Tamsui Oxford Journal of Mathematical Sciences **23(3)** (2007) 305-315
Aletheia University

On 2-dissection and 4-dissection of Ramanujan's Cubic Continued Fraction and Identities*

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Received September 6, 2005, Accepted October 19, 2005.

Abstract

We have proved identities for Ramanujan's cubic continued fraction which are analogous to the identities for Rogers-Ramanujan continued fraction *R(q)*.

Keywords and Phrases: *q-Hypergeometric series, Continued fraction, Dissection.*

1. Introduction

The celebrated Rogers-Ramanujan continued fraction is defined by

$$R(q) = 1 + \frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\cdots}\ , \qquad |q| < 1. \tag{1.1}$$

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* *2000 Mathematics Subject Classification*. 11A55.

On page 366 of his "lost" notebook [8] Ramanujan investigated another continued fraction

$$G(q) = 1 + \frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\cdots} \tag{1.2}$$

and claimed that many results of $G(q)$ are analogous to $R(q)$. This continued fraction is called Ramanujan's cubic continued fraction.
The firsts proofs that

$$\frac{(q^3;q^6)_\infty^2}{(q;q^6)_\infty (q^5;q^6)_\infty} = 1 + \frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\cdots} \tag{1.3}$$

were given by Watson [10] in 1929 and Selberg [9, p. 19] in 1936. Gordon [4] in 1965 and Andrews [1] in 1968. Hirschhorn [5] has recently shown that (1.3) can be deduced from Ramanujan's continued fraction

$$F(a,b,\lambda,q) = 1 + \frac{aq+\lambda q}{1+}\frac{bq+\lambda q^2}{1+}\frac{aq^2+\lambda q^3}{1+\cdots} \ . \tag{1.4}$$

Entry 54 [2, p 59] is

$$\frac{f(-q,-q^5)}{f(-q^3,-q^3)} = \frac{1}{1+}\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\cdots}, \qquad |q| < 1.$$

To quote Andrews et al [2] "Beneath this continued fraction, Ramanujan writes

$$Num? = \frac{\varphi(-q^3)}{f(-q)} \text{ and } Dem? = \frac{\psi(q^3)}{f(-q^2)}.$$

In fact he (Ramanujan) inverted the identification of the numerator and denominator on the left side of Entry 54, and they have shown that Ramanujan has mistakenly confused the roles of the 'numerator' and 'denominator'. I feel that in Ramanujan's mind the continued fraction was

$$1 + \frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+\cdots}$$

and for this continued fraction he wrote ‘numerator’ and ‘denominator’ correctly.

Following Ramanujan’s observation we give in this paper six identities which are analogous to the identities for $R(q)$ given by Ramanujan and proved by Andrews [2, pp. 189-190 and 194]. Andrews has used Hecke’s operator and quintuple identity in proving the identities. I have given 2-dissection and 4-dissection for Ramanujan’s cubic continued fraction $G(q)$ and using these have given the analogous identities for $G(q)$. The proof is simple and more direct.

2. Notation

We shall use the following notations :
For q a complex number, $q \neq 0$, $|q| < 1$,

$$[z;q]_\infty := (z;q)_\infty (z^{-1}q;q)_\infty, \quad z \neq 0 \ ,$$

$$(z;q)_\infty := \prod_{n=0}^{\infty} (1 - zq^n),$$

$$[z_1, z_2, \cdots, z_n; q]_\infty := [z_1;q]_\infty [z_2;q]_\infty \cdots [z_n;q]_\infty .$$

The following results follow easily :

$$[z^{-1};q]_\infty = -z^{-1}[z;q]_\infty = [zq;q]_\infty, \tag{2.1}$$

$$[z, zq; q^2]_\infty = [z;q]_\infty, \tag{2.2}$$

$$[z, -z; q]_\infty = [z^2; q^2]_\infty, \tag{2.3}$$

$$[z^{-1}q;q]_\infty = [z;q]_\infty, \tag{2.4}$$

$$[-1;q]_\infty [q;q^2]_\infty = 2. \tag{2.5}$$

3. The Identities

Theorem 1.

$$\sum_{n=0}^{\infty} c_{4n} q^n = \frac{[q,q^4,q^4,q^5;q^{12}]_\infty}{[q^3,q^3,q^3,q^3;q^{12}]_\infty}$$

$$\sum_{n=0}^{\infty} c_{4n+1} q^n = q^2 \frac{[q^2,q^4,q^4,q^4;q^{12}]_\infty}{[q^3,q^3,q^3,q^3;q^{12}]_\infty}$$

$$\sum_{n=0}^{\infty} c_{4n+2} q^n = q^4 \frac{[q^2,q^2,q^2,q^2,q^4,q^4;q^{12}]_\infty}{[q,q^3,q^3,q^3,q^3,q^5;q^{12}]_\infty}$$

$$\sum_{n=0}^{\infty} c_{4n+3} q^n = -q^6 \frac{[q,q,q^5,q^5;q^{12}]_\infty}{[q^3,q^3,q^3,q^3;q^{12}]_\infty}.$$

Theorem 2.

$$\sum_{n=0}^{\infty} c_{2n} q^n = \frac{[q^2,q^2,q^4,q^4;q^{12}]_\infty}{[q,q^3,q^3,q^5;q^{12}]_\infty}$$

$$\sum_{n=0}^{\infty} c_{2n+1} q^n = q \frac{[q,q^5;q^{12}]_\infty}{[q^3,q^3;q^{12}]_\infty}.$$

For proving these theorems we shall first give 2-dissection and 4-dissection of *G(q)*. For the dissections I shall require the following theorem of Lewis and Liu [7].

Theorem of Lewis and Liu [7]

Suppose $a_1, a_2, \cdots, a_n; b_1, b_2, \cdots b_n \in C \setminus \{0\}$ satisfy

(i) $a_i \neq q^n a_j$ for $i \neq j$ and $n \in z$

(ii) $a_1 a_2 \cdots a_n = b_1 b_2 \cdots b_n$.

Then

$$\sum_{i=1}^{n} \frac{\prod_{j=1}^{n} [a_i^{-1} b_j ; q]_\infty}{\prod_{j=1,\, j\neq i}^{n} [a_i^{-1} a_j ; q]_\infty} = 0\,. \tag{3.1}$$

4. 2-Dissections of G(*q*)

We give the definition of dissection :

Definition. *The n-dissection of the power series* $P = \sum_{n=0}^{\infty} a_n q^n$ *is the presentation of P as* $P = P_0 + P_1 + \cdots + P_{m-1}$ *where* $P_k = \sum_{n=0}^{\infty} a_{mn+k} q^{mn+k}$.

Theorem 3. *In the following theorem we give a 2-dissection for G(q).*

$$G(q) = \frac{[q^4, q^8; q^{24}]_\infty^2}{[q^6; q^{24}]_\infty^2 [q^2, q^{10}; q^{24}]_\infty} + q \frac{[q^2, q^{10}; q^{24}]_\infty}{[q^6; q^{24}]_\infty^2} \,. \tag{4.1}$$

Proof.

$$G(q) = \frac{[q^3; q^6]_\infty}{[q; q^6]_\infty}$$

$$= \frac{[q^3, q^3; q^{12}]_\infty}{[q, q^5; q^{12}]_\infty} \quad \text{, by (2.2)}$$

$$= \frac{[q^3, q^3; q^{12}]_\infty}{[q, q^7; q^{12}]_\infty}$$

$$= \frac{1}{[q^2;q^{24}]_\infty} \frac{[q^3,-q,q^3;q^{12}]_\infty}{[q,q^7;q^{12}]_\infty} \quad \text{, by (2.3).} \tag{4.2}$$

Define

$$G(q) = \alpha(q^2) + q\beta(q^2) \tag{4.3}$$

where $\alpha(q)$ and $\beta(q)$ are power series.

Now we determine $\alpha(q^2)$ and $\beta(q^2)$.

$$\alpha(q^2) = \frac{1}{2}\left[G(q) + G(-q)\right]$$

$$= \frac{1}{2[q^2;q^{24}]_\infty}\left(\frac{[q^3,-q,q^3;q^{12}]_\infty}{[q^7;q^{12}]_\infty} + \frac{[-q^3,q,-q^3;q^{12}]_\infty}{[-q^7;q^{12}]_\infty}\right).$$

Taking $(a_1,a_2,a_3;b_1,b_2,b_3) = (1,-1,q^7;q^3,-q;q^3)$ and $q \to q^{12}$ in (3.1), we have

$$\frac{[q^3,-q,q^3;q^{12}]_\infty}{[-1,q^7;q^{12}]_\infty} + \frac{[-q^3,q,-q^3;q^{12}]_\infty}{[-1,-q^7;q^{12}]_\infty} + \frac{[q^{-4},-q^{-6},-q^{-4};q^{12}]_\infty}{[q^{-7},-q^{-7};q^{12}]_\infty} = 0$$

or

$$\frac{[q^3,-q,q^3;q^{12}]_\infty}{[q^7;q^{12}]_\infty} + \frac{[-q^3,q,-q^3;q^{12}]_\infty}{[-q^7;q^{12}]_\infty} = 2\frac{[q^4,q^4;q^{12}]_\infty}{[q^6;q^{12}]_\infty[q^{10};q^{24}]_\infty} \quad \text{, by (2.1) and (2.5)}$$

Hence by (4.4)

$$\alpha(q^2) = \frac{1}{[q^2;q^{24}]_\infty} \frac{[q^4,q^4;q^{12}]_\infty}{[q^6;q^{12}]_\infty[q^{10};q^{24}]_\infty}$$

$$= \frac{[q^4,q^8;q^{24}]_\infty^2}{[q^6;q^{24}]_\infty^2[q^2,q^{10};q^{24}]_\infty} \ .$$

Similarly

$$\beta(q^2)=\frac{1}{[q^2;q^{24}]_\infty}\left[G(q)-G(-q)\right].$$

Taking $(a_1,a_2,a_3;b_1,b_2,b_3)=(1,-1,q^5;q^3,-q^{-1};q^3)$ and $q\to q^{12}$ in (3.1) and proceeding as earlier, we have

$$\beta(q^2)=\frac{[q^2,q^{10};q^{24}]_\infty}{[q^6;q^{24}]_\infty^2}. \tag{4.5}$$

Hence

$$G(q)=\frac{[q^4,q^4,q^8,q^8;q^{24}]_\infty}{[q^2,q^6,q^6,q^{10};q^{24}]_\infty}+q\frac{[q^2,q^{10};q^{24}]_\infty}{[q^6,q^6;q^{24}]_\infty},$$

in simplifying we have used (2.1), (2.2) and (2.5), which proves theorem 3.

5. 4-Dissection of *G(q)*

In the following theorem we give a 4-dissection for *G(q)*.

Theorem 4.

$$G(q)=\frac{[q^4,q^{16},q^{16},q^{20};q^{48}]_\infty}{[q^{12},q^{12},q^{12},q^{12};q^{48}]_\infty}+q^2\frac{[q^8,q^8,q^{16},q^{16};q^{48}]_\infty}{[q^{12},q^{12},q^{12},q^{12};q^{48}]_\infty}$$

$$+q^4\frac{[q^8,q^8,q^8,q^8,q^{16},q^{16};q^{48}]_\infty}{[q^4,q^{12},q^{12},q^{12},q^{12},q^{20};q^{48}]_\infty}-q^6\frac{[q^4,q^4,q^{20},q^{20};q^{48}]_\infty}{[q^{12},q^{12},q^{12},q^{12};q^{48}]_\infty} \tag{5.1}$$

Proof. Let

$$a(q^2)+qb(q^2) = \frac{[q^2,q^2,q^4,q^4;q^{12}]_\infty}{[q,q^3,q^3,q^5;q^{12}]_\infty} = A(q) \qquad \text{(say)} \tag{5.2}$$

and

$$c(q^2)+qd(q^2) = \frac{[q,q^5;q^{12}]_\infty}{[q^3,q^3;q^{12}]_\infty} = B(q) \qquad \text{(say)} . \tag{5.3}$$

Hence by (4.1)

$$G(q) = a(q^4)+qc(q^4)+q^2b(q^4)+q^3d(q^4) . \tag{5.4}$$

By (5.2)

$$a(q^2) = \frac{1}{2}\left[A(q)+A(-q)\right]$$

$$= \frac{1}{2}\frac{[q^2,q^4;q^{12}]_\infty^2}{[q^2,q^6,q^{10};q^{24}]_\infty}\left(\frac{[-q,-q^3,-q^5;q^{12}]_\infty}{[q^9;q^{12}]_\infty}+\frac{[q,q^3,q^5;q^{12}]_\infty}{[-q^9;q^{12}]_\infty}\right).$$

Taking $(a_1,a_2,a_3;b_1,b_2,b_3) = (1,-1,q^9;-q,-q^3;-q^5)$ and $q \to q^{12}$ in (3.1), we have

$$\frac{[-q,-q^3,-q^5;q^{12}]_\infty}{[-1,q^9;q^{12}]_\infty}+\frac{[q,q^3,q^5;q^{12}]_\infty}{[-1,-q^9;q^{12}]_\infty}+\frac{[-q^{-8},-q^{-6},-q^{-4};q^{12}]_\infty}{[q^{-9},-q^{-9};q^{12}]_\infty} = 0 ,$$

using (2.1), (2.2) and (2.5), the above expression simplifies to

$$\frac{[-q,-q^3,-q^5;q^{12}]_\infty}{[q^9;q^{12}]_\infty}+\frac{[q,q^3,q^5;q^{12}]_\infty}{[-q^9;q^{12}]_\infty} = \frac{2[-q^8,-q^4;q^{12}]_\infty}{[q^6;q^{12}]_\infty[q^6;q^{24}]_\infty}$$

$$= \frac{2}{[q^4,q^4q^6,q^6,q^6;q^{24}]_\infty} .$$

Therefore

$$a(q^2) = \frac{[q^2,q^2,q^4,q^4;q^{12}]_\infty}{[q^2,q^6,q^{10};q^{24}]_\infty[q^4,q^4,q^6,q^6,q^6;q^{24}]_\infty}$$

$$= \frac{[q^2,q^8,q^8,q^{10};q^{24}]_\infty}{[q^6,q^6,q^6,q^6;q^{24}]_\infty}. \qquad (5.5)$$

Similarly

$$qb(q^2) = \frac{1}{2}\left[A(q) - A(-q)\right]$$

$$= \frac{1}{2}\frac{[q^2,q^4;q^{12}]_\infty^2}{[q^2,q^6,q^{10};q^{24}]_\infty}\left(\frac{[-q,-q^3,-q^5;q^{12}]_\infty}{[q^9;q^{12}]_\infty} - \frac{[-q,-q^3,-q^5;q^{12}]_\infty}{[-q^9;q^{12}]_\infty}\right).$$

Taking

$(a_1,a_2,a_3;b_1,b_2,b_3) = (1,-1,q^3;-q,-q^{-3},-q^5)$ and $q \to q^{12}$ in (3.1), we have

$$\frac{[-q,-q^{-3},-q^5;q^{12}]_\infty}{[-1,q^3;q^{12}]_\infty} + \frac{[q,q^{-3},q^5;q^{12}]_\infty}{[-1,-q^3;q^{12}]_\infty} + \frac{[-q^{-2},-q^{-6},-q^2;q^{12}]_\infty}{[q^{-3},-q^{-3};q^{12}]_\infty} = 0,$$

using (2.1), (2.2) and (2.5) , gives

$$b(q^2) = \frac{[q^4,q^4,q^4,q^4,q^8,q^8;q^{24}]_\infty}{[q^2,q^6,q^6,q^6,q^6,q^{10};q^{24}]_\infty}. \qquad (5.5)$$

Similarly taking $(a_1,a_2,a_3;b_1,b_2,b_3) = (1,-1,q^9;q,-q^3,q^5)$ and $q \to q^{12}$, we have

$$C(q^2) = \frac{[q^4,q^4,q^8,q^8;q^{24}]_\infty}{[q^6,q^6,q^6,q^6;q^{24}]_\infty}. \qquad (5.6)$$

and taking $(a_1,a_2,a_3;b_1,b_2,b_3) = (1,-1,q^3;q,-q^{-3},q^5)$, we have

$$d(q^2) = -\frac{[q^2,q^2,q^{10},q^{10};q^{24}]_\infty}{[q^6,q^6,q^6,q^6;q^{24}]_\infty}. \tag{5.7}$$

Hence by (5.4)

$$G(q) = \frac{[q^4,q^{16},q^{16},q^{20};q^{48}]_\infty}{[q^{12},q^{12},q^{12},q^{12};q^{24}]_\infty} + q^2\frac{[q^8,q^8,q^{16},q^{16};q^{48}]_\infty}{[q^{12},q^{12},q^{12},q^{12};q^{48}]_\infty}$$

$$+q^4\frac{[q^8,q^8,q^8,q^8,q^{16},q^{16};q^{48}]_\infty}{[q^4,q^{12},q^{12},q^{12},q^{12},q^{20};q^{48}]_\infty} - q^6\frac{[q^4,q^4,q^{20},q^{20};q^{48}]_\infty}{[q^{12},q^{12},q^{12},q^{12};q^{48}]_\infty}, \tag{5.8}$$

which proves Theorem 4.

Proof of the Identities

By the definition of the dissection and writing q for q^4 in (5.8), we have Theorem 1. Similarly by writing q for q^2 in (4.1), we have Theorem 2.

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