1}\right)^n . \tag{28}$$

It is therefore evident that we can write the generating function (18), extended to the associated Laguerre as

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{n!}\, L_n^{(\alpha)}(x) L_n^{(\beta)}(y) = \\ &= \left(1 - \frac{\partial}{\partial x}\right)^{\alpha} \left(1 - \frac{\partial}{\partial y}\right)^{\beta} \exp(t)\; C_{0,0}(xt, yt, xyt) . \end{aligned} \tag{29}$$

The explicit action of the derivative can also be obtained quite straightforwardly, by noting indeed that we can write the two-variable Tricomi function as an expansion in terms of two variable Hermite polynomials as follows

$$\begin{aligned} &C_{0,0}(xt, yt, xyt) = \sum_{r,s=0}^{\infty} x^r\, y^s (r!)^2\, (s!)^2\; h_{r,s}(-t, -t\,|\,t) , \\ &h_{r,s}(x, y\,|\,\tau) = r!\, s! \sum_{p=0}^{[r,s]} \tau^p\, x^{r-p}\, y^{s-p} p!\, (r-p)!\, (s-p)! , \end{aligned} \tag{30}$$

In a forthcoming investigation we will discuss the extension to the arbitrary product case and derive the relevant closed formulae in terms of multi-variable Tricomi functions.

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