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# Connectedness Based on Strongly Preclosed $L$-Sets*

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**Abstract**

In this paper, the concepts of SP1-connectedness and SP2-connectedness of $L$-sets in $L$-topological spaces are introduced by means of strongly preclosed $L$-sets. SP1-connectedness and SP2-connectedness preserve some fundamental properties of connectedness in general topology. Especially, the famous K. Fan's Theorem can be generalized to $L$-set theory.

**Keywords and Phrases:** *L-topology, Strongly preopen L-set, Strongly preclosed L-set, SP-separated L-sets, SP1-connectedness, SP2-connectedness.*

## 1. Introduction

Connectivity is one of the most important notions in general topology. It has been generalized to $L$-fuzzy set theory in terms of many forms (see [1, 2, 4, 5, 6, 10, 12, 13, 16, 17, 18], and other related works). In [15], M. K. Singal and N. Prakash introduced the concepts of fuzzy preopen sets and fuzzy preclosed sets and researched their some properties. In [6], Bai introduced

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the concept of P-connectivity by means of preclosed $L$-set. In [7], Krateska introduced the concepts of fuzzy strongly preopen set and strongly preclosed set in [0,1]-topological spaces.

In this paper, we shall introduce the concepts of SP1-connectivity and SP2-connectivity in terms of strongly preclosed $L$-sets. They preserve many good properties of connectedness in general topology. Especially, the famous K. Fan's Theorem can be extended to $L$-set theory for SP1-connectivity and SP2-connectivity.

## 2. Preliminaries

In this paper, $L$ denotes a completely distributive de Morgan algebra. 0 and 1 denote the smallest element and the largest element in $L$ respectively. An element $a$ in $L$ is called a

$\vee$-irreducible element if $a = b \vee c$ implies $a = b$ or $a = c$. The set of all nonzero $\vee$-irreducible elements in $L$ is denoted by $M(L)$.

For a nonempty set $X$, $L^X$ denotes the set of all $L$-fuzzy subsets (or $L$-subsets for short) on $X$. $\underline{0}$ and $\underline{1}$ respectively denote the smallest element and the largest element in $L^X$. It is easy to see that $M(L^X) = \{x_\alpha \mid x \in X, a \in M(L)\}$ is exactly the set of all nonzero $\vee$-irreducible elements in $L^X$.

An $L$-topological space is a pair $(X, \tau)$, where $\tau$ is a subfamily $L^X$ which contains $\underline{0}, \underline{1}$ and is closed for any suprema and finite infima. $\tau$ is called an $L$-topology on $X$. Each member of $\tau$ is called an open $L$-set and its quasi-complementation is called a closed $L$-set.

**Definition 2.1** ([16]). *In an $L$-space $(X, \tau)$, two $L$-fuzzy sets $A, B$ are called separated if $cl(A) \wedge B = A \wedge cl(B) = \underline{0}$.*

**Definition 2.2** ([16]). *In an $L$-topological space $(X, \tau)$, an $L$-fuzzy set $D$ is called connected if $D$ cannot be represented as a union of two separated non-null sets.*

**Definition 2.3** ([15]). *Let $(X, \tau)$ be an $L$-topological space, $A \in L^X$. Then*

*(1) $A$ is called a preopen $L$-set if and only if $A \leq int(cl(A))$;*

*(2) $A$ is called a preclosed $L$-set if and only if $cl(int(A)) \leq A$.*

**Definition 2.4** ([15]). *Let $(X, \tau)$ be an $L$-topological space, $A \in L^X$. Define:*

*(1)* $int_p(A) = \bigvee\{B \mid B \leq A, B$ *is a preopen $L$-set of $X$*$\}$;

*(2)* $cl_p(A) = \bigwedge\{B \mid B \geq A, B$ *is a preclosed $L$-set of $X$*$\}$.
$int_p(A)$ *and* $cl_p(A)$ *are respectively called preinterior and preclosure of* $A$.

**Definition 2.5** ([6]). *In an $L$-topological space $(X,\tau)$, two $L$-fuzzy sets $A, B$ are called $P$-separated if* $cl_p(A) \wedge B = A \wedge cl_p(B) = \underline{0}$.

**Definition 2.6** ([6]). *In an $L$-topological space $(X,\tau)$, an $L$-fuzzy set $D$ is called $P$-connected if $D$ cannot be represented as a union of two $P$-separated non-null sets.*

In [7], the concepts of strongly preopen and strongly preclosed sets were introduced in [0,1]-fuzzy set theory by Biljana Krateska. They can easily be extended to $L$-sets as follows:

**Definition 2.7**. *Let $(X,\tau)$ be an $L$-topological space, $A \in L^X$. Then*

*(1) $A$ is called a strongly preopen $L$-set if and only if* $A \leq int(cl_p(A))$;

*(2) $A$ is called a strongly preclosed $L$-set if and only if* $cl(int_p(A)) \leq A$.

The set of all strong preopen $L$-sets and the set of all strong preclosed $L$-sets in $L^X$ are respectively denoted as $\mathbf{SPO}(X)$ and $\mathbf{SPC}(X)$.

**Definition 2.8**. *Let $(X,\tau)$ be an $L$-topological space, $A, B \in L^X$. Define:*

*(1)* $int_{sp}(A) = \bigvee\{B \mid B \leq A, B$ *is a strongly preopen $L$-set of $X$*$\}$;

*(2)* $cl_{sp}(A) = \bigwedge\{B \mid B \geq A, B$ *is a strongly preclosed $L$-set of $X$*$\}$.
$int_{sp}(A)$ *and* $cl_{sp}(A)$ *are called strong preinterior and strong preclosure of $A$, respectively.*

**Theorem 2.9.** *Let $(X,\tau)$ be an $L$-topological space, $A \in L^X$. Then*

*(1) $A$ is a strongly preopen $L$-set if and only if* $A = int_{sp}(A)$;

*(2) $A$ is a strongly preclosed $L$-set if and only if* $A = cl_{sp}(A)$.

**Lemma 2.10**([17]). *Let $A, B \in L^X$ and $A \not\leq B$. If $1 \in M(L)$, then* $A' \vee B \neq \underline{1}$.

## 3. SP1-connectedness

In this section, we shall research SP1-connectedness in $L$-topological spaces by means of SP-separated $L$-sets.

**Definition 3.1.** *Let $(X,\tau)$ be an $L$-topological space and $A, B \in L^X$. Then $A, B$ are said to be SP-separated if $cl_{sp}(A) \wedge B = A \wedge cl_{sp}(B) = \underline{0}$.*

The following theorem is obvious.

**Theorem 3.2.** *Let $(X,\tau)$ be an $L$-topological space and $A, B \in L^X$. If $A$ and $B$ are SP-separated and $C \leq A$, $D \leq B$. Then $C$ and $D$ are also SP-separated.*

**Definition 3.3.** *Let $(X,\tau)$ be an $L$-topological space and $G \in L^X$. $G$ is called SP1-connected if $G$ can't be represented as a union of two non-null SP-separated $L$-sets. When $G = \underline{1}$ is SP1-connected, we call $(X,\tau)$ an SP1-connected $L$-topological space.*

**Theorem 3.4**. *Let $(X,\tau)$ be an $L$-topological space, $G \in L^X$. Then the following conditions are equivalent:*

*(1) $G$ is SP1-connected;*

*(2) There don't exist two strongly preclosed $L$-sets $A$, $B$ such that*

$$A \wedge G \neq \underline{0}, B \wedge G \neq \underline{0}, G \leq A \vee B \ \text{ and } \ A \wedge B \wedge G = \underline{0};$$

*(3) There don't exist two strongly preclosed $L$-sets $A$, $B$ such that*

$$G \not\leq A, G \not\leq B, G \leq A \vee B \ \text{ and } \ A \wedge B \wedge G = \underline{0}.$$

**Proof.** (1)⇒(2) Suppose that $G$ is SP1-connected and there exist two strongly preclosed $L$-sets $A$, $B$ such that

$$A \wedge G \neq \underline{0}, B \wedge G \neq \underline{0}, G \leq A \vee B, A \wedge B \wedge G = \underline{0}.$$

Then obviously $(A \wedge G) \vee (B \wedge G) = (A \vee B) \wedge G = G$. We can prove $cl_{sp}(A \wedge G) \wedge (B \wedge G) = \underline{0}$ from the following fact:

$$cl_{sp}(A \wedge G) \wedge (B \wedge G) \leq cl_{sp}(A) \wedge (B \wedge G) = A \wedge B \wedge G = \underline{0}.$$

Similarly we have $(A \wedge G) \wedge cl_{sp}(B \wedge G) = \underline{0}$. This shows that $G$ is not SP1-connected, which is a contradiction.

(2)⇒(3) Suppose that there exist two strongly preclosed $L$-sets $A$, $B$ such that

$$G \not\leq A, G \not\leq B, G \leq A \vee B, A \wedge B \wedge G = \underline{0}.$$

We easily prove that $A \wedge G \neq \underline{0}$ and $B \wedge G \neq \underline{0}$. This is a contradiction.

(3)⇒(1) Suppose that (3) is true and $G$ is not SP1-connected. Then there are $C, D \neq \underline{0}$ such that $G = C \vee D$ and $cl_{sp}(C) \wedge D = C \wedge cl_{sp}(D) = \underline{0}$. Let $A = cl_{sp}(C)$, $B = cl_{sp}(D)$. Then $G = C \vee D \leq cl_{sp}(C) \vee cl_{sp}(D) = A \vee B$ and by

$$\begin{aligned} cl_{sp}(C) \wedge cl_{sp}(D) \wedge G &= cl_{sp}(C) \wedge cl_{sp}(D) \wedge (C \vee D) \\ &= (cl_{sp}(C) \wedge cl_{sp}(D) \wedge C) \vee (cl_{sp}(C) \wedge cl_{sp}(D) \wedge D) \\ &= (cl_{sp}(D) \wedge C) \vee (cl_{sp}(C) \wedge D) \\ &= \underline{0} \vee \underline{0} = \underline{0} \end{aligned}$$

we know $A \wedge B \wedge G = \underline{0}$. Moreover we have that $G \not\leq A$ and $G \not\leq B$. In fact, if $G \leq A$, then $B \wedge G = B \wedge (G \wedge A) = \underline{0}$, i.e., $cl_{sp}(D) \wedge G = \underline{0}$. Therefore $D = D \wedge G \leq cl_{sp}(D) \wedge G = \underline{0}$. This is a contradiction. Analogously we have that $G \not\leq B$. This contradicts (3).

**Corollary 3.5.** *An $L$-topological space $(X, \tau)$ is SP1-connected if and only if there don't exist two non-null strongly preclosed $L$-sets $A$, $B$ such that $A \vee B = \underline{1}$ and $A \wedge B = \underline{0}$.*

**Theorem 3.6.** *Let $(X, \tau)$ be an $L$-topological space, $G \in L^X$. Then the following conditions are equivalent:*

*(1) $G$ is SP1-connected;*

*(2) If $A, B \in L^X$ are SP-separated and $G \leq A \vee B$, then $G \wedge A = \underline{0}$ or $G \wedge B = \underline{0}$;*

*(3) If $A, B \in L^X$ are SP-separated and $G \leq A \vee B$, then $G \leq A$ or $G \leq B$.*

**Proof.** (1)⇒(2) If $A, B \in L^X$ are SP-separated and $G \leq A \vee B$, then by Theorem 3.2 we know that $G \wedge A$ and $G \wedge B$ are SP-separated. Since $G$ is SP1-connected and $G = G \wedge (A \vee B) = (G \wedge A) \vee (G \wedge B)$, one of $G \wedge A$ and $G \wedge B$ equals to $\underline{0}$.

(2)⇒(3) Suppose $G \wedge A = \underline{0}$, then $G = G \wedge (A \vee B) = (G \wedge A) \vee (G \wedge B) = G \wedge B$. So $G \leq B$. Similarly $G \wedge B = \underline{0}$ implies $G \leq A$.

(3)⇒(1) Suppose that $A, B$ are SP-separated and $G = A \vee B$. By (3) we have that $G \leq A$ or $G \leq B$. If $G \leq A$, then $B = B \wedge G \leq B \wedge A \leq B \wedge cl_{sp}(A) = \underline{0}$ since $A, B$ are SP-separated. Similarly if $G \leq B$, then $A = \underline{0}$. So $G$ can't be represented as a union of two SP-separated non-null $L$-subsets. Therefore $G$ is SP1-connected.

**Corollary 3.7.** *Each element in $M(L^X)$ is SP1-connected.*

**Theorem 3.8.** *Let $(X, \tau)$ be an $L$-topological space and $G$ be SP1-connected. If $G \leq H \leq cl_{sp}(G)$, then $H$ is also SP1-connected.*

**Proof.** Suppose that $H$ is not SP1-connected. Then there exist two strongly preclosed $L$-sets $A$ and $B$ such that

$$H \not\leq A, H \not\leq B, H \leq A \vee B, H \wedge A \wedge B = \underline{0}.$$

By $G \leq H$, we obtain $G \wedge A \wedge B = \underline{0}$ and $G \leq A \vee B$. Now we prove that $G \not\leq A$ and $G \not\leq B$. In fact, if $G \leq A$, then $cl_{sp}(G) \leq A$, therefore $H \leq cl_{sp}(G) \leq A$, which is a contradiction. Hence $G \not\leq A$. Similarly we have $G \not\leq B$. This contradicts that $G$ is SP1-connected.

**Theorem 3.9.** *Let $(X, \tau)$ be an $L$-topological space, both $G$ and $H$ be SP1-connected. If $G$ and $H$ are not SP-separated. Then $G \vee H$ is SP1-connected.*

**Proof.** Suppose that $G \vee H$ is not SP1-connected. Then there exist two strongly preclosed $L$-sets $A, B$ such that

$$G \vee H \not\leq A, G \vee H \not\leq B, G \vee H \leq A \vee B, (G \vee H) \wedge A \wedge B = \underline{0}.$$

By $G \vee H \not\leq A$ we have that $G \not\leq A$ or $H \not\leq A$. If $G \not\leq A$, then by SP1-connectedness of $G$, we have $G \leq B$. Hence $H \not\leq B$ and $H \leq A$. This implies that $A \wedge G \leq A \wedge B \wedge (G \vee H) = \underline{0}$. Therefore $cl_{sp}(H) \wedge G \leq cl_{sp}(A) \wedge G = A \wedge G = \underline{0}$. Similarly $H \wedge cl_{sp}(G) = \underline{0}$. This shows that $G$ and $H$ are SP-separated, a contradiction.

**Theorem 3.10.** *Let $(X, \tau)$ be an $L$-topological space and $\{G_i\}_{i\in I}$ be a family of SP1-connected $L$-sets. If there is $j \in I$ such that $G_i$ and $G_j$ are not SP-separated for each $i \neq j$, then $\bigvee_{i\in I} G_i$ is SP1-connected.*

**Proof.** Suppose that $\bigvee_{i\in I} G_i$ is not SP1-connected. Then there exist two strongly preclosed $L$-sets $A, B$ such that

$$\bigvee_{i\in I} G_i \not\leq A,\ \bigvee_{i\in I} G_i \not\leq B,\ \bigvee_{i\in I} G_i \leq A \vee B,\ (\bigvee_{i\in I} G_i) \wedge A \wedge B = \underline{0}.$$

Thus there exist $r, s \in I$ such that

$$G_r \vee G_s \vee G_j \not\leq A,\ G_r \vee G_s \vee G_j \not\leq B,\ G_r \vee G_s \vee G_j \leq A \vee B,\ (G_r \vee G_s \vee G_j) \wedge (A \wedge B) = \underline{0}.$$

This shows that $G_r \vee G_s \vee G_j$ is not SP1-connected. By Theorem 3.9 we obtain a contradiction.

**Corollary 3.11.** *Let $(X, \tau)$ be an $L$-topological space and $\{G_i\}_{i\in I}$ be a family of SP1-connected $L$-sets. If $\bigwedge_{i\in I} G_i \neq \underline{0}$, then $\bigvee_{i\in I} G_i$ is SP1-connected.*

**Theorem 3.12.** *Let $(X, \tau)$ be an $L$-topological space and $G \in L^X$. Then $G$ is SP1-connected if and only if for any two nonzero $\vee$-irreducible elements $a, b$ in $G$, there exists an SP1-connected $L$-set $H$ such that $a, b \leq H \leq G$.*

**Proof.** The necessity is obvious. Now we prove the sufficiency. Suppose that $G$ is not SP1-connected in $(X, \tau)$. Then there exist two strongly preclosed $L$-sets $A, B \in L^X$ such that

$$G \not\leq A,\ G \not\leq B,\ G \leq A \vee B \text{ and } G \wedge A \wedge B = \underline{0}.$$

Take two nonzero $\vee$-irreducible elements $a, b \leq G$ such that $a \not\leq A$ and $b \not\leq B$. Let $H$ be a SP1-connected $L$-set satisfying
$a, b \leq H \leq G$. We have that

$$H \not\leq A,\ H \not\leq B,\ H \leq A \vee B \text{ and } H \wedge A \wedge B = \underline{0}.$$

This shows that $H$ is not SP1-connected, a contradiction.

In [8], Biljana Krateska introduce the concept of fuzzy SP-irresolute mapping. Now we extend it to $L$-sets.

**Definition 3.13.** *Let $(X,\tau)$ and $(Y,\mu)$ be two $L$-topological spaces and $f: X \to Y$ be a mapping. An $L$-value Zadeh's type mapping $f_L^{\to}: L^X \to L^Y$ is called an SP-irresolute mapping if $f_L^{\leftarrow}(B)$ is strongly preopen in $(X,\tau)$ for each strongly preopen $L$-set $B$ in $(Y,\mu)$.*

**Theorem 3.14.** *If $f_L^{\to}: L^X \to L^Y$ is SP-irresolute, then $cl_{sp}(f_L^{\leftarrow}(B)) \leq f_L^{\leftarrow}(cl_{sp}(B))$ for each $B \in L^Y$.*

**Proof.** Analogous to the proof of Theorem 3.1(4) in [8].

**Theorem 3.15.** *Let $(X,\tau)$ and $(Y,\mu)$ be two $L$-topological spaces and $f_L^{\to}: L^X \to L^Y$ be SP-irresolute. If $G$ is SP1-connected in $(X,\tau)$, then so is $f_L^{\to}(G)$ in $(Y,\mu)$.*

**Proof.** Suppose that $f_L^{\to}(G)$ is not SP1-connected in $(Y,\mu)$. Then there exist two strongly preclosed $L$-sets $A, B$ such that

$$f_L^{\to}(G) \not\leq A,\ f_L^{\to}(G) \not\leq B,\ f_L^{\to}(G) \leq A \vee B,\ f_L^{\to}(G) \wedge A \wedge B = \underline{0}.$$

Hence we have that

$$G \not\leq f_L^{\leftarrow}(A),\ G \not\leq f_L^{\leftarrow}(B),\ G \leq f_L^{\leftarrow}(A) \vee f_L^{\leftarrow}(B)$$

and

$$G \wedge f_L^{\leftarrow}(A) \wedge f_L^{\leftarrow}(B) \leq f_L^{\leftarrow}(f_L^{\to}(G)) \wedge f_L^{\leftarrow}(A) \wedge f_L^{\leftarrow}(B) = f_L^{\leftarrow}(f_L^{\to}(G) \wedge A \wedge B) = \underline{0}.$$

This implies that $G$ is not SP1-connected, a contradiction. Therefore $f_L^{\to}(G)$ is SP1-connected in $(Y,\mu)$.

**Corollary 3.16.** *Let $(X,\tau)$ and $(Y,\mu)$ be two $L$-topological spaces and $f_L^{\to}$ be an SP-irresolute mapping from $L^X$ onto $L^Y$. If $(X,\tau)$ is SP1-connected , then so is $(Y,\mu)$.*

Now, K. Fan's Theorem will be extended to $L$-topology.

In [14], the concept of remote neighborhood mapping is introduced. Analogously we introduce the following definition:

**Definition 3.17.** *Let $(X,\tau)$ be an $L$-topological space and $G \in L^X$. A mapping $P: M(G) \to$ **SPC**$(X)$ is called an SP-remote neighborhood mapping in*

*$G$ if $e \not\leq P(e)$ for every $e \in M(G)$, where $M(G)$ denotes the set of all nonzero $\vee$-irreducible elements contained in $G$.*

**Theorem 3.18.** *Let $(X, \tau)$ be an $L$-topological space. Then $G$ is SP1-connected if and only if for each pair $a, b$ in $M(G)$ and each SP-remote neighborhood mapping $P : M(G) \to \mathbf{SPC}(L^X)$, there exists a finite number of point $x_1 = a, x_2, \cdots, x_n = b$ in $M(G)$ such that*

$$G \not\leq P(x_i) \vee P(x_{i+1}), i = 1, 2, \cdots, n-1.$$

**Proof.** Sufficiency. Suppose that $G$ is not SP1-connected. Then there are two strongly preclosed $L$-sets $A, B \in L^X$ such that

$$G \not\leq A, G \not\leq B, G \leq A \vee B \text{ and } G \wedge A \wedge B = \underline{0}.$$

Define an SP-remote neighborhood mapping $P : M(G) \to \mathbf{SPC}(X)$ as follows:

$$\forall x \in M(G) \;\; P(x) = \begin{cases} B, & \text{if} \quad x \leq A, \\ A, & \text{if} \quad x \not\leq A. \end{cases}$$

Take $a, b \in M(G)$ such that $a \leq A$ and $b \leq B$. Since for arbitrary finite many elements $x_1 = a, x_2, \cdots, x_n = b$ in $M(G)$, one and only one of $x_i \leq A$ and $x_i \leq B (i = 1, 2, \cdots, n)$ is true, we have that $P(x_i) = B$ or $P(x_i) = A$. But $P(x_1) = B$ and $P(x_n) = A$, hence there exists $j (1 \leq j \leq n-1)$ such that $P(x_j) = B$ and $P(x_{j+1}) = A$. This shows that

$$G \leq A \vee B = P(x_j) \vee P(x_{j+1}),$$

which is a contradiction. Thus the sufficiency is proved.

Necessity. Suppose that condition of theorem is not true, i.e, there are two elements $a, b \in M(G)$ and an SP-remote neighborhood mapping $P : M(G) \to \mathbf{SPC}(X)$, such that

$$G \not\leq P(x_i) \vee P(x_{i+1}), i = 1, 2, \cdots, n-1.$$

is not true for arbitrary finite many elements $a = x_1, x_2, \cdots, x_n = b \in M(G)$. For the sake of convenience, we follow the agreement that $a$ and $b$ can be linked if there are finite many elements $a = x_1, x_2, \cdots, x_n = b \in M(G)$ such that

$$G \not\leq P(x_i) \vee P(x_{i+1}), i = 1, 2, \cdots, n-1.$$

Otherwise, $a$ and $b$ can not be linked. Let

$$\begin{aligned}\Phi &= \{x \in M(G) \mid a \text{ and } x \text{ can be linked}\};\\ \Psi &= \{x \in M(G) \mid a \text{ and } x \text{ can not be linked}\}.\end{aligned}$$

Then for any $c \in \Phi$ and for any $d \in \Psi$, we have $G \leq P(c) \vee P(d)$. Let

$$A = \bigwedge\{P(c) \mid c \in \Phi\}, \quad B = \bigwedge\{P(d) \mid d \in \Psi\}.$$

Then

$$\begin{aligned}A \vee B &= (\bigwedge\{P(c) \mid c \in \Phi\}) \vee (\bigwedge\{P(d) \mid d \in \Psi\})\\ &= \bigwedge\{(P(c) \vee P(d)) \mid c \in \Phi,\ d \in \Psi\} \geq G\end{aligned}$$

Obviously, $a$ and $a$ can be linked. So $a \in \Phi$. Since $a$ and $b$ can't linked, we have $b \in \Psi$, hence $G \not\leq A$, $G \not\leq B$. Moreover it is obvious that $G \wedge A \wedge B = \underline{0}$ and by definition of $A, B$ we know that $A, B$ are strongly preclosed $L$-sets. This shows that $G$ is not SP1-connected, a contradiction.

## 4. SP2-connectedness

In this section, we shall introduce the concept of SP2-connectedness by means of strongly preclosed $L$-sets. Since the proof of many results is analogous to the proof of some results in last section, we omitted them.

**Definition 4.1.** *Let $(X, \tau)$ be an $L$-topological space and $G \in L^X$. $G$ is called SP2-connected if there don't exist strongly preclosed $L$-sets $A, B$ such that*

$$G \not\leq A, G \not\leq B, G' \vee A \vee B = \underline{1} \text{ and } G \wedge A \wedge B = \underline{0}.$$

**Theorem 4.2.** *Let $(X, \tau)$ be an $L$-topological space and $G \in L^X$. Then the following conditions are equivalent:*

*(1) $G$ is SP2-connected;*

*(2) There don't exist two strongly preclosed $L$-sets $A$, $B$ such that*

$$A \wedge G \neq \underline{0},\ B \wedge G \neq \underline{0},\ G' \vee A \vee B = \underline{1} \text{ and } A \wedge B \wedge G = \underline{0};$$

*(3) There don't exist strongly preopen $L$-sets $A, B$ such that*

$$G \not\leq A,\ G \not\leq B,\ G^{'} \vee A \vee B = \underline{1} \text{ and } G \wedge A \wedge B = \underline{0};$$

*(4) There don't exist two strongly preopen $L$-sets $A$, $B$ such that*

$$G \wedge A \neq \underline{0}, G \wedge B \neq \underline{0}, G^{'} \vee A \vee B = \underline{1} \text{ and } A \wedge B \wedge G = \underline{0}.$$

**Theorem 4.3.** *Let $(X, \tau)$ be an $L$-topological space and $G$ be SP2-connected. If $G \leq H \leq cl_{sp}(G)$, then $H$ is also SP2-connected.*

**Theorem 4.4.** *Let $(X, \tau)$ be an $L$-topological space, $G$ and $H$ be SP2-connected. If $G$ and $H$ are not SP-separated, then $G \vee H$ is SP2-connected.*

**Theorem 4.5.** *Let $(X, \tau)$ be an $L$-topological space and $\{G_i\}_{i \in I}$ be a family of SP2-connected $L$-sets. If there is $j \in I$ such that $G_i$ and $G_j$ are not SP-separated for each $i \neq j$, then $\bigvee\limits_{i \in I} G_i$ is SP2-connected.*

**Corollary 4.6.** *Let $(X, \tau)$ be an $L$-topological space and $\{G_i\}_{i \in I}$ be a family of SP2-connected $L$-sets. If $\bigwedge\limits_{i \in I} G_i \neq \underline{0}$, then $\bigvee\limits_{i \in I} G_i$ is SP2-connected.*

**Theorem 4.7.** *Let $(X, \tau)$ be an $L$-topological space and $G \in L^X$. Then $G$ is SP2-connected if and only if for any two nonzero $\vee$-irreducible elements $a, b$ in $G$, there exists an SP2-connected $L$-set $H$ such that $a, b \leq H \leq G$.*

**Theorem 4.8.** *Let $(X, \tau)$ and $(Y, \mu)$ be two $L$-topological spaces and $f_L^{\rightarrow} : L^X \rightarrow L^Y$ be SP-irresolute. If $G$ is SP2-connected in $(X, \tau)$, then so is $f_L^{\rightarrow}(G)$ in $(Y, \mu)$.*

**Theorem 4.9.** *Let $(X, \tau)$ be an $L$-topological space and $G \in L^X$. Then $G$ is SP2-connected if and only if for each pair $a, b$ in $M(G)$ and each SP1-remote neighborhood mapping $P : M(G) \rightarrow \mathbf{SPC}(X)$. There exists a finite number of elements $a = x_1, x_2, \cdots, x_n = b$ in $M(G)$ such that*

$$G^{'} \vee P(x_i) \vee P(x_{i+1}) \neq \underline{1}, i = 1, 2, \cdots, n-1.$$

# 5. The Relation Among SP1-connectedness, SP2-connectedness, P-connectedness and Connectedness

The following fact is obvious.

**Theorem 5.1.** *In an $L$-topological space, a P-connected $L$-set is SP1-connected. An SP1-connected $L$-set is connected.*

**Remark 5.2.** The inverse of Theorem 5.1 is not true. This can be seen from the following examples.

**Example 5.3.** Let $X = \{x_1, x_2\}$, $L = \{0, a, b, c, d, 1\}$, where $a' = a, b' = b, c' = d, d' = c, 1' = 0, 0' = 1$; $0 < d < a < c < 1, 0 < d < b < c < 1$, $a$ and $b$ are incomparable. $\forall \lambda, \mu \in L$ we define fuzzy set $C(\lambda, \mu) : X \to L$ such that

$$C(\lambda,\mu)(x) = \begin{cases} \lambda, & \text{if} \quad x = x_1, \\ \mu, & \text{if} \quad x = x_2. \end{cases}$$

Let $(X, \tau)$ be an $L$-topological space, where

$$\tau = \{C(0,0), C(a,c), C(b,d), C(d,d), C(c,c), C(1,1)\}$$

and $\Omega$ be the set of all preclosed $L$-sets. Then

$$\begin{aligned}\Omega = \{&C(0,0), C(0,a), C(0,b), C(0,c), C(0,d), C(0,1), C(a,0), C(a,a), C(a,b),\\ &C(a,d), C(b,0), C(b,c), C(b,1), C(c,0), C(c,c), C(c,1), C(d,0), C(d,a),\\ &C(d,b), C(d,c), C(d,d), C(d,1), C(1,0), C(1,c), C(1,1)\}.\end{aligned}$$

Let $\Phi$ be the set of all strongly preclosed $L$-sets, then

$$\Phi = \{C(0,0), C(a,d), C(b,c), C(c,c), C(d,a), C(d,b), C(d,c), C(d,d), C(1,1)\}.$$

For $L$-set $C(b,b)$, we have that

$$C(b,b) = C(b,0) \vee C(0,b), \text{ and } cl_p(C(0,b)) \wedge C(b,0) = C(0,b) \wedge cl_p(C(b,0)) = \underline{0},$$

hence $C(b,b)$ is not P-connected, but there don't exist strongly preclosed $L$-sets $A, B \in \Phi$ such that

$$C(b,b) \not\leq A, \ C(b,b) \not\leq B, \ C(b,b) \leq A \vee B \text{ and } C(b,b) \wedge A \wedge B = \underline{0},$$

hence $C(b,b)$ is SP1-connected.

From Example 5.3 we also see that $C(b,b)$ is SP2-connected, but $C(b,b)$ is not P-connected and hence P-connected and SP2-connected are different concepts.

**Example 5.4**. Let $X = \{x_1, x_2\}$, $L = \{0, a, b, 1\}$, where $a^{'} = a, b^{'} = b, 1^{'} = 0, 0^{'} = 1$; $0 < a < 1, 0 < b < 1$, $a \wedge b = 0, a \vee b = 1$, $a$ and $b$ are incomparable. $\forall \lambda, \mu \in L$ we define fuzzy set $C(\lambda, \mu) : X \to L$ such that

$$C(\lambda,\mu)(x) = \begin{cases} \lambda, & \text{if} \quad x = x_1, \\ \mu, & \text{if} \quad x = x_2. \end{cases}$$

Let $(X, \tau)$ be an $L$-topological space, where

$$\tau = \{C(0,0), C(1,0), C(1,1)\},$$

then we obtain that $C(0,1)$ is connected $L$-set. Now we show that $C(0,1)$ is not SP1-connected. In fact, let $\Omega$ be the set of all preclosed $L$-sets. Then

$$\begin{aligned} \Omega \;=\; & \{C(0,0), C(0,a), C(0,b), C(0,1), C(a,0), C(a,a), C(a,b), C(a,1), C(b,0), \\ & C(b,a), C(b,b), C(b,1), C(1,1)\}. \end{aligned}$$

By easy computations is follows that $C(0,a)$ and $C(0,b)$ are strong preclosed and hence by $C(0,1) \not\leq C(0,a), C(0,1) \not\leq C(0,b)$, $C(0,1) \leq C(0,a) \vee C(0,b)$ and $C(0,1) \wedge C(0,a) \wedge C(0,b) = \underline{0}$, we know that $C(0,1)$ is not SP1-connected.

**Theorem 5.5.** *Let $(X, \tau)$ be an $L$-topological space and $G \in L^X$. If $1 \in M(L)$ and $G$ is SP1-connected, then $G$ is SP2-connected.*

**Proof**. Suppose that $G$ is not SP2-connected. Then there exist two strongly preclosed $L$-sets $A, B$ such that

$$G \wedge A \neq \underline{0}, G \wedge B \neq \underline{0}, G^{'} \vee A \vee B = \underline{1}, G \wedge A \wedge B = \underline{0}.$$

By $G^{'} \vee A \vee B = \underline{1}$ and Lemma 2.10, we can obtain that $G \leq A \vee B$. This is a contradiction.

**Theorem 5.6.** *Let $(X, \tau)$ be an $L$-topological space and $G$ is a crisp subset. Then $G$ is SP1-connected if and only if it is SP2-connected.*

**Proof.** The proof is easy and omitted.

In general, SP2-connectedness doesn't imply SP1-connectedness. This can be seen from the following example.

**Example 5.7.** Let $X = \{x_1, x_2\}$, $L = \{0, a, b, c, d, 1\}$, where $a^{'} = b, b^{'} = a, c^{'} = d, d^{'} = c, 1^{'} = 0, 0^{'} = 1$, $0 < d < a < c < 1, 0 < d < b < c < 1$, $a$ and $b$ are incomparable.

$\forall \lambda, \mu \in L$ we define fuzzy set $C(\lambda, \mu) : X \to L$ such that

$$C(\lambda,\mu)(x) = \begin{cases} \lambda, & \text{if} \quad x = x_1, \\ \mu, & \text{if} \quad x = x_2. \end{cases}$$

Let $(X, \tau)$ be an $L$-topological space,
where $\tau = \{C(0,0), C(b,a), C(b,1), C(1,a), C(1,1)\}$ and $\Omega$ be the set of all preopen $L$-sets. In this case, we have that

$$\begin{aligned} \Omega = \; & \{C(0,0), C(0,a), C(0,c), C(0,1), C(a,a), C(a,c), C(a,1), C(b,0), C(b,a), \\ & C(b,b), C(b,c), C(b,d), C(b,1), C(c,0), C(c,a), C(c,b), C(c,c), C(c,d), C(c,1), \\ & C(d,a), C(d,c), C(d,1), C(1,0), C(1,a), C(1,b), C(1,c), C(1,d), C(1,1)\}. \end{aligned}$$

Let $\Phi$ be the set of all strongly preclosed $L$-sets, then

$$\Phi = \{C(0,0), C(0,b), C(0,d), C(a,0), C(a,b), C(a,d), C(d,0), C(d,a), C(d,b), C(1,1)\}.$$

For $L$-set $C(a,b)$, we have that

$$C(a,b) \not\leq C(a,0), C(a,b) \not\leq C(0,b), C(a,b) \leq C(a,0) \vee C(0,b)$$

$$\text{and } \; C(a,b) \wedge C(a,0) \wedge C(0,b) = \underline{0},$$

hence $C(a,b)$ is not SP1-connected, but there don't exist strongly preclosed $L$-sets $A, B \in \Phi$ such that

$$C(a,b) \not\leq A, C(a,b) \not\leq B, (C(a,b))^{'} \vee A \vee B = \underline{1} \; \text{ and } \; C(a,b) \wedge A \wedge B = \underline{0}.$$

Hence $G$ is SP2-connected.

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