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Subclasses of Analytic Functions Defined by Carlson - Shaffer Linear Operator*

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Abstract

We introduce the subclass $\mathcal{L}_{\mathcal{T}}(a,c;\alpha,\beta)$ of analytic functions with negative coefficients defined by the linear operator $\mathcal{L}(a,c)f(z)$ which introduced and studied by Carlson and Shaffer [4]. Coefficient inequalities, distortion theorems, closure theorems, radii of close-to-convexity, starlikeness, and convexity for functions belonging to the class $\mathcal{L}_{\mathcal{T}}(a,c;\alpha,\beta)$ are obtained. Finally, we determine fractional calculus for functions belonging to this class.

Keywords and Phrases: *Analytic functions, Linear operator, Coefficient inequalities, Distortion theorems, Hadamard product, Fractional calculus.*

1. Introduction and Definitions.

Let $\mathcal{A}$ denote the class of functions of the form :

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

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which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. For two functions $f(z)$ and $g(z)$ given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = z + \sum_{n=2}^{\infty} d_n z^n \tag{1.2}$$

their Hadamard product (or convolution) is defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n d_n z^n. \tag{1.3}$$

For a complex parameters $b_{1,\ldots}, b_q$ and $c_{1,\ldots}, c_s$ $(c_j \neq 0, -1, -2, \ldots; j = 1, \ldots, s)$, we define the generalized hypergeometric function ${}_qF_s(b_{1,\ldots}, b_q\ ; c_{1,\ldots}, c_s; z)$ by

$$ {}_qF_s(b_{1,\ldots}, b_q\ ; c_{1,\ldots}, c_s; z) = \sum_{n=0}^{\infty} \frac{(b_1)_n \ldots (b_q)_n}{(c_1)_n \ldots (c_s)_n} \frac{z^n}{n!} \tag{1.4}$$

$$(q \leq s + 1;\ q, s \in N_0 = N \cup \{0\};\ z \in \mathcal{U})$$

where $(\lambda)_n$ is the Pochhammer symbol given, in terms of the Gamma function Γ, by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1, & (n = 0), \\ \lambda(\lambda + 1)(\lambda + 2) \ldots (\lambda + n - 1), & (n \in N). \end{cases} . \tag{1.5}$$

Corresponding to a function $h(b_{1,\ldots}, b_q\ ; c_{1,\ldots}, c_s; z)$ defined by

$$h(b_{1,\ldots}, b_q\ ; c_{1,\ldots}, c_s; z) = z {}_qF_s(b_{1,\ldots}, b_q\ ; c_{1,\ldots}, c_s; z)$$

Dziok and Srivastava [1] (see also [2,3,9]) consider a linear operator $\mathcal{H}(b_{1,\ldots}, b_q\ ; c_{1,\ldots}, c_s) : \mathcal{A} \to \mathcal{A}$ defined by the convolution

$$\mathcal{H}(b_{1,\ldots}, b_q\ ; c_{1,\ldots}, c_s) f(z) = h(b_{1,\ldots}, b_q\ ; c_{1,\ldots}, c_s; z) * f(z). \tag{1.6}$$

We observe that, for a function of the form (1.1), we have

$$\mathcal{H}(b_{1,\ldots}, b_q\ ; c_{1,\ldots}, c_s) f(z) = z + \sum_{n=2}^{\infty} \Gamma_n a_n z^n \tag{1.7}$$

where

$$\Gamma_n = \frac{(b_1)_{n-1} \dots (b_q)_{n-1}}{(c_1)_{n-1} \dots (c_s)_{n-1}(n-1)!}. \tag{1.8}$$

The linear operator $\mathcal{H}(b_{1,\dots}, b_q\ ; c_{1,\dots}, c_s)$ includes various other linear operators which were considered in earlier works. In particular for $s = 1$, and $q = 2$, we obtain the linear operator:

$$\mathcal{F}(b_{1,}b_2\ , c_1)f(z) = \mathcal{H}(b_{1,}b_2\ , c_1)f(z), \tag{1.9}$$

which was defined by Hoholv [5]. Putting, moreover, $b_1 = b$, $b_2 = 1$ and $c_1 = c$, we obtain the Carlson-Shaffer operator:

$$\mathcal{L}(b,c)f(z) := \mathcal{H}(b,1,c)f(z) := z + \sum_{n=1}^{\infty} \frac{(b)_n}{(c)_n} z^{n+1} a_{n+1} z^{n+1} \quad (z \in \mathcal{U}), \tag{1.10}$$

which was introduced by Carlson and Shaffer [4]. Note that $\mathcal{L}(1,1)f(z) = f(z)$, $\mathcal{L}(2,1)f(z) = zf'(z)$ and $\mathcal{L}(3,1)f(z) = zf'(z) + \frac{1}{2}z^2 f''(z)$.

Using the above Carlson-Shaffer operator, we introduce the following subclasses of analytic and univalent functions defined as follows:

Definition 1. *For $-1 \leq \alpha < 1$, $\beta \geq 0$ and $b \geq c$,we let $\mathcal{L}(b,c;\alpha,\beta)$ consist of functions f in $\mathcal{A}$ satisfying the condition*

$$\mathrm{Re}\left\{\frac{b\mathcal{L}(b+1,c)f(z)}{\mathcal{L}(a,c)f(z)} - (b-1)\right\} > \beta \left|\frac{b\mathcal{L}(b+1,c)f(z)}{\mathcal{L}(b,c)f(z)} - b\right| + \alpha, \quad (z \in \mathcal{U}). \tag{1.11}$$

The family $\mathcal{L}(b,c;\alpha,\beta)$ is of special interest for it contains many well-known as well as many new classes of analytic univalent functions. For $\mathcal{L}(1,1;\alpha,0)$, we obtain the family $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha(0 \leq \alpha < 1)$ and $\mathcal{L}(2,1;\alpha,0)$ is the family $\mathcal{C}(\alpha)$ of convex functions of order $\alpha(0 \leq \alpha < 1)$. For $\mathcal{L}(1,1;0,\beta)$ and $\mathcal{L}(2,1;0,\beta)$, we obtain the classes $\beta-\mathcal{ST}$ and $\beta-\mathcal{UCV}$ of uniformly β- starlike functions and uniformly β- convex functions, respectively, introduced by Kanas and Winsiowska [6, 7](see also the work of Kanas and Srivastava [8], Goodman [11, 12], Rønning [16, 17], Ma and Minda [13] and Gangadharan et al.[10]).

Let $\mathcal{T}$ denotes the subclass of $\mathcal{A}$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \qquad (a_n \geq 0) \tag{1.12}$$

Further, we define the class $\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$ by

$$\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta) = \mathcal{L}(b, c; \alpha, \beta) \cap \mathcal{T}. \tag{1.13}$$

In the present paper, we prove various coefficient inequalities, distortion theorems, closure theorems, radii of close-to-convexity, starlikeness, and convexity for functions belonging to the class $\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$. We also determine fractional calculus for functions belonging to this class.

2. Coefficient Inequalities

A necessary and sufficient condition for a function $f(z)$ to be in the class $\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$ is given by

Theorem 1. *Let the function $f(z)$ be defined by (1.12). Then $f(z) \in \mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{(1+\beta)(b)_n + [1-\alpha-b(1+\beta)](b)_{n-1}}{(c)_{n-1}} |a_n| \leq 1-\alpha. \tag{2.1}$$

where $-1 \leq \alpha < 1$, $\beta \geq 0$. The result (2.1) is sharp.

Proof. Assume that the inequality (2.1) holds true. It suffices to show that

$$\beta \left| \frac{b\mathcal{L}(b+1, c)f(z)}{\mathcal{L}(b, c)f(z)} - b \right| - \mathrm{Re}\left\{ \frac{b\mathcal{L}(b+1, c)f(z)}{\mathcal{L}(b, c)f(z)} - b \right\} \leq 1-\alpha$$

We have

$$\begin{aligned} & \beta \left| \frac{b\mathcal{L}(b+1, c)f(z)}{\mathcal{L}(b, c)f(z)} - b \right| - \mathrm{Re}\left\{ \frac{b\mathcal{L}(b+1, c)f(z)}{\mathcal{L}(b, c)f(z)} - b \right\} \\ \leq \ & (1+\beta) \left| \frac{b\mathcal{L}(b+1, c)f(z)}{\mathcal{L}(b, c)f(z)} - b \right| \end{aligned}$$

$$\leq \quad \frac{(1+\beta)\sum\limits_{n=2}^{\infty}\left(\frac{b(b+1)_n - b(b)_{n-1}}{(c)_{n-1}}\right)|a_n|\,|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}\frac{(b)_{n-1}}{(c)_{n-1}}|a_n|\,|z|^{n-1}}$$

$$\leq \quad \frac{(1+\beta)\sum\limits_{n=2}^{\infty}\left(\frac{(a)_n - b(b)_{n-1}}{(c)_{n-1}}\right)|a_n|}{1-\sum\limits_{n=2}^{\infty}\frac{(b)_{n-1}}{(c)_{n-1}}|a_n|}.$$

The last expression is bounded above by $1-\alpha$ if

$$\sum_{n=2}^{\infty}\frac{(1+\beta)(b)_n + [1-\alpha-b(1+\beta)](b)_{n-1}}{(c)_{n-1}}|a_n| \leq 1-\alpha.$$

Thus we have the inequality (2.1).

Conversely, assume that the function $f(z)$ is in the class $\mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$ and z is real then we have

$$\frac{b-\sum\limits_{n=2}^{\infty}\left(\frac{b(b+1)_n}{(c)_{n-1}}\right)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\frac{(b)_{n-1}}{(b)_{n-1}}a_n z^{n-1}} - (b-1) - \alpha \geq \frac{\sum\limits_{n=2}^{\infty}\beta\frac{b(b+1)_n - b(b)_{n-1}}{(c)_{n-1}}a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}\frac{(b)_{n-1}}{(b)_{n-1}}a_n z^{n-1}}.$$

Letting $z \to 1^-$ along the real axis, we obtain the inequality (2.1).

Finally, the function $f(z)$ given by

$$f(z) = z - \frac{(1-\alpha)(c)_{n-1}}{(1+\beta)(b)_n + [1-\alpha-b(1+\beta)](b)_{n-1}}z^n \qquad (n \geq 2) \qquad (2.2)$$

is an extremal function for the assertion of Theorem 1.

Remark 1. *Taking different choices of* b, c, α *and* β *as stated in Section 1, Theorem 1 leads to necessary and sufficient*

condition for a function f *to be in the classes* $\mathcal{S}^*(\alpha)$, $\mathcal{C}(\alpha)$, β–$\mathcal{ST}$ *and* β–$\mathcal{UCV}$.

Corollary 1. *Let the function* $f(z)$ *be defined by* (1.8) *be in the class* $\mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$. *Then*

$$a_n \leq \frac{(1-\alpha)(c)_{n-1}}{(1+\beta)(b)_n + [1-\alpha-b(1+\beta)](b)_{n-1}} \qquad (n \geq 2) \qquad (2.3)$$

The equality in (2.3) is attained for the function $f(z)$ given by (2.2).

For the notational convenience we shall henceforth denote

$$\sigma_n(b,c;\alpha,\beta) := \frac{(1+\beta)(b)_n + [1-\alpha-b(1+\beta)](b)_{n-1}}{(c)_{n-1}} \tag{2.4}$$

3. Growth and Distortion Theorems

Theorem 2. *Let the function $f(z)$ be defined by (1.12) be in the class $\mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$. If $\{\sigma_n(b,c;\alpha,\beta)\}_{n=2}^{\infty}$ is a non-decreasing sequence, then, for $|z| = r < 1$*

$$r - \frac{1-\alpha}{\sigma_2(b,c;\alpha,\beta)} r^2 \le |f(z)| \le r + \frac{1-\alpha}{\sigma_2(b,c;\alpha,\beta)} r^2 \tag{3.1}$$

and *if $\{\sigma_n(b,c;\alpha,\beta)/n\}_{n=2}^{\infty}$ is a non-decreasing sequence, then, for $|z| = r < 1$*

$$1 - \frac{2(1-\alpha)}{\sigma_2(b,c;\alpha,\beta)} r \le |f'(z)| \le 1 + \frac{2(1-\alpha)}{\sigma_2(b,c;\alpha,\beta)} r. \tag{3.2}$$

The results (3.1) and (3.2) are sharp for the function $f(z)$ given

$$f(z) = z - \frac{1-\alpha}{\sigma_2(b,c;\alpha,\beta)} z^2 \qquad (z = \pm r) \tag{3.3}$$

Proof. In view of Theorem 1, we note that

$$\sigma_2(b,c;\alpha,\beta) \sum_{n=2}^{\infty} a_n \le \sum_{n=2}^{\infty} \sigma_n(b,c;\alpha,\beta) a_n \le 1-\alpha, \tag{3.4}$$

Thus, we have

$$|f(z)| \ge |z| - \sum_{n=2}^{\infty} a_n |z|^n \ge r - r^2 \sum_{n=2}^{\infty} a_n \ge r - \frac{1-\alpha}{\sigma_2(b,c;\alpha,\beta)} r^2 \tag{3.5}$$

Similarly,

$$|f(z)| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n \leq r + r^2 \sum_{n=2}^{\infty} a_n \leq r + \frac{1-\alpha}{\sigma_2(b,c;\alpha,\beta)} r^2. \tag{3.6}$$

Also from Theorem 1, we have

$$\frac{\sigma_2(b,c;\alpha,\beta)}{2} \sum_{n=2}^{\infty} n a_n \leq \sum_{n=2}^{\infty} \sigma_n(b,c;\alpha,\beta) a_n \leq 1-\alpha. \tag{3.7}$$

Thus,

$$|f'(z)| \geq 1 - \sum_{n=2}^{\infty} n a_n |z|^{n-1} \geq 1 - r \sum_{n=2}^{\infty} n a_n \geq 1 - \frac{2(1-\alpha)}{\sigma_2(b,c;\alpha,\beta)} r. \tag{3.8}$$

On the other hand,

$$|f'(z)| \leq 1 + \sum_{n=2}^{\infty} n a_n |z|^{n-1} \leq 1 + r \sum_{n=2}^{\infty} n a_n \leq 1 + \frac{2(1-\alpha)}{\sigma_2(b,c;\alpha,\beta)} r. \tag{3.9}$$

This completes the proof.

Corollary 2. *The disk $|z| < 1$ is mapped onto a domain that contains the disk $|w| < \frac{\sigma_2(b,c;\alpha,\beta)-(1-\alpha)}{\sigma_2(b,c;\alpha,\beta)}$ by any $f(z) \in \mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$. The theorem is sharp with extermal function $f(z)$ given by (3.3).*

Proof. The proof follow upon letting $r \to 1$ in (3.1).

4. Closure Theorems

In this section, we shall prove that the class $\mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$ is closed under convex linear combinations.

Theorem 3. *Let the function $f_i(z)$, $i = 1,2,\ldots,m$, defined by*

$$f_i(z) = z - \sum_{n=2}^{\infty} a_{n,i} z^n \qquad (a_{n,i} \geq 0) \tag{4.1}$$

for $z \in \mathcal{U}$, be in the class $\mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{n=2}^{\infty} \left(\frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right) z^n \tag{4.2}$$

also belongs to the class $\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$.

Proof. Let $f_i(z) \in \mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$,it follows from Theorem 1, that

$$\sum_{n=2}^{\infty} \sigma_n(b, c; \alpha, \beta) a_{n,i} \leq 1 - \alpha \qquad (\ i = 1, 2, \ldots, m). \tag{4.3}$$

Therefore,

$$\sum_{n=2}^{\infty} \sigma_n(b, c; \alpha, \beta) \left(\frac{1}{m} \sum_{i=1}^{m} a_{n,i} \right) \tag{4.4}$$

$$= \frac{1}{m} \sum_{i=1}^{m} \left(\sum_{n=2}^{\infty} \sigma_n(b, c; \alpha, \beta) a_{n,j} \right) \leq 1 - \alpha. \tag{4.5}$$

Hence by Theorem 1, $h(z) \in \mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$.

With the aid of Theorem 1, we can prove the following

Theorem 4. *Let the functions* $f_i(z)$ *be defined by (4.1) be in the class* $\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$ *for every* $i = 1, 2, \ldots, m$. *Then the functions*

$$h(z) = \sum_{i=1}^{m} c_i f_i(z) \qquad (c_i \geq 0) \tag{4.6}$$

is also in the same class $\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$ *where* $\sum\limits_{i=1}^{m} c_i = 1$.

As a consequence of Theorem 4, the extreme points of the class $\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$ are giving by

Theorem 5. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z - \frac{1 - \alpha}{\sigma_n(b, c; \alpha, \beta)} z^k \qquad (n \geq 2) \tag{4.7}$$

for $0 \leq \alpha < 1$. *Then* $f(z)$ *is in the class* $\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$ *if and only if it can be expressed in the form* $f(z) = \sum\limits_{n=1}^{\infty} t_n z^n$ *where* $t_n \geq 0$ $(n \geq 1)$ and $\sum\limits_{n=1}^{\infty} t_n = 1$.

5. Radii of Close-to-Convexity, Starlikeness, and Convexity

Theorem 6. *Let the function $f(z)$ be defined by* (1.12) *be in the class $\mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$. Then $f(z)$ is close-to-convex of order $\sigma(0 \leq \sigma < 1)$ in $|z| < r_1$, where*

$$r_1 = \inf_n \left[\frac{\sigma_n(b,c;\alpha,\beta)(1-\sigma)}{1-\alpha}\right]^{1/(n-1)} \qquad (n \geq 2). \tag{5.1}$$

The result is sharp, the extermal function $f(z)$ being given by (2.2).

Proof. We must show that $|f'(z)-1| \leq 1-\sigma$ for $|z| < r_1$,where r_1 is given by (5.1). From (1.8) we find that

$|f'(z)-1| \leq \sum\limits_{n=2}^{\infty} na_n |z|^{n-1}$. Thus $|f'(z)-1| \leq 1-\sigma$ if

$$\sum_{n=2}^{\infty} \left(\frac{n}{1-\sigma}\right) a_n |z|^{n-1} \leq 1. \tag{5.2}$$

But, by Theorem 1, (5.2) will be true if

$$\left(\frac{n}{1-\sigma}\right) |z|^{n-1} \leq \frac{\sigma_n(b,c;\alpha,\beta)}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{\sigma_n(b,c;\alpha,\beta)(1-\sigma)}{1-\alpha}\right]^{1/(n-1)} \qquad (n \geq 2). \tag{5.3}$$

Theorem 6 follows easily from (5.3).

Theorem 7. *Let the function $f(z)$ be defined by* (1.12) *be in the class $\mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$. Then $f(z)$ is starlike of order $\sigma(0 \leq \sigma < 1)$ in $|z| < r_2$, where*

$$r_2 = \inf_n \left[\frac{\sigma_n(b,c;\alpha,\beta)(1-\sigma)}{(n-\sigma)(1-\alpha)}\right]^{1/(n-1)} \qquad (n \geq 2). \tag{5.4}$$

The result is sharp, the extermal function $f(z)$ being given by (2.2).

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)} - 1\right| \leq 1-\sigma$ for $|z| < r_2$,where r_2 is given by (5.4). From (1.10) we have

$\left|\frac{zf'(z)}{f(z)}-1\right| \le \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n|z|^{n-1}}$. Thus $\left|\frac{zf'(z)}{f(z)}-1\right| \le 1-\sigma$ if

$$\sum_{n=2}^{\infty}\left(\frac{n-\sigma}{1-\sigma}\right)a_n\,|z|^{n-1} \le 1. \tag{5.5}$$

But, by Theorem 1, (5.5) will be true if

$$\left(\frac{n-\sigma}{1-\sigma}\right)|z|^{n-1} \le \frac{\sigma_n(b,c;\alpha,\beta)}{1-\alpha},$$

that is, if

$$|z| \le \left[\frac{\sigma_n(b,c;\alpha,\beta)(1-\sigma)}{(n-\sigma)(1-\alpha)}\right]^{1/(n-1)} \qquad (n\ge 2). \tag{5.6}$$

Theorem 7 follows easily from (5.6).

Corollary 3. *Let the function $f(z)$ be defined by* (1.12) *be in the class $\mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$. Then $f(z)$ is convex of order $\sigma(0\le\sigma<1)$ in $|z|<r_3$, where*

$$r_3 = \inf_n\left[\frac{\sigma_n(b,c;\alpha,\beta)(1-\sigma)}{n(n-\sigma)(1-\alpha)}\right]^{1/(n-1)} \qquad (n\ge 2). \tag{5.7}$$

The result is sharp, the extermal function $f(z)$ being given by (2.5).

6. Fractional Calculus

In this section, we find it to be convenient to recall here the following of fractional calculus which were introduced by Owa ([14], [15]).

Definition 2. *The fractional integral of order δ is defined, for a function $f(z)$, by*

$$D_z^{-\delta}f(z) = \frac{1}{\Gamma(\delta)}\int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}}d\zeta \qquad (\delta>0), \tag{6.1}$$

where the function $f(z)$ is analytic in a simply-connected region of the z-plane containing the origin and the multiplicity of the function $(z-\zeta)^{\delta-1}$is removed by requiring the function $\log(z-\zeta)$ *to be real when* $z-\zeta>0$.

Definition 3. *The fractional derivative of order δ is defined, for a function $f(z)$, by*

$$D_z^{\delta} f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\delta}} d\zeta \qquad (0 \leq \delta < 1), \tag{6.2}$$

where the function $f(z)$ is constrained, and the multiplicity of the function $(z-\zeta)^{-\delta}$ is removed as in Definition 2.

Definition 4. *Under the hypotheses of Definition 2, the fractional derivative of order $m+\delta$ is defined by*

$$D_z^{m+\delta} f(z) = \frac{d^m}{dz^m} D_z^{\delta} f(z) \qquad (0 \leq \delta < 1;\, m \in N_0). \tag{6.3}$$

Remark 2. From Definition 2, we have $D_z^0 f(z) = f(z)$, which in view of Definition 4 yields

$$D_z^{m+0} f(z) = \tfrac{d^m}{dz^m} D_z^0 f(z) = f^{(m)}(z).$$

Thus,

$$\lim_{\delta \to 0} D_z^{-\delta} f(z) = f(z) \text{ and } \lim_{\delta \to 0} D_z^{1-\delta} f(z) = f'(z)\ .$$

Theorem 8. *Let the function $f(z)$ be defined by (1.12) be in the class $\mathcal{L}_{\mathcal{T}}(b, c; \alpha, \beta)$. If $\{\sigma_n(b, c; \alpha, \beta)\}_{n=2}^{\infty}$ is a non-decreasing sequence, then*

$$\left| D_z^{-\delta} f(z) \right| \geq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left\{ 1 - \frac{1-\alpha}{(2+\delta)\sigma_2(b, c; \alpha, \beta)} |z| \right\} \tag{6.4}$$

and

$$\left| D_z^{-\delta} f(z) \right| \leq \frac{|z|^{1+\delta}}{\Gamma(2+\delta)} \left\{ 1 + \frac{1-\alpha}{(2+\delta)\sigma_2(b, c; \alpha, \beta)} |z| \right\} \tag{6.5}$$

for $\delta > 0$, and $z \in \mathcal{U}$. The result is sharp.

Proof. Let

$$\begin{aligned} F(z) &= \Gamma(2+\delta) z^{-\delta} D_z^{-\delta} f(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} a_n z^n = z - \sum_{n=2}^{\infty} \Delta(n) a_n z^n, \end{aligned} \tag{6.6}$$

where

$$\Delta(n) = \frac{\Gamma(n+1)\Gamma(2+\delta)}{\Gamma(n+1+\delta)} \qquad (n \geq 2). \tag{6.7}$$

It is easy to see that

$$0 < \Delta(n) \leq \Delta(2) = \frac{2}{2+\delta}. \tag{6.8}$$

Therefore, by using (3.4) and (6.8), we can see that

$$|F(z)| \geq |z| - \Delta(2)\,|z|^2 \sum_{n=2}^{\infty} a_n \geq |z| - \frac{1-\alpha}{(2+\delta)\sigma_2(b,c;\alpha,\beta)}\,|z|^2 \tag{6.9}$$

$$|F(z)| \leq |z| + \Delta(2)\,|z|^2 \sum_{n=2}^{\infty} a_n \leq |z| + \frac{1-\alpha}{(2+\delta)\sigma_2(b,c;\alpha,\beta)}\,|z|^2 \tag{6.10}$$

which prove the inequality of Theorem 8. Further, equalities (6.4) and (6.5) are attained for the function $f(z)$ defined by

$$D_z^{-\delta} f(z) = \frac{z^{1+\delta}}{\Gamma(2+\delta)} \left\{ 1 + \frac{1-\alpha}{(2+\delta)\sigma_2(b,c;\alpha,\beta)} z \right\} \tag{6.11}$$

Theorem 9. *Let the function $f(z)$ be defined by (1.12) be in the class $\mathcal{L}_{\mathcal{T}}(b,c;\alpha,\beta)$. If $\{\sigma_n(b,c;\alpha,\beta)/n\}_{n=2}^{\infty}$ is a non-decreasing sequence, then*

$$\left| D_z^{\delta} f(z) \right| \geq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left\{ 1 - \frac{2(1-\alpha)}{(2-\delta)\sigma_2(b,c;\alpha,\beta)}\,|z| \right\} \tag{6.12}$$

and

$$\left| D_z^{\delta} f(z) \right| \leq \frac{|z|^{1-\delta}}{\Gamma(2-\delta)} \left\{ 1 + \frac{2(1-\alpha)}{(2-\delta)\sigma_2(b,c;\alpha,\beta)}\,|z| \right\} \tag{6.13}$$

for $0 \leq \delta < 1$, and $z \in \mathcal{U}$. The result is sharp.

Proof. Let

$$\begin{aligned} H(z) &= \Gamma(2-\delta) z^{\delta} D_z^{\delta} f(z) \qquad (6.14) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} a_n z^n = z - \sum_{n=2}^{\infty} n\Omega(n) a_n z^n, \end{aligned}$$

where

$$\Omega(n) = \frac{\Gamma(n)\Gamma(2-\delta)}{\Gamma(n+1-\delta)} \qquad (n \geq 2). \tag{6.15}$$

Since

$$0 < \Omega(n) \leq \Omega(2) = \frac{1}{2-\delta}. \tag{6.16}$$

Therefore, by using (3.7) and (6.16), we can see that

$$|H(z)| \geq |z| - \Omega(2)\,|z|^2 \sum_{k=2}^{\infty} na_n \geq |z| - \frac{2(1-\alpha)}{(2-\delta)\sigma_2(b,c;\alpha,\beta)}\,|z|^2 \tag{6.17}$$

$$|H(z)| \leq |z| + \Omega(2)\,|z|^2 \sum_{n=2}^{\infty} na_n \leq |z| + \frac{2(1-\alpha)}{(2-\delta)\sigma_2(b,c;\alpha,\beta)}\,|z|^2 \tag{6.18}$$

which give the inequalities of Theorem 10. Further, since the equalities (6.12) and (6.13) are attained for the function $f(z)$ defined by

$$D_z^{\delta} f(z) = \frac{z^{1-\delta}}{\Gamma(2-\delta)} \left\{1 + \frac{2(1-\alpha)}{(2-\delta)\sigma_2(b,c;\alpha,\beta)} z\right\} \tag{6.19}$$

we see that the result is sharp.

Remark 3. *Letting $\delta = 0$ in Theorem 8, we have (3.1) of Theorem 3, and letting $\delta \longrightarrow 1$ in Theorem 9, we have (3.2) in Theorem 3.*

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