Tamsui Oxford Journal of Mathematical Sciences **23(2)** (2007) 235-242
Aletheia University

On Sharp Perturbed Trapezoidal Inequalities for The Harmonic Sequence of Polynomials*

Dah-Yan Hwang†

Center for General Education, Kuang Wu Institute of Technology, Peito, Taipei, 11271 TAIWAN.

and

Gou-Sheng Yang

Department of Mathematics, Tamkang University, Tamsui, 25137 TAIWAN.

Received Septemer 21, 2005, Accepted December 12, 2006.

Abstract

The main purpose of this paper is to use a variant of Grüss inequality to obtain some perturbed trapezoid inequality with bounded derivatives of n-th order.

Keywords and Phrases: *Grüss inequality, Trapezoid inequality, Perturbed trapezoid inequality, Harmonic sequence of polynomials, n-convex function.*

1. Introduction

Let $f(x)$ be a convex function on the closed interval $[a, b]$. The inequality

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2} \tag{1.1}$$

---

*2001 *Mathematics Subject Classification.* 26D15, 41A55.

†E-mail: dyhuang@tsint.edu.tw

is well known in the literature as the Hermite-Hadamard inequality [16].

A function $f(x)$ is said to be r-convex on $[a,b]$ with $r \geq 2$ if and only if $f^{(r)}(x)$ exists and $f^{(r)}(x) \geq 0$.

In terms of a trapezoid formula for a real function $f(x)$ defined and integerable on $[a,b]$, using the first and second Euler-Maclaurin summation formulas, inequality (1.1) was generalized for $(2r)$-convex function functions on $[a,b]$ with $r \geq 1$ in [2, 6].

In [5], Lj. Dedić et al. established the following trapezoidal Grüss type inequality for n-time differentiable function:

Let $f : [a,b] \to R$ be such that $f^{(n)}$ is a continuous function for some $n \geq 1$ and

$$m_n \leq f^{(n)}(t) \leq M_n, \quad t \in [a,b], \quad m_n,\ M_n \in R.$$

Then, we have

$$\left| \int_a^b f(t)dt - \frac{b-a}{2}[f(a)+f(b)] - S_n^T(a,b) \right| \leq \frac{1}{2}(b-a)^{n+1}(M_n - m_n)\sqrt{\frac{|B_{2n}|}{(2n)!}}. \quad (1.2)$$

where B_{2n} is the Bernoulli numbers, $S_1^T(a,b) = 0$ and

$$S_n^T(a,b) = -\sum_{j=1}^{[n/2]} \frac{(b-a)^{2j}}{(2j)!} B_{2j}[f^{(2j-1)}(b) - f^{(2j-1)}(a)],$$

for $n \geq 2$. The other trapezoidal Grüss type inequality for n-time differentiable function, see [9, 11, 12, 14, 17]. In this paper, using concept of the harmonic sequence of polynomials, we shall establish some new generalizations of trapezoidal Grüss type for n-time differentiable function.

2. Definitions and Lemmas

Definition 1. *A sequence of polynomials $\{P_i(t)\}_{i=0}^{\infty}$ is called harmonic if it satisfies the following condition*

$$P_i'(t) = P_{i-1}(t) \quad (2.1)$$

and $P_0(t) = 1$ for all defined t and $i \in N$.

It is well-known that Bernoulli's polynomials $B_i(t)$ can be defined by the following expansion

$$\frac{xe^{tx}}{e^x - 1} = \sum_{i=0}^{\infty} \frac{B_i(t)}{i!} x^i, \quad |x| < 2\pi, \quad t \in R,$$

and are uniquely determined by the following formulae

$$B_i'(t) = iB_{i-1}(t), \quad B_0(t) = 1; \tag{2.2}$$
$$B_i(t+1) - B_i(t) = it^{i-1}. \tag{2.3}$$

Similarly, Euler's polynomials can be defined by

$$\frac{2e^{tx}}{e^x + 1} = \sum_{i=0}^{\infty} \frac{E_i(t)}{i!} x^i, \quad |x| < \pi, \quad t \in R,$$

and are uniquely determined by the following properties

$$E_i'(t) = iE_{i-1}(t), \quad E_0(t) = 1; \tag{2.4}$$
$$E_i(t+1) + E_i(t) = 2t^i. \tag{2.5}$$

For further details about Bernoulli's polynomials and Euler's polynomials, please refer to [1, 23.1.5 and 23.1.6] or [18]. Moreover, some new generalizations of Bernoulli's numbers and polynomials can be found in [10, 13].

If i is a nonegative integer, t, s, $\theta \in R$ and $s \neq \theta$, then

$$P_{i,E}(t) = \frac{(s-\theta)^i}{i!} E_i\Big(\frac{t-\theta}{s-\theta}\Big)$$

is a harmonic sequences of polynomials.

As usual, let $B_i = B_i(0)$, $i \in N$, denote Bernoulli's numbers. From properties (2.2) and (2.3), (2.4) and (2.5) of Bernoulli's and Euler's polynomials respectively, we can obtain easily that, for $i \geq 1$,

$$B_{i+1}(0) = B_{i+1}(1) = B_{i+1}, \quad B_1(0) = -B_1(1) = -\frac{1}{2}, \quad B_2(0) = B_2 = \frac{1}{6} \tag{2.6}$$

and, for $j \in N$,

$$E_j(0) = -E_j(1) = -\frac{2}{j+1}(2^{j+1} - 1)B_{j+1}. \tag{2.7}$$

It is also a well known fact that $B_{2i+1}=0$ for all $i\in N$.

In 1935, G. Gruss proved the following integral inequality which gives an approximation for the integral of the product of two functions in terms of the product of the integrals of the two functions [15, P.296].

Let $f,\ g:[a,b]\to R$ be two integrable functions such that $\phi\le f(x)\le\Phi$ and $\gamma\le g(x)\le\Gamma$ for all $x\in[a,b]$, where ϕ, Φ, γ and Γ are real numbers. Then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\cdot\frac{1}{b-a}\int_a^b g(x)dx\right|\le\frac{1}{4}(\Phi-\phi)(\Gamma-\gamma),$$

and the inequality is sharp, in the sense that the constant $\frac{1}{4}$ can not be replaced by a smaller one.

The above inequality is well known in the literature as Grüss inequality. In [4], X. L. Cheng and J. Sun proved the following variant of the Grüss inequality.

Lemma 2. *Let $f,\ g:[a,b]\to R$ be two integrable functions such that $\gamma\le g(x)\le\Gamma$ for all $x\in[a,b]$, where $\gamma,\ \Gamma\in R$ are constants. Then*

$$\begin{aligned}&\left|\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\cdot\int_a^b g(x)dx\right|\\ \le\ &\frac{(\Gamma-\gamma)}{2}\int_a^b\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|dx. \qquad (2.8)\end{aligned}$$

Further, Cerone and Dragomir [3] *proved that the constant $\frac{1}{2}$ in* (2.8) *is sharp.*

3. Mail Results

Theorem 3. *Let $\{P_i(t)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials, let $f(t)$ be n-time differentiable on the closed interval $[a,b]$ such that $m_n\le f^{(n)}(t)\le M_n$ for $t\in[a,b]$, $n\in N$ and $m_n,\ M_n\in R$. Then*

$$\begin{aligned}&\Bigg|(-1)^n\int_a^b f(t)dt+\sum_{i=1}^{n}(-1)^{n+i}[P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)]\\ &\quad-\frac{1}{b-a}[P_{n+1}(b)-P_{n+1}(a)][f^{(n-1)}(b)-f^{(n-1)}(a)]\Bigg|\\ \le\ &\frac{(M_n-m_n)}{2}\int_a^b\left|P_n(t)-\frac{1}{b-a}[P_{n+1}(b)-P_{n+1}(a)]\right|dt. \qquad (3.1)\end{aligned}$$

Proof. By successive integrabtion by parts and mathematical induction, we have

$$(-1)^n \int_a^b P_n(t) f^{(n)}(t)dt - \int_a^b f(t)dt = \sum_{i=1}^{n} (-1)^i [P_i(b) f^{(i-1)}(b) - P_i(a) f^{(i-1)}(a)]. \tag{3.2}$$

Using the definition of the harmonic sequence of polynomials yields

$$\int_a^b P_n(t)dt = P_{n+1}(b) - P_{n+1}(a), \tag{3.3}$$

By Lemma 2, we have

$$\begin{aligned} & \left| \int_a^b P_n(t) f^{(n)}(t)dt - \frac{1}{b-a} \int_a^b P_n(t)dt \int_a^b f^{(n)}(t)dt \right| \\ \leq \ & \frac{(M_n - m_n)}{2} \int_a^b \left| P_n(t) - \frac{1}{b-a} \int_a^b P_n(x)dx \right| dt. \end{aligned} \tag{3.4}$$

From combining of (3.2), (3.3) and (3.4) we obtain (3.1). This completes the proof.

Remark 4. *If taking $P_1(t) = t$ and $n = 1$ in (3.1), then we obtain*

$$\left| \int_a^b f(t)dt - \frac{b-a}{2}[f(a) + f(b)] \right| \leq \frac{(M_1 - m_1)}{8}(b-a)^2. \tag{3.5}$$

The constant $\frac{1}{8}$ in inequality (3.5) is better than the constant $\frac{1}{4\sqrt{3}}$ in inequality (1.2) for $n = 1$. In fact, the constant $\frac{1}{8}$ is sharp (see [7], [8]).

4. Application

Using Theorem 3, we have the following Theorem.

Theorem 5. *Let $\{E_i(t)\}_{i=0}^{\infty}$ be the Euler's polynomials and $\{B_i\}_{i=0}^{\infty}$ the Bernoulli's numbers. Let $f(t)$ be n-time differentiable on the closed interval $[a, b]$ such that $m_n \leq f^{(n)}(t) \leq M_n$ for $t \in [a, b]$, $n \in N$ and m_n, $M_n \in R$. Then*

$$\begin{aligned}
&\Big|(-1)^n\int_a^b f(t)dt+2\sum_{i=1}^{[\frac{n+1}{2}]}(-1)^n(1-4^i)\frac{(b-a)^{2(i-1)}}{(2i)!}[f^{2(i-1)}(a)+f^{2(i-1)}(b)]B_{2i}\\
&\quad-\frac{4(2^{n+2}-1)(b-a)^nB_{n+2}}{(n+2)!}[f^{(n-1)}(b)-f^{(n-1)}(a)]\Big|\\
\leq\ &\frac{(M_n-m_n)(b-a)^n}{2n!}\int_a^b\Big|E_n\Big(\frac{t-a}{b-a}\Big)-\frac{4(2^{n+2}-1)}{(n+2)(n+1)}B_{n+2}\Big|dt
\end{aligned} \tag{3.6}$$

where $[x]$ denotes the Gauss function, whose value is the largest integer not more than x.

Proof. Let

$$P_i(t)=P_{i,E}(t;b;a)=\frac{(b-a)^i}{i!}E_i\Big(\frac{t-a}{b-a}\Big) \tag{3.7}$$

Then, we have

$$\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a}=\frac{4(2^{n+2}-1)(b-a)^nB_{n+2}}{(n+2)!}. \tag{3.8}$$

Using formula (2.7) and straightforward calculating yields

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^{n+i}[P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)]\\
=\ &\sum_{i=1}^{n}(-1)^{n+i}\frac{(b-a)^i}{i!}[E_i(1)f^{(i-1)}(b)-E_i(0)f^{(i-1)}(a)]\\
=\ &\sum_{i=1}^{n}(-1)^{n+i}\frac{(b-a)^i}{i!}E_i(1)[f^{(i-1)}(a)+f^{(i-1)}(b)]\\
=\ &2\sum_{i=1}^{n}(-1)^{n+i}\frac{(b-a)^i}{(i+1)!}[f^{(i-1)}(a)+f^{(i-1)}(b)](2^{i+1}-1)B_{i+1}\\
=\ &2\sum_{i=1}^{[\frac{n+1}{2}]}(-1)^n(1-4^i)\frac{(b-a)^{2^{i-1}}}{(2i)!}[f^{2(i-1)}(a)+f^{2(i-1)}(b)]B_{2i}.
\end{aligned} \tag{3.9}$$

Substituting (3.7), (3.8) and (3.9) into (3.1) lead to (3.6). The proof is complete.

Remark 6. *If taking $E_1(t) = t - \frac{1}{2}$ and $n = 1$ in (3.6), then we recapture (3.5) again.*

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formula, Graphs, Tables, National Bureau of Standards*, Applied Mathematics Series 55, 4^{th} printing, Washington, 1965.

[2] G. Allasia, C. Diodano, and J. Pecaric, Hadamard-type inequalities for $(2r)$-convex functions with application, *Atti Accad. Sci. Torino Cl. Sci. Fis. Mat. Natur.* **133** (1999), 187-200.

[3] P. Cerone and S. S. Dragomir, A refinement of Gruss inequality and applications, *RGMIA Res. Rep. Coll.* **5** (2) (2002), Article 14. Available on line at: http://rgmia.vu.edu.au/v5n2.html

[4] Xiao-liang Cheng and Jie Sun, Note on the perturbed trapezoid inequality, *Journal of Inequalities in Pure and Applied Mathematics*, **3** (2) (2002), Article 29. Available on line at: http://jipam.vu.edu.au

[5] Lj. Dedić, M. Matić, and J. Pečarić, Some inequalities of Euler-Grüss type, *Computers Math. Applic.* **41** (2001), 843-856.

[6] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Type Inequalities and Applications*, RGMIA Monographs, 2000. Available on line at:
http://rgmia.vu.edu.au/monographs/hermite_hadamard.html

[7] S. S. Dragomir, Improvements of Ostrowski and generalized trapezoid inequality in terms of the upper and lower bounds of the first derivative. *Tamkang J. Math.* **34** (2003), no. 3, 213-222.

[8] S. S. Dragomir and Th. M. Rassias (ED.), *Ostrowski Type Inequalities and Applications in Numerical Integration*, Kluwer Academic Publishers, Dordrecht, 2002.

[9] Hillel Gauchman, Some integeral inequalities involving Taylor's Remainder II, *Journal of Inequalities in pure and Applied Mathematics* **4** (1) (2003), Article 1. [http://jipam.vu.edu.au/]

[10] Bai-Ni Guo and Feng Qi, Generalization of Bernoulli polynomials, *Internat. J. Math. Ed. Sci. Tech.* **33** (2002), no. 3, 428-431.

[11] Dah-Yan Hwang, Improvements of some integral inequalities involving Taylor's remainder. *J. Appl. Math. Comput.* **16** (2004), no. 1-2, 151-163.

[12] Dah-Yan Hwang, Kuei-Lin Tseng, and Gou-Sheng Yang, Improvements and generalizations of some Euler Gruss type inequalities and applications, *Tamsui Oxford Journal of Mathematical Sciences, submitted.*

[13] Oiu-Ming Luo, Bai-Ni, Feng Qi, and Lokenath Debnath, Generalizations of Bernoulli numbers and polynomials. *Int. J. Math. Math. Sci.* (2003), no. 59, 3769-3776.

[14] M. Matić, J. Pečarić, and N. Ujević, Improvement and further generalization of some inequalities of Ostrowski-Gruss type, *Computers Math. Applic.* **39** (3/4) (2000), 161-175.

[15] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht/Boston/New York, 1993.

[16] J. E. Pečarić, F. Proschan, and Y. L. Tong, *Convex Function, Partial Ordering and Statistical Applications*, Academic Press, New York, 1991.

[17] Feng Qi, Zong- Li Wei, and Qiao Yang, Generalizations and refinements of Hermite-Hadamard's inequality. *Rocky Mountain J. Math.* **35** (2005), no. 1, 235-251.

[18] Zhu-Xi Wang and Dun-Ren Guo, Teshu Hanshu Gailun (introduction to Sepecial Function), The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)