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Convolution Integral Equations Involving the $\overline{\text{H}}$-Function*

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Abstract

The object of the present paper is to generalize the Srivastava – Buschman's solution of the Convolution Integral Equation involving the Fox H-function kernel to the case of Generalized H-function kernel. There are some novel functions that form special cases of the generalized H-function but not Fox H-function. In the present paper, we shall give the solution of convolution equations involving two of such novel functions as kernels.

Keywords and Phrases: *Convolution integral equation, Generalized H-function (The* $\overline{\text{H}}$ *-function), Laplace transform.*

1. Introduction

Srivastava and Buschman established the following theorem giving the solution of the Convolution integral equation whose kernel is Fox H-function.

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Theorem 1(Srivastava and Buschman [5, 6]**).** *The convolution Integral Equation*

$$\int_0^x (x-t)^{\rho-1}\, H_{p,q}^{1,n}\left((x-t) \middle| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (0,1),\ (b_j,\beta_j)_{2,q} \end{matrix} \right) f(t)\, dt = g(x)\,, \qquad (1.1)$$

where f is an unknown function and g is a prescribed functions such that $g^{(u)}(0)=0$ for $0 \le u \le \ell - 1$, *Re(* ρ *)>0, with suitable restrictions on the parameters has the solution given by*

$$f(x) = \int_0^x (x-t)^{\ell-k-\rho-1}\; R_{\ell\text{-}k\text{-}\rho}\left((x-t) \middle| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{2,q} \end{matrix} \right) D_t^{\ell}\,\{g(t)\}\, dt, \qquad (1.2)$$

where

$$R_{\ell\text{-}k\text{-}\rho}\left(t \middle| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{2,q} \end{matrix} \right) = \sum_{\lambda=0}^{\infty} \frac{c_\lambda t^{\lambda}}{\Gamma(\lambda - k + \ell - \rho)}.$$

The details about the determination of c_λ can be referred in the theorem given in the reference 5 mentioned above.This theorem was extended to the case of the Srivastava-Panda multivariable H-function kernel by Srivastava, Koul and Raina [8]. In the present paper, we generalize Theorem 1 by taking the kernel as generalized H-function popularly known as $\overline{H}$ -function.

The $\overline{H}$ -function occurring in the paper is defined and represented as follows [4]

$$\overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n}\left(z \middle| \begin{matrix} (a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q} \end{matrix} \right)$$

$$= \frac{1}{2\pi\omega} \int_{-\omega\infty}^{\omega\infty} \overline{\phi}(\xi)\, z^{\xi}\, d\xi, \qquad (1.3)$$

where

$\omega = \sqrt{-1}$,

$$\bar{\phi}(\xi) = \frac{\prod_{j=1}^{m} \Gamma(b_j - \beta_j \xi) \prod_{j=1}^{n} \{\Gamma(1 - a_j + \alpha_j \xi)\}^{A_j}}{\prod_{j=m+1}^{q} \{\Gamma(1 - b_j + \beta_j \xi)\}^{B_j} \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j \xi)}. \tag{1.4}$$

Here $(a_j)_{1,p}$ and $(b_j)_{1,q}$ are complex parameters, $(\alpha_j)_{1,p}$, $(\beta_j)_{1,q}$ are non-negative real numbers and $(A_j)_{1,n}, (B_j)_{m+1,q}$ are positive real numbers. The sufficient conditions for the absolute convergence of the integral have been established by Buschman and Srivastava [1, p. 4708, eq.(6) and (7)]. When all the exponents A_j and B_j take the value 1, the $\bar{H}$-function reduces to the popular Fox H-function [7]

The following series representation for the $\bar{H}$-function can easily be obtained by adopting the standard procedure:

$$\bar{H}_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} (a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p} \\ (b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q} \end{matrix} \right) = \sum_{t=0}^{\infty} \sum_{h=1}^{m} \bar{\theta}(S_{t,h})\, z^{S_{t,h}}, \tag{1.5}$$

where,

$$\bar{\theta}(S_{t,h}) = \frac{\prod_{\substack{j=1 \\ j\neq h}}^{m} (\Gamma(b_j - \beta_j S_{t,h}) \prod_{j=1}^{n} \{\Gamma(1 - a_j + \alpha_j S_{t,h})\}^{A_j}}{\prod_{j=m+1}^{q} \{\Gamma(1 - b_j + \beta_j S_{t,h})\}^{B_j} \prod_{j=n+1}^{p} \Gamma(a_j - \alpha_j S_{t,h})} \frac{(-1)^t}{t!\beta_h}, \quad S_{t,h} = \frac{b_h + t}{\beta_h}. \tag{1.6}$$

The Laplace transform occurring in the paper will be defined in the following usual manner:

$$\hat{f}(s) = L\{f(t);s\} = \int_0^{\infty} e^{-st} f(t)\, dt, \tag{1.7}$$

The following Laplace transform will be required in the sequel. It can be computed from the defining integral (1.3) of the $\overline{\mathrm{H}}$ -function by following the standard procedure.

$$L\left\{t^{\rho-1}\overline{\mathrm{H}}_{p,q}^{m,n}\left(t^{\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},\ (a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},(b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right);s\right\}$$

$$= s^{-\rho}\,\overline{\mathrm{H}}_{p+1,q}^{m,n+1}\left(s^{-\sigma}\left|\begin{matrix}(1-\rho,\sigma;1),\ (a_j,\alpha_j;A_j)_{1,n},\ (a_j,\alpha_j)_{n+1,p}\\(b_j,\beta_j)_{1,m},\ (b_j,\beta_j;B_j)_{m+1,q}\end{matrix}\right.\right) \tag{1.8}$$

$$\textit{where}\ \min\left\{\min_{1\le j\le m}\mathrm{Re}\left(\rho+\sigma\frac{b_j}{\beta_j}\right),\mathrm{Re}(s),\sigma\right\}>0.$$

Theorem 2. *The convolution integral equation*

$$\int_0^x (x-t)^{\rho-1}\,\overline{\mathrm{H}}_{p,q}^{1,n}\left((x-t)^{\sigma}\left|\begin{matrix}(a_j,\alpha_j;A_j)_{1,n},(a_j,\alpha_j)_{n+1,p}\\(0,1),\ (b_j,\beta_j;B_j)_{2,q}\end{matrix}\right.\right)f(t)\,dt = g(x) \tag{1.9}$$

has the solution given by :

$$f(x) = \int_0^x (x-t)^{\ell-\sigma k-\rho-1}\sum_{\lambda=0}^{\infty}\frac{C_{\lambda}(x-t)^{\sigma\lambda}}{\Gamma(\ell-\sigma k+\sigma\lambda-\rho)}D_t^{\ell}\{g(t)\}\,dt, \tag{1.10}$$

where $\min\ \mathrm{Re}(\rho,\sigma)>0$, ℓ *is a positive integer such that* $\mathrm{Re}(\ell-\sigma k-\rho)>0$,

and k denotes the least ν for which $C_{\nu}^{'}\neq 0$, where

$$C'_\nu = \frac{\Gamma(\rho+\sigma\nu)\prod_{j=1}^{n}\{\Gamma(1-a_j+\alpha_j\nu)\}^{A_j}}{\prod_{j=2}^{q}\{\Gamma(1-b_j+\beta_j\nu)\}^{B_j}\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\nu)}\frac{(-1)^\nu}{\nu!},\quad \nu=0,1,2,\cdots \tag{1.11}$$

g is prescribed such that $g^{(u)}(0)=0$ for $0\le u\le \ell-1$

and C_λ *are given by the recurrence relation*

$$C_0=\frac{1}{C'_k} \text{ and for } \nu>0, \sum_{\lambda=0}^{\nu} C_\lambda C'_{\nu+k-\lambda}=0 \tag{1.12}$$

or by

$$C_\lambda=(-1)^\lambda\left(C'_k\right)^{-\lambda-1}\det\begin{bmatrix} C'_{k+1} & C'_k & 0 & 0 & \cdots & 0 \\ C'_{k+2} & C'_{k+1} & C'_k & 0 & \cdots & 0 \\ \vdots & & & & & \\ C'_{k+\lambda} & C'_{k+\lambda-1} & & & \cdots & C'_{k+1} \end{bmatrix}. \tag{1.13}$$

Proof of Theorem 2:

To solve the convolution integral equation (1.9), we take the Laplace transform of its both sides and then apply the well known convolution theorem for the Laplace transform to the left hand side , we easily obtain the following result with the help of (1.8)

$$s^{-\rho}\,\overline{H}^{1,n+1}_{p+1,q}\left(s^{-\sigma}\,\middle|\,\begin{matrix}(1-\rho,\sigma;1),\,(a_j,\alpha_j;A_j)_{1,n},\,(a_j,\alpha_j)_{n+1,p}\\(0,1),\,(b_j,\beta_j;B_j)_{2,q}\end{matrix}\right)\hat{f}(s)=\hat{g}(s). \tag{1.14}$$

Now, expressing the $\overline{H}$-function involved in the above equation (1.14) in terms of series, with the help of (1.5), we have:

$$s^{-\rho}\;\hat{f}(s)\sum_{\nu=0}^{\infty}C'_\nu\,s^{-\sigma\nu}=\hat{g}(s), \tag{1.15}$$

where C'_ν is given by (1.11) and min Re(ρ,s) > 0.

The other details of proof would run parallel to those given already in reference 5 and so we omit them here.

2. Special Cases

If we take $\sigma = 1$ and $A_j(j = 1, 2, \cdots, n) = B_j(j = 2, \cdots, q) = 1$ in Theorem 2, we at once arrive at Theorem 1. Now we list two special cases of Theorem 2 that cannot be obtained as special cases from Theorem 1.

Result 1

If we reduce the $\overline{H}$-function involved in (1.9), to the generalized Riemann Zeta function, $\phi\left((x\text{-}t)^{\sigma}, \mu, \eta\right)$,[2,p.27,§1.11,eq.(1); 3,pp.314 and 315,eqs.(1.6) and (1.7)], we arrive at the following interesting result:
The convolution integral equation:

$$\int_0^x (x-t)^{\rho-1}\, \phi\left((x\text{-}t)^{\sigma}, \mu, \eta\right) f(t)\, dt = g(x) \tag{2.1}$$

has the solution given by

$$f(x) = \int_0^x (x-t)^{\ell-\rho-1} \sum_{\lambda=0}^{\infty} \frac{C_{\lambda}(x-t)^{\sigma\lambda}}{\Gamma(\ell+\sigma\lambda-\rho)} D_t^{\ell}\{g(t)\}\, dt \tag{2.2}$$

provided that min $\mathrm{Re}(\rho, \sigma, \ell-\rho) > 0$, ℓ *is a positive integer*

and C_{λ} is given by (1.12) and (1.13) where

$$C_{\nu}^{'} = \frac{\Gamma(\rho+\sigma\nu)}{(\eta+\nu)^{\mu}} \quad , \quad \nu = 0,1,2,... \quad \text{Also } g^{(u)}(0) = 0 \quad \textit{for} \quad 0 \le u \le \ell - 1$$

Result 2

If we reduce the $\overline{H}$-function involved in (1.9) to the Polylogarithm function $F(t, \mu)$ of order μ [2,p.30, § 1.11,eq.(14) ;3, p.315, eq.(1.9)], we get the following interesting result which is believed to be new.

The convolution integral equation:

$$\int_0^x (x-t)^{\rho-1}\, F\left((x\text{-}t)^{\sigma}, \mu\right) f(t)\, dt = g(x) \tag{2.3}$$

has the solution given by

$$f(x) = \int_0^x (x-t)^{\ell-\rho-\delta-1} \sum_{\lambda=0}^{\infty} \frac{C_{\lambda}(x-t)^{\sigma\lambda}}{\Gamma(\ell+\sigma\lambda-\rho-\sigma)} D_t^{\ell}\{g(t)\}\, dt \tag{2.4}$$

provided that min $\text{Re}(\rho, \sigma, \ell-\rho-\sigma) > 0$, ℓ *is a positive integer*

and C_{λ} is given by (1.12) and (1.13) where

$C'_{\nu} = \dfrac{\Gamma(\rho+\sigma+\sigma\nu)}{(1+\nu)^{\mu}}$, $\nu = 0,1.2...$ Also $g^{(u)}(0) = 0$ *for* $0 \le u \le \ell - 1$

It may be noted that the results 1 and 2 cannot be obtained as special cases from Theorem 1. Many other results of practical utility which do not follow as special cases of Theorem 1, can also be obtained as special cases of Theorem 2. But we omit the details.

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