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Generalized Difference Sequence Spaces Defined by Orlicz Functions*

N. Subramanian†

Department of Mathematics, Shanmugha Arts,
Science,Technology & Research Academy,
Deemed University, Tanjore-613 402, India

Y. Altin‡ and Ç. Bektas§

Deemed University, Tanjore-613 402, INDIA
Department of Mathematics, Firat University , 23119, Elaziğ- Turkey

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Abstract

The idea of difference sequence spaces was introduced by Kizmaz [1] and then this subject has been studied and generalized by various mathematicians. In this paper we define some difference sequence spaces by Orlicz space of entire sequences and establish some inclusion relations. Some properties of these spaces are studied.

Keywords and Phrases: *Difference sequence , Entire sequence , Analytic sequence , Orlicz function.*

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† E-mail: nsmaths@yahoo.com
‡ E-mail: yaltin23 @yahoo.com

1. Introduction

A complex sequence, whose k^{th} term is x_k is denoted by $\{x_k\}$ or simply. Let Φ be the set of all finite sequences. A sequence $x = \{x_k\}$ is said to be analytic if $\sup_k |x_k|^{1/k} < \infty$. The vector space of all analytic sequences will be denoted by $\wedge$. A sequence x is called entire sequence if $\lim_{k\to\infty} |x_k|^{1/k} = 0$. The vector space of all entire sequences will be denoted by Γ .
Throughout the article $\Gamma_M, \wedge_M$ denote the Orlicz space entire and analytic sequences respectively.
Throughout m denotes an arbitrary positive integer . Kizmaz [1] introduced the notation of difference
sequence spaces as follows: $X(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\}$; for $X = \ell_\infty, c, c_0$, where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$. Later on the notion was generalized by Et and Colak [2] as follows: $X(\Delta^m) = \{x = (x_k) : (\Delta^m x_k) \in X\}$ for $X = \ell_\infty, c, c_0$,where $m \in N, \Delta^0 x = (x_k)$ and $\Delta^m x = (\Delta^m x_k) = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1})$
$= \sum_{v=0}^{m} (-1)^v \binom{m}{v} x_{k+v}$ for all $k \in N$.

Later on difference sequence spaces have been studied by Et [3], Et and Nuray [4], Çolak *et al* [5], Işık [6], Altin and Et [7] and many others.

Orlicz [8] used the idea of Orlicz function to construct the space (L^M). Lindenstrauss and Tzafriri [9] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space ℓ_M contains a subspace isomorphic to ℓ_p $(1 \le p < \infty)$. Subsequently different classes of sequence spaces defined by Parashar and Choudhary [10], Mursaleen *et al* [11], Bektas and Altin [12], Tripathy *et al.* [13], Rao and Subramanian [14] and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in Ref [15].

Recall ([8],[15]) an Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by

§ E-mail: cbektas@firat.edu.tr

$M(x+y) \le M(x) + M(y)$, then this function is called modulus function , defined and discussed by Ruckle [16] and Maddox [17] .

Let (Ω, Σ, μ) be a finite measure space. We denote by $E(\mu)$ the space of all (equivalence classes of) $\Sigma -$ measurable functions *x* from Ω into $[0, \infty)$. Given an Orlicz function *M*, we define on $E(\mu)$ a convex functional I_M by

$$I_M(x) = \int_\Omega M(x(t))d\mu,$$

and an Orlicz space $L^M(\mu)$ by $L^M(\mu) = \{x \in E(\mu) : I_M(\lambda x) < +\infty \text{ for some } \lambda > 0\}$, (For detail see [8], [15]).

Lindenstrauss and Tzafriri [9] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \textit{ for some } \rho > 0 \right\}$$

where $w = \{\text{all complex sequences}\}$.

The space ℓ_M with the norm

$$\|x\| = \inf\left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \le 1 \right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p, 1 \le p < \infty,$ the spaces ℓ_M coincide with the classical sequence space ℓ_p .

Given a sequence $x = \{x_k\}$ its n^{th} section is the sequence
$x^{(n)} = (x_1, x_2, \cdots, x_n, 0, 0, \cdots)$

$\delta^{(n)} = (0,0,\cdots, 1,0,0,\cdots.)$, 1 in the n^{th} place and zero's else where ; An FK-space (Frechet coordinate space) is a Frechet space which is made up of numerical sequences and has the property that the coordinate functionals
$p_k(x) = x_k \ (k = 1,2,\cdots.)$ are continuous.

An FK-space or a metric space X is said to have AK-property if $(\delta^{(n)})$ is a Schauder basis for X or equivalently $x^{(n)} \to x$ (AK stands for Abschnitts Konvergenz or Sectional Convergence). The space is said to have AD (or be an AD space) if Φ is dense in X.

We note that AK implies AD by [18].
If X is a sequence space, we define

(i) $X^{©}$ = the continuous dual of X.

(ii) $X^{\alpha} = \{a = (a_k) : \sum_{k=1}^{\infty} |a_k x_k| < \infty, \text{for each } x \in X\};$

(iii) $X^{\beta} = \{a = (a_k) : \sum_{k=1}^{\infty} a_k x_k \text{ is convergent, for each } x \in X\};$

(iv) $X^{\gamma} = \{a = (a_k) : \sup_n \left| \sum_{k=1}^{n} a_k x_k \right| < \infty, \text{for each } x \in X\}.$

(v) Let X be an FK-space and $X \supset \Phi$. Then $X^{f} = \{f(\delta^{(n)}) : f \in X^{©}\}$.

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called the $\alpha-, \beta-$ and $\gamma-$ dual of X, respectively.

Note that $X^{\alpha} \subset X^{\beta} \subset X^{\gamma}$. If $X \subset Y$ then $Y^{\mu} \subset X^{\mu}$, for $\mu = \alpha, \beta$, or γ.

Lemma 1.1. *(See(9,Theorem 7.2.7)) Let X be an FK space and* $X \supset \Phi$. *Then*

(i) $X^{\gamma} \subset X^{f}$.

(ii) If X has AK, $X^{\beta} = X^{f}$.

(iii) If X has AD, $X^{\beta} = X^{\gamma}$.

We note that $\Gamma^{\alpha} = \Gamma^{\beta} = \Gamma^{\gamma} = \wedge$.

Definition 1.2. *The space consisting of all those sequences* x *in* w *such that* $M\left(\frac{|x_k|^{1/k}}{\rho}\right) \to 0$ as $k \to \infty$ *for some arbitrary fixed* $\rho > 0$ *is denoted by* Γ_M, *M being an Orlicz function. In other words* $\left\{M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right\}$ *is a null sequence.* Γ_M *is called the Orlicz space of entire sequences. The space* Γ_M *is a metric space* *with the metric* $d(x,y)=\sup_k M\left(\frac{|x_k-y_k|^{1/k}}{\rho}\right)$ *for all* $x=\{x_k\}$ *and* $y=\{y_k\}$ *in* Γ_M .

Definition 1.3. *If* M *is a convex function and* $M(0)=0$, then $M(\lambda x)\le \lambda M(x)$ for all λ with $0<\lambda<1$.

Definition 1.4. *The space consisting of all those sequences* x *in* w *such that* $\left(\sup_k\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)\right)<\infty$ *for some arbitrarily fixed* $\rho>0$ *is denoted by* $\wedge_M$, M *being an Orlicz function. In other words* $\left(M\left(\frac{|x_k|^{1/k}}{\rho}\right)\right)$ *is a bounded sequences .* $\wedge_M$ *is called the Orlicz space of bounded sequence.*

Definition 1.5. *A sequence space E is said to be solid or normal if* $(\alpha_k x_k)\in E$ *whenever* $(x_k)\in E$ *and for all sequences of scalars* (α_k) with $|\alpha_k|\le 1$*, [20].*

Let $p=(p_k)$ be a sequence of positive real numbers with $0<p_k<\sup p_k=G$ and let $D=Max(1,2^{G-1})$. Then for $a_k,b_k\in C,$ the set of complex numbers for all $k\in N$, we have

$$|a_k+b_k|^{p_k}\le D\left\{|a_k|^{p_k}+|b_k|^{p_k}\right\} \tag{1}$$

In this paper , we define the following sequence spaces.

Let M be an Orlicz function , X be locally convex Hausdorff topological linear space whose topology is determined by a set Q of continuous semi norms q . The symbol $\wedge(X),\Gamma(X)$ denotes the space of all analytic and entire sequences defined over X . We define the following sequence spaces:

$$\wedge_M(\Delta^m,p,q)=\left\{x\in\wedge(X):\sup_n\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{(\Delta^m x_k)^{1/k}}{\rho}\right)\right)\right]^{p_k}<\infty,\text{ for some }\rho>0\right\}$$

$$\Gamma_M\left(\Delta^m, p, q\right) = \left\{ x \in \Gamma(X) : \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } \rho > 0 \right\}$$

2. Main Results

In this section we examine some topological properties of spaces $\Gamma_M\left(\Delta^m, p, q\right)$ and $\wedge_M\left(\Delta^m, p, q\right)$ and investigate some inclusion relations between these spaces.

Proposition 2.1. *If* M *is an Orlicz function, then* $\Gamma_M\left(\Delta^m, p, q\right)$ *is a linear set over the set of complex numbers C.*

Proof. Let $x, y \in \Gamma_M\left(\Delta^m, p, q\right)$ and $\alpha, \beta \in C.$ In order to prove the result, we need to find some ρ_3 such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m\left(\alpha x_k + \beta y_k\right)\right)^{1/k}}{\rho_3}\right)\right)\right]^{p_k} \to 0 \; as \; n \to \infty. \tag{2.1}$$

Since $x, y \in \Gamma_M\left(\Delta^m, p, q\right)$, there exist some positive ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_1}\right)\right)\right]^{p_k} \to 0 \; as \; n \to \infty \tag{2.2}$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m y_k\right)^{1/k}}{\rho_2}\right)\right)\right]^{p_k} \to 0 \; as \; n \to \infty. \tag{2.3}$$

Since M is a non-decreasing modulus function, q is a seminorm and Δ^m is linear then

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m\left(\alpha x_k + \beta y_k\right)\right)^{1/k}}{\rho_3}\right)\right)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{(\alpha)^{1/k}\left(\Delta^m x_k\right)^{1/k}}{\rho_3} + \frac{(\beta)^{1/k}\left(\Delta^m y_k\right)^{1/k}}{\rho_3}\right)\right)\right]^{p_k}$$

$$\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m(\alpha x_k+\beta y_k)\right)^{1/k}}{\rho_3}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{(\alpha)\left(\Delta^m x_k\right)^{1/k}}{\rho_3}+\frac{(\beta)\left(\Delta^m y_k\right)^{1/k}}{\rho_3}\right)\right)\right]^{p_k}$$

Take ρ_3 such that $\dfrac{1}{\rho_3}=\min\left\{\dfrac{1}{|\alpha|}\dfrac{1}{\rho_1},\dfrac{1}{|\beta|}\dfrac{1}{\rho_2}\right\}$

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m(\alpha x_k+\beta y_k)\right)^{1/k}}{\rho_3}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_1}+\frac{\left(\Delta^m y_k\right)^{1/k}}{\rho_2}\right)\right)\right]^{p_k}$$

$$\le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_1}\right)\right)^{p_k}+M\left(q\left(\frac{\left(\Delta^m y_k\right)^{1/k}}{\rho_2}\right)\right)^{p_k}\right]$$

$$\le D\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_1}\right)\right)\right]^{p_k}+D\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m y_k\right)^{1/k}}{\rho_2}\right)\right)\right]$$

$$\to 0\ (n\to\infty)$$

By (2.2) and (2.3). Hence $\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\alpha\Delta^m x_k+\beta\Delta^m y_k\right)^{1/k}}{\rho_3}\right)\right)\right]^{p_k}\to 0$ as $n\to\infty$. So $(\alpha x+\beta y)\in\Gamma_M\left(\Delta^m,p,q\right)$. Therefore $\Gamma_M\left(\Delta^M,p,q\right)$ is a linear space.
This completes the proof.

Proposition 2.2. $\Gamma_M\left(\Delta^m,p,q\right)$ *are para normed spaces (not totally paranormed) with*

$$g_\Delta{}^*(x)=\inf\left\{\rho^{p_m/H}:\sup_{k\ge1}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k}\le 1;\rho>0\right\},\ \textit{where}\ H=\max\left(1,\sup_k p_k\right)$$

Proof. Clearly $g_\Delta(x) \geq 0, g_\Delta(x) = g_\Delta(-x)$ *and* $g_\Delta\left(\bar{\theta}\right) = 0$, where θ is the zero sequence of X.

Let $(x_k),(y_k) \in \Gamma_M\left(\Delta^m, p, q\right)$. Let ρ_1 and $\rho_2 > 0$ be such that

$$\sup_{k\geq 1}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_1}\right)\right)\right]^{p_k} \leq 1 \text{ and } \sup_{k\geq 1}\left[M\left(q\left(\frac{\left(\Delta^m y_k\right)^{1/k}}{\rho_2}\right)\right)\right]^{p_k} \leq 1.$$

Then

$$\sup_{k\geq 1}\left[M\left(q\left(\frac{\left(\Delta^m (x_k + y_k)\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \leq \left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{k\geq 1}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_1}\right)\right)\right]^{p_k} +$$

$$\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{k\geq 1}\left[M\left(q\left(\frac{\left(\Delta^m y_k\right)^{1/k}}{\rho_2}\right)\right)\right]^{p_k} \leq 1.$$

$\leq 1.$

Hence

$$g_\Delta(x+y) \leq \inf\left\{\left(\rho_1+\rho_2\right)^{p_m/H} : m \in N\right\}$$

$$\leq \inf\left\{\left(\rho_1\right)^{p_m/H} : \sup_{k\geq 1}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_1}\right)\right)\right]^{p_k} \leq 1, \rho_1 > 0, m \in N\right\}$$

$$+\inf\left\{\left(\rho_2\right)^{p_m/H} : \sup_{k\geq 1}\left[M\left(q\left(\frac{\left(\Delta^m y_k\right)^{1/k}}{\rho_2}\right)\right)\right]^{p_k} \leq 1, \rho_2 > 0, m \in N\right\}$$

Thus we have $g_\Delta(x+y) \leq g_\Delta(x) + g_\Delta(y)$. Hence g_Δ satisfies the triangle inequality.

$$g_\Delta(\lambda x) = \inf\left\{\left(\rho\right)^{p_m/H} : \sup_{k\geq 1}\left[M\left(q\left(\frac{\left(\lambda\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \leq 1, \rho > 0, m \in N\right\}$$

$$= \inf\left\{ \left(r|\lambda|\right)^{p_m/H} : \sup_{k\geq 1}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{r}\right)\right)\right]^{p_k} \leq 1, r>0, m\in N\right\}, \text{ where } r=\frac{\rho}{|\lambda|}.$$

Hence $\Gamma_M\left(\Delta^m, p, q\right)$ is a paranormed space.
This completes the proof.

Propostiton 2.3. Let M_1 and M_2 be two Orlicz function .
Then $\Gamma_{M_1}\left(\Delta^m, p, q\right) \cap \Gamma_{M_2}\left(\Delta^m, p, q\right) \subseteq \Gamma_{M_1+M_2}\left(\Delta^m, p, q\right)$.

Proof. Let $x \in \Gamma_{M_1}\left(\Delta^m, p, q\right) \cap \Gamma_{M_2}\left(\Delta^m, p, q\right)$.
Then there exist
ρ_1 and ρ_2 such that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_1}\right)\right)\right]^{p_k} \to 0 \text{ as } n\to\infty. \tag{3.1}$$

and

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m y\right)_k^{1/k}}{\rho_2}\right)\right)\right]^{p_k} \to 0 \text{ as } n\to\infty. \tag{3.2}$$

Let $\rho = \min\left(\frac{1}{\rho_1}, \frac{1}{\rho_2}\right)$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[\Gamma_{M_1+M_2}\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \leq D\left[\frac{1}{n}\sum_{k=1}^{n}\left[\Gamma_{M_1}\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_1}\right)\right)\right]^{p_k}\right] + D\left[\frac{1}{n}\sum_{k=1}^{n}\left[\Gamma_{M_2}\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho_2}\right)\right)\right]^{p_k}\right].$$

$$\to 0 \text{ as } n\to\infty$$

by (3.1) and (3.2) . Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[\Gamma_{M_1+M_2}\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty.$$

Therefore $x \in \Gamma_{M_1+M_2}\left(\Delta^m, p, q\right)$.

This completes the proof.

Proposition 2.4. *Let* $m \geq 1$. *Then we have the following inclusions .*

(i) $\Gamma_M\left(\Delta^{m-1}, p, q\right) \subseteq \Gamma_M\left(\Delta^m, p, q\right)$,

(ii) $\wedge_M\left(\Delta^{m-1}, p, q\right) \subseteq \wedge_M\left(\Delta^m, p, q\right)$.

Proof. We prove the case (i) only. The other cases follow in a similar way. Let $x \in \Gamma_M\left(\Delta^{m-1}, p, q\right)$. Then we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^{m-1} x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty, \text{ for some } \rho > 0.$$

Since M is non-decreasing convex function and q is a semi norm , we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^{m} x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \leq \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}\right)^{1/k}}{\rho}\right)\right)\right]^{p_k}$$

$$\leq D\left\{\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^{m-1} x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} + \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^{m-1} x_{k+1}\right)^{1/k}}{\rho}\right)\right)\right]^{p_k}\right\} \to 0 \text{ as } n \to \infty.$$

Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^{m} x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty.$$

Hence $x \in \Gamma_M\left(\Delta^m, p, q\right)$.

This completes the proof.

Proposition 2.5. Let $0 \leq p_k \leq r_k$ and let $\left\{\frac{r_k}{p_k}\right\}$ be bounded. Then $\Gamma_M\left(\Delta^m, r, q\right) \subset \Gamma_M\left(\Delta^m, p, q\right)$.

Proof. Let $x \in \Gamma_M\left(\Delta^m, r, q\right)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{r_k} \to 0 \quad \text{as} \quad n \to \infty \quad . \tag{5.1}$$

Let $t_k = \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{q_k}$ and $\lambda_k = \frac{p_k}{r_k}$. Since $p_k \le r_k$, we have $0 \le \lambda_k \le 1$. Take $0 < \lambda < \lambda_k$.

Define

$$u_k = \begin{cases} t_k & (t_k \ge 1) \\ 0 & (t_k < 1) \end{cases} \quad \text{and} \quad v_k = \begin{cases} 0 & (t_k \ge 1) \\ t_k & (t_k < 1) \end{cases} \tag{5.2}$$

$$t_k = u_k + v_k \quad , \quad t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k} \quad .$$

Now it follows that $u_k^{\lambda_k} \le u_k \le t_k \quad , \quad v_k^{\lambda_k} \le v_k^{\lambda}$

since $t_k^{\lambda_k} = u_k^{\lambda_k} + v_k^{\lambda_k}$, then $t_k^{\lambda_k} \le t_k + v_k^{\lambda}$

$$\frac{1}{n}\sum_{k=1}^{n}\left[\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{r_k}\right]^{\lambda_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{r_k}$$

$$\Rightarrow \quad \frac{1}{n}\sum_{k=1}^{n}\left[\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{r_k}\right]^{p_k/r_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{r_k}$$

$$\Rightarrow \quad \frac{1}{n}\sum_{k=1}^{n}\left[\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{r_k}\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{r_k}$$

But $\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{r_k} \to 0$ as $n \to \infty$ (by (5.1)).

Therefore

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)^{r_k}\right]^{p_k} \to 0 \quad \text{as} \quad n\to\infty.$$

Hence $x \in \Gamma_M\left(\Delta^m, p, q\right)$. From (5.1), we get $\Gamma_M\left(\Delta^m, r, q\right) \subset \Gamma_M\left(\Delta^m, p, q\right)$.
This completes the proof .

Proposition 2.6. (a) *Let* $0 < \inf p_k \le p_k \le 1$. *Then* $\Gamma_M\left(\Delta^m, p, q\right) \subset \Gamma_M\left(\Delta^m, q\right)$

(b) *Let* $1 \le p_k \le \sup p_k < \infty$. *Then* $\Gamma_M\left(\Delta^m, q\right) \subset \Gamma_M\left(\Delta^m, p, q\right)$

Proof. **(a)** Let $x \in \Gamma_M\left(\Delta^m, p, q\right)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n\to\infty \tag{6.1}$$

Since $0 < \inf p_k \le p_k \le 1$,

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right] \le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \tag{6.2}$$

From (6.1) and (6.2) it follows that , $x \in \Gamma_M\left(\Delta^m, q\right)$. Thus $\Gamma_M\left(\Delta^m, p, q\right) \subset \Gamma_M\left(\Delta^m, q\right)$
We have thus proven (a) .

(b) Let $p_k \ge 1$ for each k and $\sup p_k < \infty$ and let $x \in \Gamma_M\left(\Delta^m, q\right)$. Then

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right] \to 0 \text{ as } n\to\infty . \tag{6.3}$$

Since $1 \le p_k \le \sup p_k < \infty$, we have

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]$$

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n\to\infty . \text{(by using (6.3))}$$

Therefore $x \in \Gamma_M\left(\Delta^m, p, q\right)$.

This completes the proof .

Proposition 2.7.

$\Gamma \subset \Gamma_M\left(\Delta^m, p, q\right)$, with the hypothesis that $\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \leq \left|x_k\right|^{1/k}$.

Proof. Let $x \in \Gamma$. Then we have the following implications:

$$\left|x_k\right|^{1/k} \to 0 \text{ as } k \to \infty \quad . \tag{7.1}$$

But $\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \leq \left|x_k\right|^{1/k}$, by our assumption, implies that

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty \text{ by } (7.1)$$

Then $x \in \Gamma_M\left(\Delta^m, p, q\right)$ and $\Gamma \subset \Gamma_M\left(\Delta^m, p, q\right)$.
This completes the proof.

Proposition 2.8. $\Gamma_M\left(\Delta^m, p, q\right)$ has AK where M is an Orlicz function.

Proof.

Let $x = \left(x_k\right) \in \Gamma_M\left(\Delta^m, p, q\right)$, but then $\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \in \Gamma$, and hence

$$\sup_{k \geq n+1}\left\{\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k}\right\} \to 0 \text{ as } n \to \infty . \tag{8.1}$$

Take the n^{th} sectional sequence of x, $x^{(n)} = \left(x_1, x_2, x_3, \ldots, x_n, 0, \ldots\right)$. By using (8.1),

$$d\left(x, x^{(n)}\right) = \sup_{k \geq n+1}\left[\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]\right]^{p_k} \to 0 \text{ as } n \to \infty \quad ,$$

which implies that $x^{(n)} \to x$ as $n \to \infty$, implying that $\Gamma_M(\Delta^m, p, q)$ has AK.

This completes the proof .

Proposition 2.9. $\Gamma_M(\Delta^m, p, q)$ is solid.

Proof. Let $|x_k| \le |y_k|$ and let $y = (y_k) \in \Gamma_M(\Delta^m, p, q)$. Because M is non-decreasing

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{(\Delta^m x_k)^{1/k}}{\rho}\right)\right)\right]^{p_k} \le \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{(\Delta^m x_k)^{1/k}}{\rho}\right)\right)\right]^{p_k}$$

And because $y \in \Gamma_M(\Delta^m, p, q)$

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{(\Delta^m x_k)^{1/k}}{\rho}\right)\right)\right]^{p_k} \in \Gamma ,$$

That is,

$$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{(\Delta^m y_k)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty \text{ and } \frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{(\Delta^m x_k)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0 \text{ as } n \to \infty .$$

There fore $x = \{x_k\} \in \Gamma_M(\Delta^m, p, q)$.
This completes the proof .

Proposition 2.10. $[\Gamma_M(\Delta^m, p, q)]^\beta = \wedge$.

Proof .
Step 1:
$\Gamma \subset \Gamma_M(\Delta^m, p, q)$ by Proposition 2.7, this implies that $[\Gamma_M(\Delta^m, p, q)]^\beta \subset \Gamma^\beta = \wedge$.

Therefore

$$\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^\beta \subset \wedge . \tag{10.1}$$

Step 2: Let $y \in \wedge$. Then $\left|y_k\right| < \mathrm{T}^k$ for all k and for some constant $\mathrm{T} > 0$.

Let $x \in \Gamma_M\left(\Delta^m, p, q\right)$. Then $\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} \to 0$ as $n \to \infty$. Hence

$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} < \varepsilon$ for given $\varepsilon > 0$ for sufficiently large k. Take $\varepsilon = \frac{1}{2\mathrm{T}}$

so that

$\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k\right)^{1/k}}{\rho}\right)\right)\right]^{p_k} < \frac{1}{(2\mathrm{T})^k}$. But then $\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k y_k\right)}{\rho}\right)\right)\right]^{p_k} < \frac{1}{2^k}$

so that

$\sum_{k=1}^{\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k y_k\right)}{\rho}\right)\right)\right]^{p_k}\right]$ converges.

Therefore $\sum_{k=1}^{\infty}\left[\frac{1}{n}\sum_{k=1}^{n}\left[M\left(q\left(\frac{\left(\Delta^m x_k y_k\right)}{\rho}\right)\right)\right]^{p_k}\right]$

converges. Hence $\sum_{k=1}^{\infty} x_k y_k$ converges so that $y \in \left[\Gamma_M\left(\Delta^m, p, q\right)\right]^\beta$. Thus

$$\wedge \subset \left[\Gamma_M\left(\Delta^m, p, q\right)\right]^\beta \tag{10.2}$$

Step 3. From (10.1) and (10.2), we obtain $\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^\beta = \wedge$.

This completes the proof .

Proposition 2.11. $\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\mu} = \wedge$ for $\mu = \alpha, \beta, \gamma, f$.

Proof.

Step 1:
$\Gamma_M\left(\Delta^m, p, q\right)$ has AK by proposition 2.8. Hence by Lemma 1.1 (ii) we get $\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\beta} = \left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{f}$. But $\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\beta} = \wedge$. Hence

$$\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{f} = \wedge. \tag{11.1}$$

Step 2:
Since AK implies AD, hence by Lemma 1.1 (iii) we get $\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\beta} = \left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\gamma}$
Therefore

$$\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\gamma} = \wedge. \tag{11.2}$$

Step 3:
$\Gamma_M\left(\Delta^m, p, q\right)$ is normal by Proposition 2.9 Hence by [20, proposition 2.7], we get

$$\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\alpha} = \left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\gamma} = \wedge . \tag{11.3}$$

From (11.1), (11.2) and (11.3), we have

$$\left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\alpha} = \left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\beta} = \left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{\gamma} = \left[\Gamma_M\left(\Delta^m, p, q\right)\right]^{f} = \wedge .$$

This completes the proof.

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