Tamsui Oxford Journal of Mathematical Sciences **23(3)** (2007) 243-267
Aletheia University

A Model on Two Storage Inventory System Under Stock Dependent Selling Rate Incorporating Marketing Decisions and Transportation Cost with Optimum Release Rule

B. Mondal*

Department of Mathematics, Raja N. L Khan Womens' College
Midnapore, W. B., India

A. K. Bhunia

Department of Mathematics, The University of Burdwan
Burdwan -713104, W. B., India

and

M. Maiti

Department of Applied Mathematics, Vidyasagar University
Midnapore-721102, W. B., India

Received January 15, 2005, Accepted March 18, 2005.

Abstract

In the retail business, an additional storage (warehouse) is very essential to reduce the lost sale. Particularly, when the area of the existing

* E-mail: abm_mondal@rediffmail.com

storage (Owned warehouse, OW) in an important market place (like, super market, corporation market etc.) is relatively small, the need of the additional storage can be released. In this paper, a single item inventory model is developed considering two separate storage facilities (owned warehouse, OW and rented warehouse, RW) due to insufficient space of existing warehouse. The stocks of additional storage RW are transferred to OW in bulk release pattern and the associated transportation cost is taken into account. Here, we consider the effect of different stock-dependency and marketing policies such as the price variation and the frequency of advertisement on the selling rate of an item. Shortages are not allowed. To replenish the item to the storage/warehouse, transportation cost is considered explicitly. In most situations, this cost is assumed to be fixed and are, therefore, included in the ordering cost or variable and included in the cost of the item. Here, it is assumed that this transportation cost is depended on the lot-size as well as the distance between the source and the destination. Different cases of the system have been mentioned and developed. A numerical example is given to illustrate the solution procedure of the model. Finally, based on this example, a sensitivity analysis is done for the effect of different parameters on the optimal profit and cycle length.

Scope and purpose

The existing two storage models were developed based on the assumption that storage capacity of the rented warehouse (RW) is unlimited. Obviously, this is an unrealistic assumption as the capacity of storage is always limited. In this paper, a two storage inventory model with frequency of advertisement, selling price and stock-dependent selling rate is developed considering limited storage capacity of the rented warehouse. This model is solved by a gradient method based mathematical program. This methodology of model development and its solution are quite general and it can be applied to inventory models of any item whose selling rate is dependent on stock-level and marketing decisions. The purpose of this paper is to report an economic replenishment policy in which decision-makers can know when and how much amount to replenish as well as which system to choose for storing the replenished items.

Keywords and Phrases: *Stock-dependent selling rate, Inventory, Two storage, Transportation, Bulk released pattern*.

1. Introduction

Now-a-days, an item is well known to the people in the modern society through advertisement in the well-known media like, Newspaper, Magazine, Radio, TV, Cinema etc. and also through the sales representatives and/or by the glamorous display of that item in large numbers with the help of modern light and electronic arrangements. This type of advertisement and attractive display of items have a motivational effect to buy more. Observing / investigating this effect around the people, marketing researchers / practitioners have recognised the fact that displayed stock-level, selling price and advertisement for certain item has an impact on the selling rate of those items. A large amount of stocks causes higher selling rate and it is reverse when the stock-level is low However, for the selling price of an item, it is commonly observed that the lower selling price causes higher selling rate and higher price has the reverse effect. Hence, it can be concluded that the selling rate of an item is a function of displayed inventory level in a show room, selling price and the frequency of advertisement of that item. On the best of my knowledge, till now, this type of selling rate is not reported in the existing literature. Most of OR researchers / practitioners have developed inventory system as an independent and self-contained system. They do not consider the marketing decisions which affects largely in selling of an item. Very few researchers studied the effects of advertising and price variation on selling rate. Among them, Kotler [1] first considered the marketing polices into inventory decisions and discussed the relationship of pricing decision with EOQ. Ladany and Sternleib [2] discussed the effect of price variation on demand, but they did not consider the effect of advertisement. After Ladany and Sternleib [2], research works related to this topic were done by Subramanyam and Kumaraswamy [3], Urban [4], Goyal and Gunasekaran [5], Abad [6], Luo [7] and others.

Again, in the last few years, very few researchers developed inventory models incorporating the selling rate which is dependent only on displayed stock level. Baker and Urban [8] first developed this type of model. They reflected the idea that the selling rate would be decline along with the displayed inventory level throughout the entire cycle. In their model, the polynomial type of selling rate was considered for the displayed stock-level dependent selling rate. Next, Mandal and Phaujdar [9] developed independently a production inventory model considering the selling rate as a general function of the on hand stock-level during stock-in and stock-out period. Datta and Pal [10] modified the model of Baker and Urban [8] by assuming that the selling rate of an item is dependent on the displayed inventory level until a given level was achieved after which the selling rate becomes constant. Since then, very few researchers have given considerable attention on inventory problems with inventory level-dependent selling rate.

To get an idea of the trends of recent research in this area, one may refer to the works of Urban [11,12], Pal et.al.[13], Giri et.al. [14], Padmanabhan and Vrat [15], Sarkar etal [16], Giri and Chaudhuri [17] and others. All these models were developed for a single warehouse under the basic assumption that the available warehouse has unlimited storage capacity. However, this assumption is not realistic. Any warehouse has finite storage capacity. On the other hand, inventory management is generally attracted for large stock for several reasons – an attractive price discount for bulk purchase; the replenishment cost including transportation cost is higher than the inventory related cost; the demand of an item is very high and so on. Therefore one (or sometimes more than one) warehouse(s) are required to keep large stocks.

In the last few years, several researchers have discussed a two – warehouse inventory system. This system has two warehouses – the own warehouse OW and the rented warehouse RW having finite and infinite storage capacity respectively. The rented warehouse RW may be located away from OW and nearer to OW. The actual service to the customer is done at OW only. As the holding cost in RW is greater than in OW, the stocks of RW are emptied first transporting the stocks from RW to OW by continuous or bulk release patterns in order to reduce the holding cost. Approximately twenty six years before, Hartely [18] first introduced the basic two warehouse problem in his book "Operations Research – A Managerial Emphasis". In his analysis, he ignored the cost of transportation for transferring the items from RW to OW and proposed a heuristic procedure for determining the optimal order quantity. After Hartely [18], a number of research papers have been published by the different authors. Among them, the works done by Sarma [19], Dave [20], Goswami and Chaudhuri [21], Pakkala and Achary [22], Bhunia and Maiti [23,24], Benkherouf [25] Kar et. al. [26] and Zhou [27] are worth mentioning. However, all these models were based on an unrealistic assumption that the rented warehouse has unlimited (infinite) storage capacity. Again, all these models were developed considering either constant or linearly time dependent selling rate. On the best of our knowledge, no article has yet been published on two storage system with displayed inventory level, selling price and frequency of advertisement dependent selling rate. It is interesting to note that the displayed inventory, selling price of an item and advertisement through different well-known media or sales representative increases the optimal order quantity. This results the procurement of large number of items. Due to space limitation of OW (show room), an additional storage is hired on rental basis to store the excess items. Therefore, the two-warehouse system is closely related to the proposed type of selling in the sales environment.

In inventory control, it is very much important to include the transportation cost for replenishing the items should be taken into account with other inventory related costs. In the existing literature, most researchers either considered the transportation cost as fixed and included it in the replenishment cost or considered it as variable and included in the unit cost of the items. However, it is well known that different transportation alternatives have different speed, reliability and cost characteristic. Hence, the transportation cost is not independent of the order quantity. As a result, we can not ignore this cost from the analysis of the inventory system. Recently, very few researchers incorporated this cost into the lot-size determination analysis. The first serious consideration of this cost was perhaps given by Baumol and Vinod [28]. They considered an inventory model of freight transport with constant transportation cost per unit.

In this paper, we have developed a deterministic inventory model with two warehouses (one is OW and other is RW) by removing the unrealistic assumption regarding the storage capacity of the rented warehouse. Shortages are not allowed. The selling rate of the system is dependent on the selling price, advertisement of an item and displayed inventory level in OW within a range, beyond this range, the rate is constant with respect to the displayed inventory level. The stocks of RW are transferred to OW under bulk released pattern and the associated transportation cost is taken into account. Also, for replenishment of items, the transportation cost is considered and it is assumed to be dependent on the lot-size as well as the distance from the supplier's godown to the show-room. Under instantaneous replenishment with constant lead time, the model is formulated for L_1 and L_2 system separately (L_1 – system and L_2 system refer to the system with single and two warehouses facilities respectively). Different cases of the system are clearly discussed. To find out the solution of each case and the whole system, two separate algorithms are developed. A numerical example is given to illustrate the solution procedure of the model. Finally, based on this example, the effect of different parameters on the optimal profit is observed by sensitivity analysis taking one or more parameters at a time.

2. Assumptions and Notations

The following assumptions and notations are used to derive the proposed mathematical model:

1. Replenishments are instantaneous with a known constant lead time.
2. The inventory planning horizon is infinite and the inventory system involves only one item and one stocking point.
3. Only a single order will be placed at the beginning of each cycle and the entire lot is delivered in one batch.
4. Shortages are not allowed.
5. The replenishment cost (ordering cost) is constant and does not include the transportation cost for replenishing the items.
6. There is no quantity discount.

7. The storage capacity of owned warehouse (OW) is W and that of rented warehouse (RW) is Q_R.
8. Q and S be the order quantity for L_1 &L_2- System respectively.
9. The items of RW are transferred to OW in n shipments of which K ($K \leq W$) units are to be transported in each shipment except the last. In the last shipment, $S'(\leq K)$ units are transported.
10. T be the total time period (cycle length).
11. The inventory carrying costs per unit per unit time in OW and RW are respectively H and F such that $F > H$.
12. If the lot-size S is less than the storage capacity of OW, the entire lot is kept in OW. This type of inventory system is assumed as L_1-System. Otherwise, first OW will be filled completely and the excess amount will be stored in RW. In this case, an additional transportation cost is incurred for special dispatch of goods to RW. This is known as L_2-System
13. The selling price p is determined by a mark-up (λ) over the purchase cost C_1 i.e. $p = \lambda C_1$ $(\lambda > 1)$.
14. A be the frequency of advertisement per cycle and G the cost per advertisement.
15. For transferring the items from RW to OW, P is the maximum number of units which can be transported under a fixed charge a' and for every additional unit after P, a variable charge b' is to be paid.
16. t_1 be the consumption period of K units.
17. t'_i $(i = 1,2,\cdots,n)$ be the consumption period of the first iK units i.e. $t'_i = it_1$.
18. t_2 be the consumption period of the last W-$K + S'$ units in OW.
19. The value of H, a, b, p and c are so choosen that H(a-bp)> c holds.
20. The value of $F, n, H, A, \gamma, S_0, K, a, b, p, c$, *are so choosen that* $2Fn^2H(a-bp)A^\gamma(a-bp+cS_0)+n^2c + Xc>(T^2nH(a-bp+c(S-nK))$ *holds.*

For replenishment of items, the following assumptions and notations are taken due to transportation.

21. The transportation cost is constant for a transport vehicle (of a given capacity) even if the quantity shipped is less than a transport vehicle load by some quantity.
22. The capacity of a transport vehicle is K' units.
23. L be distance between the show-room/shop and the source of the items/commodities from where items/commodities are to be transported.
24. C_t be the transportation cost for full load of transport vehicle and C_{tF} be the transportation cost per unit item.

    Then $C_t = C_{tj}$ and $C_{tF} = C_{tFj}$ where $L'_{j-1} \leq L \leq L'_j$ for $j = 1,2,3,\cdots$

    Here L'_{j-1} and L'_j be the lower and upper cut off distances.

25. C_{ad} be an additional transportation cost per item incurred for special despatch of goods to the rented warehouse RW.
26. U_j be the upper break point for $L'_{j-1} \leq L \leq L'_j$. Some quantity than K' but above U_j, the fixed transportation cost for the whole quantity is C_{tj}.

Hence, $U_j = [C_{tj}/C_{tFj}]$ where $[C_{tj}/C_{tFj}]$ represents the greatest integer value which is less than or equal to C_{tj}/C_{tFj}.

The selling rate function

In Baker and Urban [8], it was established that the selling rate of an item is a function of the instantaneous stock-level. They reported that this rate is not constant. It changes along with the stock-level throughout the order cycle. Therefore, the higher stock-level will cause the greater selling rate during the beginning of the cycle. According to the real life situation, this rate is not only dependent on the stock-level. It also depends on the effect of marketing policies and conditions such as the price variations and the advertisement.

The deterministic selling rate $D(A,p,q)$ of an item is a known function of marketing parameters [like, the frequency of advertisement (A), the selling price (p)] and the displayed inventory level in the showroom/shop within the inventory level S_0 to S_1 and beyond this range it becomes constant with respect to the displayed inventory level. These functional forms may be :

$$
\begin{aligned}
D(A,p,q) &= f(A,p,S_1) \quad \text{for } q > S_1 \\
&= f(A,p,q) \quad \text{for } S_0 < q \leq S_1 \\
&= f(A,p,S_0) \quad \text{for } 0 \leq q \leq S_0
\end{aligned}
$$

where $f(A,p,q)$ is a function of A, p, q. Also, it is a differentiable function of q.

The functional form of the stock-dependent selling rate will be in different form. These functional form may be power form (αq^{β}), exponential form [$\alpha \exp(\beta q)$], linear form ($\alpha + \beta q$), quadratic form ($\alpha + \beta q + \gamma q^2$) with respect to the instantaneous stock-level. The power form selling rate results zero selling when the stock-level reaches to zero. It is always remembered that the demand of an item can not be zero, Generally, the selling rate does not fully dependent on the instantaneous displayed stock-level. Due to some realistic factors, such as goodwill, good quality, genuine price-level of the goods, locality of the shop etc., a limited number (may be considered as constant) of customers arrives to purchase the goods. Hence the selling rate of an item of a particular shop will be more appropriate if the linear form of the selling rate with respect to the stock-level. In this case, we consider the linear form of the displayed inventory, the selling price of an item and polynomial form of the frequency of the advertisement for $f(A,p,q)$. Suppose $f(A,p,q) = A^{\gamma}(a-bp+cq)$ where $a, b, c, \gamma \geq 0$.

3. Transportation Costs

3.1. *Transportation Cost for shipment of items from RW to OW:*

The Transportation cost for transferring the items/goods from RW to OW in n shipment is given by

$$TC1 = (n-1)\{a'+b'(K-P)\}+a'+b'(S'-P) \quad \text{when } K, S'>P$$
$$= (n-1)\{a'+b'(K-P)\}+a' \quad \text{when } K>P, S'\leq P$$
$$= na' \quad \text{when } K, S'\leq P$$

3.2. *Transportation Cost for replenishing the order quantity*

When the order quantity is greater than one integral transport vehicle load, the order quantity, Q' ($Q' = S$ for L_2-System and $Q' = Q$ for L_1-System) can be expressed as

$$Q' = m*K' + \mu*q' \quad \text{where } m = 0,1,2,3,\cdots;\ \mu = 0 \text{ or } 1 \text{ and } q' < K'$$

In that case, two situations may arrise :

$$(i)\ m*K' < Q' \leq m*K'+U_j,\ (ii)\ m*K'+U_j < Q' \leq (m+1)*K'$$

Hence the transportation cost for L_1 – system is given by

$$TC2 = m*C_{tj} + (Q'-m*K')*C_{tFj} \quad \text{where } m*K' < Q' \leq m*K'+U_j$$
$$= (m+1)*C_{tj} \quad \text{where } m*K'+U_j < Q' \leq (m+1)*K'$$

and for L_2 – system,

$$TC3 = m*C_{tj} + (Q'-m*K')*C_{tFj} + C_{ad}(Q'-W) \quad \text{where } m*K' < Q' \leq m*K'+U_j$$
$$= (m+1)*C_{tj} + C_{ad}(Q'-W) \quad \text{where } m*K'+U_j < Q' \leq (m+1)*K'$$

The total transportation cost is given by

$$C_{tran} = TC2 \quad \text{for } L_1 \text{ – system}$$
$$= TC1 + TC3 \quad \text{for } L_2 \text{ – system}$$

4. Analysis of Two Storage System (L_2 – System)

Initially, an enterprise purchases S $(S>W)$ units and of which W units are stored in the own warehouse (OW) and the rest S-W units in the rented warehouse (RW). At first, the stocks of OW are used to meet the customer's demand until the stock-level in OW drops to (W-K) units at the end of time t_1. At this stage, K $(K \leq W)$ units from RW are transported to OW so that the stock-level of OW again becomes W to meet further demands. This process is continued for (n-1) such shipments. In the last shipment, the remaining S' $(S' \leq K)$ units in RW are transported to OW. After the last shipment, the stock-level of OW will be $(W\text{-}K+S')$ units which are used as usual to satisfy the demand during $[t_n, T]$. This entire cycle is repeated after each scheduling period T.

To analyse this model, nine different scenarios (first 3 scenario for $S_1>W$ and last 6 scenarios for $S_1<W$) may arise according to the relative size of S_0, S_1, W, the lot-size S in the beginning of the cycle, transported amount K, S' per shipment.

Scenario – 1 : $S_1 > W$, $S_0 > W\text{-}K$ and $W\text{-}K+S' > S_0$
Scenario – 2 : $S_1 > W$, $S_0 > W\text{-}K$ and $W\text{-}K+S' < S_0$
Scenario – 3 : $S_1 > W$, $S_0 < W\text{-}K$
Scenario – 4 : $S_1 < W$, $S_0 > W\text{-}K$ and $W\text{-}K+S' > S_1$
Scenario – 5 : $S_1 < W$, $S_0 > W\text{-}K$ and $S_0 < W - K+S' < S_1$
Scenario – 6 : $S_1 < W$, $S_0 > W\text{-}K$ and $W\text{-}K+S' < S_0$
Scenario – 7 : $S_1 < W$, $S_0 < W\text{-}K$ and $W\text{-}K+S' > S_1$
Scenario – 8 : $S_1 < W$, $S_0 < W\text{-}K$ and $W\text{-}K+S' < S_1$
Scenario – 9 : $S_1 < W – K$

Note that S_0 may be greater than W. In that case, the selling rate will be constant with respect to the displayed inventory in the show-room/OW. This contradicts that the proposed selling rate patterns. As a result, we shall reject the case S_0 >W.

Now, we shall study the Scenario-1 in details.

Scenario-1 : $S_1 > W$, $S_0 > W\text{-}K$ and $W\text{-}K+S' > S_0$

In this scenario, the stock-dependent selling rate is observed when the inventory level drops from W to S_0 and beyond S_0, it is constant. The pictorial representations of the inventory system in RW and OW are given in Figure-1 and Figure – 2 respectively.

As the demands are met only from the owned warehouse (OW), the stock depletion at OW is only due to the demand only of the items. Therefore, the inventory level $q(t)$ at *time* $t(0 < t < t_1)$ satisfies the following differential equations :

$$dq(t)/dt = -f(A,p,q) \quad \text{for } S_0 < q \leq W \qquad (1)$$
$$= -f(A,p,S_0) \quad \text{for } q \leq S_0 \qquad (2)$$

According to the assumptions, the relation between S, K and S' is given by

$$S - W = (n-1)\, K + S' \qquad (3)$$

Since the selling rate is a function of instantaneous displayed stock – level in OW, the time taken for the consumption of K units and the last $W - K + S'$ units in OW would depend upon the stock – level. So,

$$t_1 = \int_{S_0}^{W} \frac{dq}{f(A,p,q)} + \int_{W-K}^{S_0} \frac{dq}{f(A,p,S_0)} \qquad (4)$$

and

$$t_2 = \int_{S_0}^{W-K+S'} \frac{dq}{f(A,p,q)} + \int_{0}^{S_0} \frac{dq}{f(A,p,S_0)} \qquad (5)$$

Again, the total time period T is given by

$$T = nt_1 + t_2 \qquad (6)$$

The cost function

The total cost in a cycle consists of the following components :
(i) Ordering cost (C_4), (ii) Inventory carrying cost (C_{hol}), (iii) Transporation cost (C_{trans}), (iv) Purchase cost ($C_1 * S$), (v) Advertisement cost (C_{adv})

Inventory carrying cost : The inventory carrying cost per unit time can be expressed as the product of the inventory level and carrying cost per unit per unit time. Thus, the total inventory carrying cost is given by

$$C_{hol} = F\{n(n-1)K/2 + nS'\}t_1 + H\left[n\left\{\int_{S_0}^{W} \frac{qdq}{f(A,p,q)} + \int_{W-K}^{S_0} \frac{qdq}{f(A,p,S_0)} + (W-K)nt_1\right.\right.$$

$$+ \int_{S_0}^{W-K+S'} \frac{qdq}{f(A,\ p,q)} + \int_{0}^{S_0} \frac{qdq}{f(A,p,S_0)} \ ] \tag{7}$$

The detailed calculations are given in Appendix – B.

Advertisement Cost : The total advertisement cost is the product of the number of advertisement and the cost per advertisement i.e.

$$C_{adv} = A*G$$

The total cost of this *system* (*Scenario – 1*) *is given by*

$$TC = C_4 + C_1S + C_{hol} + C_{tran} + C_{adv} \tag{8}$$

The profit function

The net profit for the entire system (Scenario – 1) is the difference between the sales revenue per cycle and the total cost of the system i.e.,

$$X = (p - C_1)S - C_4 - C_{hol} - C_{tran} - C_{adv} \tag{9}$$

Therefore, for the fixed value of the mark– up rate, the profit function $\pi_2^{(1)}\ (m,n,S,K)$ (Average profit per unit time for the cycle) of the inventory system (Scenario – 1) is given by

$$\pi_2^{(1)}\ (m,n,S,K) = X/T \tag{10}$$

Here, the profit function is a function of two continuous variables S and K and two discrete variables m and n.

Theorem 1. *The profit function* $\pi_2^{(1)}\ (m,n,S,K)$ *is concave in S and K for fixed m and n.*

Proof. See Apendix –A

Hence, our problem is to find the optimal values of n, S and K by maximizing the profit function $\pi_2^{(1)}\ (m,n,S,K)$. For $n = 1,2,3,\cdots,$ maximizing $\pi_2^{(1)}\ (m,n,S,K)$, the values of S and K along with the value of $\pi_2^{(1)}$ are calculated. The largest value of $\pi_2^{(1)}$ for $n = 1,2,3,\cdots.$ is the optimal profit and the corresponding value of n, S and K

are the optimal solution of Scenario – 1. The optimal values of S' and T can be calculated from (3) and (6) respectively. However, for the particular value of n (like, $n=1,2,3,\cdots$), the method for finding the solution of Scenario – 1 is summarized in the following Algorithm –1.

Algorithm –1

Step – 1 : Input all the parameters except m.
Step – 2 : Find out the value of U_j.
Step – 3 : Set $m = 0$.
Step – 4 : Solved the problem on on taking the transportation cost for first situation ($m^*K' < S < m^*K' + U_j$).
Step – 5 : If $S \in [\, m^* K', m^* K'+U_j]$, then this is the optimal policy with respect to m and go to step – 6
Step – 6 : Solve the problem on tahing the transportation cost for second situation $\{m^* K' + U_j < S \le (m+1) K'\}$
Step – 7 : If $S \in [m^* K'+ U_j, (m+1)^* K']$, then this is the optimal policy with respect to m and go to step – *9*, otherwise, go to step – *8*.
Step – 8 : *Increase m* by 1 i.e., $m = m+1$ and go to step – 4.
Step – 9 : Stop.

In this Scenario – 1, let m^*, n^*, S^* and K^* be the optimal values of m, n, S and K. But, this solution is obtained without considering the capacity constrant of RW i.e., $S - W \le Q_R$ as the storage capacity of RW is limited. If $S\text{-}W \le Q_R$, it is obvious that n^*,S^* and K^* be the feasible solution of Scenario – 1 and $\pi_2^{*(1)} = \pi_2^{(1)}(m^*,n^*,S^*,K^*)$. Otherwise, $\pi_2^{(1)}$ is equal to the optimal bondary profit $\pi_2^{(1)}(n^*, K^*)$ when $S = Q_R+W$(in that case, m is fixed and easily be obtained). Now, We shall determine the optimal boundary profit. At first, fixing S by Q_R+W, then from (3), we have

$$Q_R = (n\text{-}1)K + S'$$
$$\text{i.e.,} \quad S' = Q_R - (n\text{-}1)K \tag{11}$$

In this case, the boundary profit function of Scenario – 1 is given by

$$\pi_2^{(1)}(n,K) = [(p\text{-}C_1)(Q_R+W) - C_4 - F\,\{\, nQ_R - n(n\text{-}1)\,K/2\}\; t_1$$

$$-H\{\; n\Big(\int_{S_0}^{W} \frac{qdq}{f(A,p,q)} + \int_{W\text{-}K}^{S_0} \frac{qdq}{f(A,p,S_0\,)} \Big) + (W\text{-}K)nt_1$$

$$+\int_{S_0}^{W-K+S'}\frac{qdq}{f(A,p,q)}+\int_{0}^{S_0}\frac{qdq}{f(A,p,S_0)}\ \} - C_{tran} - A^*G\ ]/T \qquad (12)$$

which depends on *n* and *K* only.

It can easily be proved that the boundary profit function $\pi_2^{(1)}(n,K)$ is concave in *K* for the fixed value of the discrete variable *n.*

For other Scenarios, the profit function , $\pi_2^{(i)}$, *i* represents the number of Scenario, are as follows :

$$\pi_2^{(i)} = [(p-C_1)S - C_4 - F\ \{n(n-1)K/2+nS'\}t_1 - H(W-K)nt_1\ \ H^*d_i - C_{tran} - A^*G\ ]/T,$$

$$i = 2,3,\cdots,9 \qquad (13)$$

where

$$d_2 = n\ \{\int_{S_0}^{W}\frac{qdq}{f(A,p,q)}+\int_{W-K}^{S_0}\frac{qdq}{f(A,p,S_0)}\ \} + \int_{0}^{W-K+S'}\frac{qdq}{f(A,p,S_0)} \qquad (14)$$

$$d_3 = n\ \{\int_{W-K}^{W}\frac{qdq}{f(A,p,q)}+\int_{S_0}^{W-K+S'}\frac{qdq}{f(A,p,q)}\ \} + \int_{0}^{S_0}\frac{qdq}{f(A,p,S_0)} \qquad (15)$$

$$d_4 = n\ \{\int_{S_1}^{W}\frac{qdq}{f(A,p,S_1)}+\int_{S_0}^{S_1}\frac{qdq}{f(A,p,q)}+\int_{W-K}^{S_0}\frac{qdq}{f(A,p,S_0)}\ \} + \int_{S_1}^{W-K+S'}\frac{qdq}{f(A,p,S_1)}$$

$$+\int_{S_0}^{S_1}\frac{qdq}{f(A.p.q)}+\int_{0}^{S_0}\frac{qdq}{f(A,p,S_0)} \qquad (16)$$

$$d_5 = n\{ \int_{S_1}^{W} \frac{qdq}{f(A,p,S_1)} + \int_{S_0}^{S_1} \frac{qdq}{f(A,p,q)} + \int_{W-K}^{S_0} \frac{qdq}{f(A,p,S_0)} \} + \int_{S_0}^{W-K+S'} \frac{qdq}{f(A,p,q)}$$

$$+ \int_{0}^{S_0} \frac{qdq}{f(A,p,S_0)} \tag{17}$$

$$d_6 = n\{ \int_{S_1}^{W} \frac{qdq}{f(A,p,S_1)} + \int_{S_0}^{S_1} \frac{qdq}{f(A,p,q)} + \int_{W-K}^{S_0} \frac{qdq}{f(A,p,S_0)} \} + \int_{0}^{W-K+S'} \frac{qdq}{f(A,p,S_0)} \tag{18}$$

$$d_7 = n\{ \int_{S_1}^{W} \frac{qdq}{f(A,p,S_1)} + \int_{W-K}^{S_1} \frac{qdq}{f(A,p,q)} + \int_{S_1}^{W-K+S'} \frac{qdq}{f(A,p,S_1)} \} + \int_{S_0}^{S_1} \frac{qdq}{f(A,p,q)}$$

$$+ \int_{0}^{S_0} \frac{qdq}{f(A,p,S_0)} \tag{19}$$

$$d_8 = n\{ \int_{S_1}^{W} \frac{qdq}{f(A,p,S_1)} + \int_{W-K}^{S_1} \frac{qdq}{f(A,p,q)} \} + \int_{S_0}^{W-K+S'} \frac{qdq}{f(A,p,q)} + \int_{0}^{S_0} \frac{qdq}{f(A,p,S_0)} \tag{20}$$

$$d_9 = n \int_{W-K}^{W} \frac{qdq}{f(A,p,S_1)} + \int_{S_1}^{W-K+S'} \frac{qdq}{f(A,p,S_1)} + \int_{S_0}^{S_1} \frac{qdq}{f(A,p,q)} + \int_{0}^{S_0} \frac{qdq}{f(A,p,S_0)} \tag{21}$$

Like Scenario – 1, it can easily be prove that the profit functions for different scenarios are concave with respect to S and K. The solutions of these Scenarios along with optimal profit $\pi_2^{(i)}$ $(i=2,3,\cdots 9)$ can be obtained by the same procedure used in scenario – 1.

Optimal solution of L_2 – System

The optimal solution in this system will be the solution corresponding to the maximum average profit (for case $S_1>W$ and $S_1< W$) of the above mentioned Scenarios. If π_2^* be the average profit then

$$\pi_2^* = \max \pi_2^{*(i)},\ i = 1,2,3 \text{ for } S_1>W$$
$$= \max \pi_2^{*(i)},\ i = 4,5,\cdots,9 \text{ for } S_1<W \tag{22}$$

5. Analysis of Single Storage System (L_1 – system)

In this system, an amount of Q units is stored in OW ($Q \le W$) at time $t = 0$. With the passage of time, this amount will be depleted gradually to meet the customers demand and at the end of the period T, the stock level will be zero. This entire cycle is repeated after each scheduling period T

To analyse the system, three different scenarios may arise according to the relative size of S_0 and S_1.

Scenario – I : $Q > S_1$
Scenario – 2 : $S_0 < Q < S_1$
Scenario – 3 : $Q < S_0$

In these Scenarios, the profit function $\pi_1^{(j)}(m,Q)$, j represents the numbers of scenario, are as follows :

$$\pi_1^{(1)}(m,Q) = [(p-C_1)Q - C_4 - H\{ \int_{S_1}^{Q} \frac{qdq}{f(A,p,S_1)} + \int_{S_0}^{S_1} \frac{qdq}{f(A,p,q)} + \int_{0}^{S_0} \frac{qdq}{f(A,p,S_0)} \}$$

$$- C_{tran} - A^*G \tag{23}$$

$$\text{where } T = \int_{S_1}^{Q} \frac{dq}{f(A,p,S_1)} + \int_{S_0}^{S_1} \frac{dq}{f(A,p,q)} + S_0/f(A,p,S_0) \tag{24}$$

$$\pi_1^{(2)}(m,Q) = [(p-C_1)Q - C_4 - H\{ \int_{S_0}^{Q} \frac{qdq}{f(A,p,q)} + \int_{0}^{S_0} \frac{qdq}{f(A,p,S_0)} \} - C_{tran} - A^*G\ ]/T$$

(25)

where $T = \int_{S_0}^{Q} \frac{dq}{f(A,p,q)} + S_0/f(A,p,S_0)$ (26)

$$\pi_1^{(3)}(m,Q) = [(p-C_1)Q - C_4 - H \int_0^Q \frac{qdq}{f(A,p,S_0)} - C_{tran} - A*G]/T \quad (27)$$

where $T = Q/f(A,p,S_0)$ (28)

The optimal solution of L_1 – System can be determined in the following way :

$$\pi_l^* = max\ \pi_1^{(j)}, j = 1,2,3 \quad (29)$$

6. Solution Procedure of the Proposed Inventory System

It may be noted that the result of L_2 – system is feasible if $S > W$. Again, if π_2^* be the maximum average profit of the L_2 – system, it has to be compared with the boundary profit of L_1 – system with $Q=W$ given by $\pi_1^*(W)$ where $\pi_1^*(W)$ can be computed from (29) by substituting $Q=W$.

We now suggest the following algorithm for finding out the optimal solution of the proposed inventory system

Algorithm – 2 :

Step – 1 : Solve the L_1 – System for m and Q
If $Q< W$, set $S_1^* = Q$, otherwise set $S_1^* = W$

Step – 2 : Solve the L_2 – system for m, n, S and K.
If $S>W$ set $S_2^* = S$, otherwise, set $S_2^* = W$

Step – 3: If $\pi_2^*(S_2^*) > \pi_1^*(S_1^*)$, then the optimal replenishment quantity will be $Q_{opt} = S_2^*$, otherwise, $Q_{opt} = S_1^*$. The corresponding solution of Q_{opt} will be the optimal solution.

Step – 4 : Stop.

7. Numerical Result

To illustrate the developed model, an example has been considered. Though the values of the model parameters have not been selected from any case study, the values considered here are feasible.

Example. Let $C_4 = 200, H = 1, F = 1.5, C_1 = 20, W = 100, Q_R = 600, a = 500, b = 0.5, C = 0.3, S_0 = 50, A = 5, G = 50, a' = 20, b' = 0.5, P' = 20,$. $= 0.2,$. $= 1.3, C_{ad} = .2, K' = 100, C_t = 100, C_{tF} = 1.25$ for $L = 65$ in appropriate units.

According to the solution procedure of L_j -System (j = 1,2), the solution of different scenarios and then the optimal solution for two different values of S_1 are obtained with the help of well known non-linear optimization package (viz., LINGO). Results are given in the Table - 1.

8. Sensitivity Analysis

The earlier numerical example is used to study the effect of under or over estimation of various parameters on optimal cycle length and profit of the inventory system. The percentage changes in cycle length and profit are taken as measure of sensitivity. The analysis is carried out by changing (increasing and decreasing) the parameters for *–20%* to *+20%*, taken are or more parameters at a time and making the other parameters their original values. The results of this analysis are given in Table – 2 which are self-explanatory. From sensitivity on the demand parameters A and p i.e. frequency of advertisement and selling price it is seen that when frequency of advertisement increases then profit increases up to certain range of increase but after that it will decrease. When selling price increases then demand will decrease but the unit price per unit item increase so profit will increase up to certain range of increase after that it will decrease.

9. Conclusions

In the present paper a deterministic two-warehouse inventory model has been developed by removing the unrealistic assumption regarding the storage capacity of the rented warehouse RW in the existing two warehouse systems which are discussed by Hartely [18], Sarma [19], Dave [20] and others. In developing the model, more realistic selling rate (dependent on selling price, advertisement, displayed inventory level in OW) and transportation cost for replenishing the items are considered. The model has been developed for items considering different scenarios depending on the level of stock dependency in selling rate, storage capacity of OW and the order quantity of the system. The impact of proposed selling rate and the optimal profit is reported. The results indicate that the effect of proposed sales on the system behaviour are significant and hence should not be ignored in developing the model

Another feature in this paper is that we have incorporated a new type selling situation which is not considered by others. The proposed selling situation can be seen to occur in cases where the customers arrive to purchase goods attracted by advertisement and glamorous display of items in a show room. This effect continues within a certain range of displayed inventory in the show room. Beyond the upper level, it will be constant (in that case for increase of displayed inventory, selling rate will not be increased with respect to that level). Also, beyond the lower level, only a limited number (may be considered as constant) of customers arrive to purchase the goods due to different factors – such as goodwill of the shop, good quality, genuine price-level of the goods, locality of the shop etc.

The two-warehouse model can be applied to many practical situations. At present, due to introduction of open market policy, the business competition becomes very high to occupy much more profit in the sales market. For this reason, in order to attract more customer a departmental store is forced to provide the customers a better purchasing-environment such as well decorated show-room with modern light and electronic arrangements and enough free space for choosing items. Again, due to the expending of market situation, there is a crisis of space in the market places specially in the super marker, corporation market etc. As a result, the management of departmental store is bounded to hire a separate warehouse on rental basis at a distance place for storing of excess items. Hence, from the economical point of view, the two-storage system is more profitable than the single storage system.

Another feature in this paper is that we have taken into account the transportation cost for replenishing the order quality in a realistic manner. In the existing literature, the transportation cost is considering either fixed and included it in the replenishment cost or variable and considered as a part of the unit cost of the item. In this paper we have explicitly the transportation costs for replenishing the order quantity to own warehouse (OW) as well as rented warehouse (RW). For feature development of research, one can extend the model developed in this article for deterioration items, multiple items, quantiity discount policies and probabilistic demand.

References

[1] P. Kotler, *Marketing Decision Making: A Model Building Approach.* Holt. Rinehart, Winston, New York 1971.

[2] S. Ladany and A. Sternleib, The intersection of economic ordering quantities and marketing policies. *AIIE Transactions* **6** (1974), 35-40.

[3] S. Subramanyam and S. Kumaraswamy, EOQ formula under varying marketing policies and conditions. *AIIE Transactions* **13** (1981), 312-314.

[4] T. C. Urban, Deterministic inventory models incorporating marketing decisions. *Computers and Industrial Engineering* **22** (1992), 85-93.

[5] S. C. Goyal and A. Gunasekaran, An integrated production – inventory marketing model for deteriorating items. *Computers and Industrial Engineering* **28 (**1995), 755 – 762.

[6] P. L. Abad, Optical pricing and lot-sizing under conditions perishability and partial backordering, *Management Science* **42** (1996), 1093-1104.

[7] W. Luo, An integrated inventory system for perishable goods with backordering, *Computers and Industrial Engineering* **34** (1998), 685 – 693.

[8] R. C. Baker and T. L. Urban, A deterministic inventory system with an inventory-level-dependent demand rate, *Journal of Operational Research Society* **39** (1998), 823-831.

[9] B. N. Mandal and S. Phaujdar, An inventory model for deteriorating items and stock-dependent consumption rate, *Journal of Operational Research Society* **4** (1989), 483-488.

[10] T. K. Datta and A. K. Pal, A note on an inventory model with inventory level-dependent demand rate, *Journal of Operational Research Society* **41** (1990), 971-975.

[11] T. L. Urban, An inventory model with an inventory-level-dependent demand rate and relaxed terminal conditions, *Journal of Operational Research Society* **43** (1992), 721-724.

[12] T. L. Urban, Inventory models with the demand rate dependent on stock and shortage levels, *International Journal Production Economics* **40 (**1995), 21-28.

[13] S. Pal, A. Goswami A, and K. S. Chaudhuri, A deterministic inventory model for deteriorating items with stock-dependent demand rate, International Journal Production Economics **32** (1993), 291-299.

[14] B. C. Giri, S. Pal, A. Goswami, and K. S. Chaudhuri. An inventory model for deteriorating items with stock-dependent demand rate, *European Journal of Operational Research* **95** (1996), 604-610.

[15] G. Padmanabhan and P. Vrat, EOQ models for perishable items under stock-dependent selling rate, *European Journal of Operational Research* **86** (1995), 281-292.

[16] B. R. Sarkar, S. Mukherjee, and C. V. Balan, An order-level lot-size inventory model with inventory-level dependent demand and deterioration, *International Journal Production Economics* **48** (1997), 227-236.

[17] B. C. Giri and K. S. Chaudhuri, Deterministic models of perishable inventory with stock-dependent demand rate and non-linear holding cost, *European Journal of Operational Research* **105** (1998), 67-474.

[18] R. V. Hartely, *Operations Research - A Managerial Emphasis*, Good Year Publishing Company, California (1976), 315-317.

[19] K. V. S. Sarma, A deterministic inventory model with two levels of storage and an optimum release rule, *Opsearch* **20** (1983), 175-180.

[20] U. Dave, On the EOQ models with two levels of storage, *Opsearch* **25** (1988), 190-196.

[21] A. Goswami and KS, Chaudhuri, An economic order quantity model for items with two level of storage for a linear trend in demand, *Journal of Operational Research Society* **43** (1992), 157-167.

[22] T. P. M. Pakkala and K. K. Achary, A deterministic inventory model for deteriorating items with two warehouses and finite replenishment rate, *European Journal of Operational Research* **57** (1992), 71-76.

[23] A. K. Bhunia and M. Maiti, A two warehouse inventory model for a linear trend in demand, *Opsearch* **31** (1994), 318-329.

[24] A. K. Bhunia and M. Maiti, A two warehouses inventory model for deteriorating items with a linear trend in demand and shortages, *Journal of Operational Research Society* **49** (1998), 289-292.

[25] L. Benkherouf, A deterministic order level inventory model for deteriorating items with two storage facilities, *International Journal Production Economics* **48** (1997), 167-175.

[26] S. K. Kar, A. K. Bhunia, and M. Maiti, Deterministic inventory model with levels of storage, a linear trend in demand and a fixed time horizon, *Computer and Operations Research* **28** (2001), 1315 – 1331.

[27] Y. W. Zhou, An optimal EOQ model for deteriorating items with two warehouses and time varying demand, *Mathematica Applicata* **10** (1998), 19 – 23.

[28] W. J. Beumol and H. C. Vinod, An inventory theoretic model of freight transport demand, *Management Science* **16 (**1970), 413 – 421.

Appendix - A

The proof of this theorem is easily done by showing that the associated Hessian matrix for maximizing the profit function given in (10). The profit function is maximum when Hassian matrix is negative definite. Hassian matrix is negative definite when its first and second order principal minors are alternately –ve and +ve.

The first principal minor with respect to S is

$$\partial^2 \pi_2^{(1)}/\partial S^2 = [\, T(H(a-bp)-c) + Xc]/T^2A^{\gamma}(a-bp+c(S-nK)) < 0 \quad [\text{ by assumption (19)}] \qquad (1A)$$

The second principal minor with respect to S, K is

$$(\partial^2 \pi_2^{(1)}/\partial S^2)(\partial^2 \pi_2^{(1)}/\partial K^2) - (\partial^2 \pi_2^{(1)}/\partial S\partial K)^2$$
$$= (\, T(H(a-bp)-c) + Xc))/(T^2A^{2\gamma}(a-bp+c(S-nK))^2\,(a-bp+cS_0))$$
$$[\, 2Fn^2H(a-bp)A^{\gamma}(a-bp+cS_0)+n^2c - (T^2nH(a-bp+c(S-nK)) + Xc\,] > 0 \quad [\text{ by assumption (20)}] \qquad (2A)$$

For our assumptions *(19)* and *(20)* the conditions *(1A)* and *(2A)* are automatically satisfied.

Appendix - B

The inventory time units in RW is

$$(S-W)t_1+(S-W-K)\, t_1+(S-W-2k)\, t_1 + \cdots + S' t_1 =\{n(n-1)K/2 + nS'\}\, t_1 \qquad \text{(B-1)}$$

and hence, the inventory carrying cost for these units in RW

$$= F\{n(n-1)K/2 + nS'\}t_1$$

Between (i-1)-th and i-th ($i=1,2,\cdots n$) shipment for transferring items/goods from RW to OW, only K units in OW are used to meet the demand and the rest (W-K) units are kept unused in OW for a period of length $(t'_i - t'_{i-1})$ *i.e,* t_1. So, the inventory carrying cost for these items in OW is

$$H\left[\int_{S_0}^{W} \frac{qdq}{f(A,p,q)} + \int_{W-K}^{S_0} \frac{qdq}{f(A,p,S_0)} + (W-K)t_1\right. \qquad \text{(B-2)}$$

Again when the last shipment arrives in OW, the on hand inventory in OW becomes $W-K+S'$. The inventory carrying cost for these units during usage in OW is

$$H\left[\int_{S_0}^{W-K+S'} \frac{qdq}{f(A,p,q)} + \int_{0}^{S_0} \frac{qdq}{f(A,p,S_0)}\right] \qquad \text{(B-3)}$$

Hence, the total inventory carrying cost in OW is

$$H\left[n\left\{\int_{S_0}^{W} \frac{qdq}{f(A,p,q)} + \int_{W-K}^{S_0} \frac{qdq}{f(A,p,S_0)}\right\} + (W-K)nt_1\right.$$

$$\left. + \int_{S_0}^{W-K+S'} \frac{qdq}{f(A,p,q)} + \int_{0}^{S_0} \frac{qdq}{f(A,p,S_0)}\right] \qquad \text{(B-4)}$$

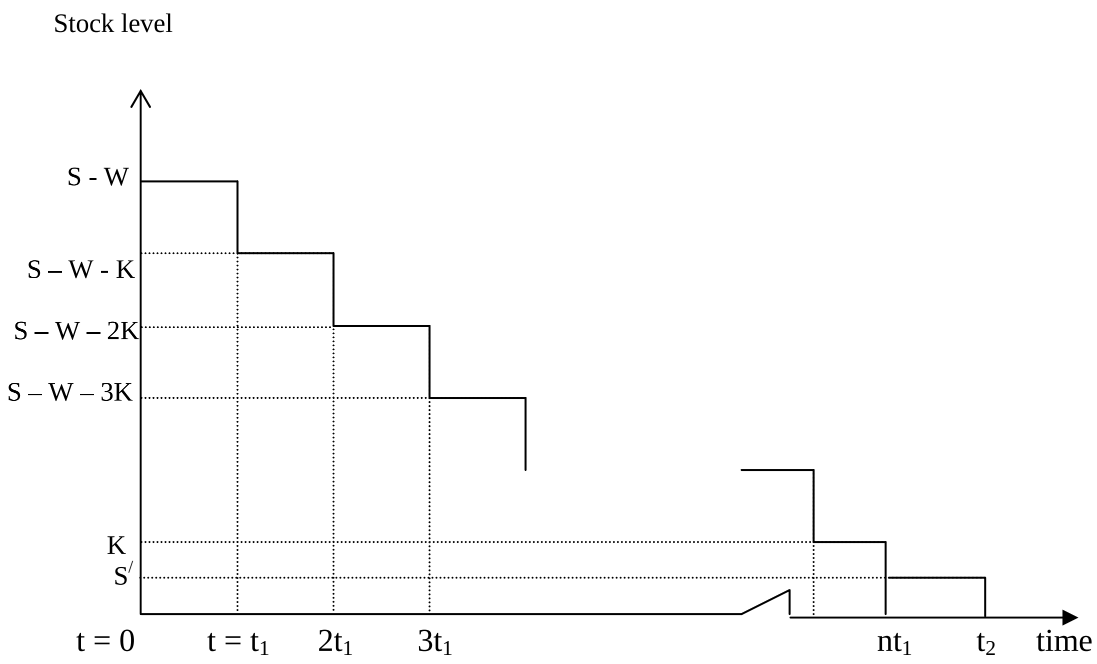

Figure – 1 : The inventory situation in RW

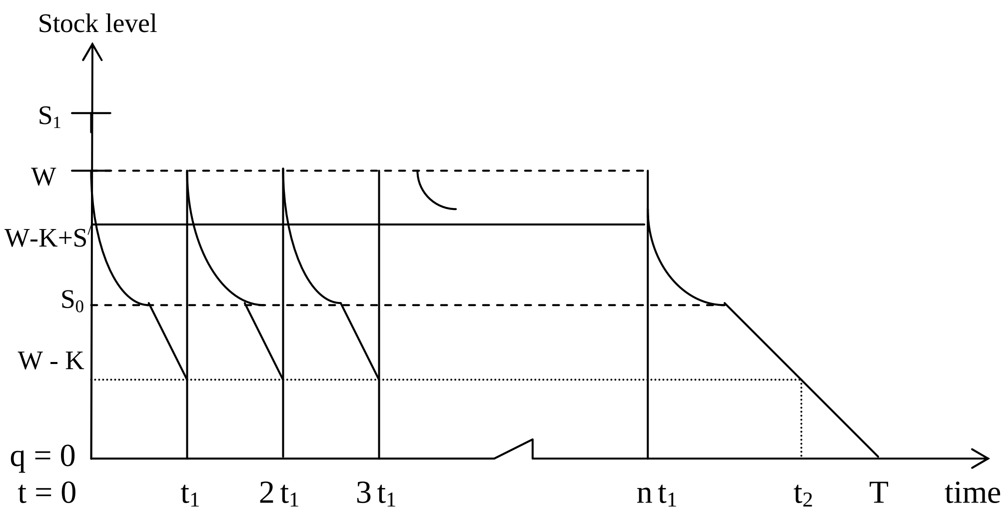

Figure – 2 : The inventory situation in OW

Table 1 : Numerical result

S_1	Case	System	Scenario	N*	N* Or q*	K*	S*	$Z_2^{*(j)}$ or $Z_1^{*(j)}$	Z^*_2 or Z^*_1	Boundary Profit	Optimal Profit	Q_{opt}	T_{opt}	Action
150	$S_1>W$	L_2	1	5	600	100	100	2071.5	2071.5	-	2071.48	600	0.86	L_2
			2	6	600	96	20	2060.3						
			3	10	600	50	50	2001.6						
		L_1	2	-	150	-	-	1279.3	1297.3	299.11				
			3	NO Feasible Solution										
75	$S_1<W$	L_2	4	5	600	100	100	2085.3	2085.3		2085.32	600	0.861	L_2
			5	6	600	91.6	41.7	2052.7						
			6	6	600	96	20	2055.7						
			7	10	600	50	50	1993.2						
			8	NO Feasible Solution										
			9	20	600	25	25	1368.4						
		L_1	1	-	580	-	-	2302.4	2302.4	101.81				
			2	NO Feasible Solution										
			3	NO Feasible Solution										

Table – 2 : Sensitivity analysis of the model

Changing parameters	**% changes**				
	-	**-20%**	**-10%**	**10%**	**20%**
Holding cost in OW (C_h)	Z	0.52 0	0.26 0	-0.26 0	-0.13 0
Holding cost in RW (C_f)	Z	3.69 16.67	1.81 0	-1.81 0	-3.42 -16.67
Unit Cost (C_p)	Z	-40.01 -0.1	-19.95 -0.26	19.85 0.26	39.59 0.52
Storing Capacity OW (W)	Z	-2.86 0.38	-1.49 0.18	0.57 -0.32	2.20 -0.46
Demand parameter (a)	Z	-23.57 3.88	-11.93 10.97	11.93 -9.00	23.87 -16.51
Demand parameter (b)	Z	0.62 -0.51	0.31 -0.26	-0.31 0.26	-0.62 0.52
Demand parameter (c)	Z	-0.39 0.74	-0.45 0.37	0.45 -0.37	0.39 -0.73
Replenishment cost (C_O)	Z	2.25 0	1.12 0	-1.12 0	-2.25 0
All holding cost (C_h, C_f)	Z	-2.28 16.67	-0.15 16.67	-4.32 0	-6.39 0
All cost parameters	Z	-19.5 -0.51	-9.72 -0.26	9.66 0.26	19.26 0.52
All demand parameters	Z	-27.61 11.09	-15.48 -4.32	16.39 -11.97	33.89 -8.84
All Transportation Cost parameters	Z	4.04 0	1.57 0	-2.51 -8.39	-3.82 0