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Eigenvalue Approach to Generalized Thermoviscoelasticity with One Relaxation Time Parameter

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Abstract

The fundamental equations of generalized thermoviscoelasticity with one relaxation time parameter have been written in the form of a vector matrix differential equation in Laplace transform domain and solve by eigenvalue approach. The resulting formulation is applied to different cases: (i) thermal shock problem, (ii) problem of Layer medium, both without heat sources and (iii) plane distribution of heat sources on whole and semi−space. Finally numerical results are given and illustrated graphically for each problem. Comparisons are made with the results predicted by both the coupled theory and with the theory of generalized thermoviscoelasticity with one relaxation time parameter.

Keywords and Phrases: *Eigenvalue, Generalized Thermoviscoelasticity, Laplace transform, Vector matrix differential equation, Numerical solution.*

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1. Introduction

Since the work of Maxwell, Boltzmen, Voigt, Kelvin and others, linear viscoelasticity remains an important area of research. Gross [1], Staverman and Schwarz [2]. Alfey and Gurnee [3] and Ferry [4] investigated the mechanical model representation of linear viscoelastic behaviour results. Solutons of boundary value problems for linear viscoelastic materials including temperature variations in both quasistatic and dynamic problems have experienced great strides in the last decades in the works of Biot [5, 6], Morland and Lee [7], Tanner [8], and Huilgal and Phan−Thien [9]. Bland [10] linked the solution of linear−viscoelasticity problems to corresponding linear elastic−solutions. An approximation method for the linear thermal viscoelastic problems given by Gurtin and Sternberg [11]. Sternberg [12], and Iiioushin [13]. One can refer to the book of Iioushin and Pobedria [14] for a formulation of the mathematical theory of thermal viscoelasticity and the solution of some boundary value problems as well as the work of probedria [15] for the coupled problems in continuum mechanics. Results of important experiments determining the mechanical properties of viscoelastic materials were involved in the book by Koltunov [16].

The classical uncoupled theory of thermoelasticity predicts two phenomena that are not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms contrary to the fact that the elastic changes produce heat effects. Second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves.

Biot[17] formulated the theory of coupled thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on temperature. The heat equations for both theories of the diffusion type predict infinite speeds of propagation for heat waves contrary to physical observations.

Several problems of generalized thermoelasticity have been solved by following either Lord and Shulman [18] (L−S theory) involves one relaxation time parameter or Green and Lindsay [19] (G−L theory) with two relaxation time parameter. In both the theories of conventional Fourier law of heat conduction has been modified to a hyperbolic type of equation which along with the equations of motion of thermoelasticity (which are hyperbolic type) are considered for the solution of the problem. As such, both theories ensures finite speed of propagation of the waves and eliminates automatically the paradox of infinite speeds of propagation inherent in both the uncoupled and coupled theories of thermoelasticity vide Chandrasekharaiah and Keshavan [20].

In dealing with coupled or generalized thermoelastic problems, the solution procedure is usually to choose a suitable thermoelastic potential function, but this approach has certain limitations as discussed by Bahar and Hetnarski [21].

In the present article we apply eigenvalue approach develop in [22] to problem of thermoviscoelasticity of one dimension with one relaxation time. The resulting formulation is applied to three different cases in the presence or absence of heat sources as Ezzat et al [23]. The solution for the cases are given in closed form in the Laplace transform domain. The inversion of the transform is carried out using a numerical inversion technique [24]. Some results are presented graphically.

2. Nomenclature

A, β, a^*	emperical constants
c_0^2	$= \frac{k}{\rho}$
C_E	Specific heat at constant strain
k	thermal conductivity
K	$= \lambda + \frac{2}{3}\mu$ Bulk modulus
χ	Poisson's ratio
Q	Intensity of applied heat source per unit mass
t	time
T	Absolute temperature
T_0	Reference temperature chosen that $\lvert T - T_0 \rvert << 1$
u_i	components of displacement vector
$R(t)$	relaxation function
α_T	Coefficient of linear thermal expansion

$\gamma \quad = 3K\alpha_T$

$\in \quad = \dfrac{\gamma}{\rho C_E}$

$\eta_0 \quad = \dfrac{\rho C_E}{k}$

λ, μ Lame's constants

ρ Density

σ_{ij} components of stress tensor

$\in_{ij}$ components of strain tensor

e_{ij} components of strain deviator tensor

S_{ij} components of stress deviator tensor

τ_0 Relaxation time parameter

3. Formulation of the Problem

We consider a thermoviscoelastic solid occuping in the region $-\infty < x < \infty$. The governing equations for generalized thermoviscoelasticity with one relaxation time parameter consists of the equation of motion

$$\sigma_{ij,j} = \rho \ddot{u}_i \tag{1}$$

The generalised heat conduction equation

$$kT_{,ii} = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right)(\rho c_E T + \gamma T_0 u_{i,i}) - \left(1 + \tau_0 \frac{\partial}{\partial t}\right)Q \tag{2}$$

The constitutive equation given by Pobedria [15] and Fung [25]

$$S_{ij} = \int_0^t R(t-\tau)\frac{\partial e_{ij}(x,\tau)}{\partial\tau}d\tau = \hat{R}(e_{ij}) \tag{3}$$

where,

$$S_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij},\ e_{ij} = \in_{ij} - \frac{e}{3}\delta_{ij},\ e = \in_{kk},\ \sigma = \sigma_{kk}$$

and R(t) is the relaxation function given by Koltunov [16] and Karamany [26]

$$R(t) = 2\mu\left[1 - A\int_0^t e^{-\beta t}t^{\alpha^*-1}dt\right] \tag{4}$$

where $0 < \alpha^* < 1$, $A > 0$ and $\beta > 0$. Ignoring the relaxation effects of the volume properties we can write for the generalized theory of thermoviscoelasticity with one relaxation time parameter.

$$\sigma = K[e - 3\alpha_T(T - T_0)] \tag{5}$$

where

$$\sigma = \frac{\sigma_{ij}}{3}, \qquad \gamma = 3\alpha_T K, \qquad K = \lambda + \frac{2}{3}\mu.$$

Using (5) in (3) we get,

$$\sigma_{ij} = \hat{R}\left(\in_{ij} - \frac{e}{3}\delta_{ij}\right) - \gamma(T - T_0)\delta_{ij} + Ke_{ij} \tag{6}$$

Equation (1) together with (6) reduces to

$$\rho\ddot{u}_i = \hat{R}\left(\frac{1}{2}\nabla^2 u_i + \frac{1}{6}e_{,i}\right) + Ke_{,i} - \gamma(T - T_0)_{,i} \tag{7}$$

The equation for one dimension becomes

$$\rho \frac{\partial^2 u}{\partial t^2} = \left(\frac{2}{3} \hat{R} + k \right) \frac{\partial^2 u}{\partial x^2} - \gamma (T - T_0)_{,x} \tag{8}$$

$$k \frac{\partial^2 T}{\partial x^2} = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \left(\rho c_E T + \gamma T_0 \frac{\partial u}{\partial x} \right) - \left(1 + \tau_0 \frac{\partial}{\partial t} \right) Q \tag{9}$$

$$\sigma_{xx} = \left(\frac{2}{3} \hat{R} + k \right) \frac{\partial u}{\partial x} - \gamma (T - T_0) \tag{10}$$

We now introduce the following non–dimensional variables

$$\left.\begin{array}{lll} x^* = c_0 \eta_0 x, & u^* = c_0 \eta_0 u, & t^* = c_0^2 \eta_0 t, \\ \tau^* = c_0^2 \eta_0 \tau, & \eta_0 = \dfrac{\rho c_E}{k}, & \sigma_{ij}^* = \dfrac{\sigma_{ij}}{K}, \\ c_0^2 = \dfrac{K}{\rho}, & R^* = \dfrac{2}{3K} R, & \theta = \dfrac{\gamma (T - T_0)}{\rho c_0^2}. \\ Q^* = \dfrac{Q}{k T_0 c_0^2 \eta_0^2} & & \end{array}\right\} \tag{11}$$

Using the non–dimensional variables to (8) – (10) and (4) becomes of the form (dropping the asterisks)

$$\frac{\partial^2 u}{\partial t^2} = (\hat{R} + 1) \frac{\partial^2 u}{\partial x^2} - \frac{\partial \theta}{\partial x} \tag{12}$$

$$\frac{\partial^2 \theta}{\partial x^2} = \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \theta + \in \left(\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2} \right) \frac{\partial u}{\partial x} - \left(1 + \tau_0 \frac{\partial}{\partial t} \right) Q \tag{13}$$

$$\sigma_{xx} = (\hat{R} + 1) \frac{\partial u}{\partial x} - \theta \tag{14}$$

$$R(t) = \frac{4\mu}{3k} \left(1 - A \int_0^t e^{-\beta t} t^{a^* - 1} dt\right) \tag{15}$$

Taking the Laplace transform defined by

$$\bar{f}(p) = \int_{o}^{\infty} e^{-pt} f(t)dt \tag{16}$$

to the equations (12) – (15) we get,

$$\frac{d^2\bar{u}}{dx^2} = \alpha p^2 \bar{u} + \alpha \frac{d\bar{\theta}}{dx} \tag{17}$$

$$\frac{d^2\bar{\theta}}{dx^2} = p(1+\tau_o p)\bar{\theta} + \in p(1+\tau_o p)\frac{d\bar{u}}{dx} - (1+\tau_o p)\bar{Q} \tag{18}$$

$$\bar{\sigma}_{xx} = \frac{1}{\alpha}\frac{d\bar{u}}{dx} - \bar{\theta} \tag{19}$$

and

$$\bar{R} = \frac{4\mu}{3kp}\left[1 - \frac{A\Gamma(a^*)}{(p+\beta)^{a^*}}\right] \tag{20}$$

where $L\left(\hat{R}\frac{\partial^2 u}{\partial x^2}\right) = p\bar{R}\frac{d^2\bar{u}}{dx^2}$, $\alpha = \frac{1}{1+pR}$ and $\Gamma(a^*)$ is a gamma function.

We also assume that at time t = 0 the body is at rest; in an undeformed and unstressed state and is maintained at the reference temperature, so the following initial conditions hold.

$$u(x,0) = \frac{\partial u(x,0)}{\partial t} = 0$$

$$\theta(x,0) = \frac{\partial \theta(x,0)}{\partial t} = 0$$

As in Das et al [22] the equations (17) and (18) can be written in vector matrix different equation as follows :

$$\frac{d}{dx}\begin{bmatrix}\bar{\theta}(x,p)\\ \bar{u}(x,p)\\ \bar{\theta}'(x,p)\\ \bar{u}'(x,p)\end{bmatrix}=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ c_{31} & 0 & 0 & c_{34}\\ 0 & c_{42} & c_{43} & 0\end{bmatrix}\begin{bmatrix}\bar{\theta}(x,p)\\ \bar{u}(x,p)\\ \bar{\theta}'(x,p)\\ \bar{u}'(x,p)\end{bmatrix}+\begin{bmatrix}0\\ 0\\ -Q(1+\tau_o p\\ 0\end{bmatrix} \tag{21}$$

$$c_{31} = p(1+\tau_o p), \qquad c_{34} = \in p\ (1+\tau_o p)$$

$$c_{42} = \alpha p^2, \qquad c_{43} = \alpha \tag{22}$$

The prime indicate differentiation with respect to x. The equation (22) can be written as

$$\frac{d}{dx}\tilde{V}(x,p)=\tilde{A}(p)\tilde{V}(x,p)+\tilde{B}(x,p) \tag{23}$$

where $\tilde{V}(x,p)=\left[\bar{\theta}(x,p),\bar{u}(x,p),\bar{\theta}'(x,p),\bar{u}'(x,p)\right]^T$

$$\tilde{B}(x,p)=-Q(1+\tau_o p)[0,0,1,0]^T$$

and

$$\tilde{A}(p)=\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ c_{31} & 0 & 0 & c_{34}\\ 0 & c_{42} & c_{43} & 0\end{bmatrix}$$

4. Solution of the Vector–Matrixequation

The rudiments of the solution methodology through eigenvalue approach as in Das et al [22]. We now proceed to solve equation (23) by eigenvalue approach.

The characteristic equation of the matrix $\tilde{A}(p)$ takes of the form

$$\lambda^4 - \lambda^2\left(c_{31} + c_{42} + c_{34}c_{43}\right) + c_{31}c_{42} = 0 \tag{24}$$

The roots of the characteristic equation (24) which are also the eigenvalues of the matrix $\tilde{A}$ are of the form

$$\lambda = \pm\lambda_1, \quad \lambda = \pm\lambda_2 \tag{25}$$

where

$$\lambda_1^2 + \lambda_2^2 = c_{31} + c_{42} + c_{34}c_{43}$$

$$\lambda_1^2\lambda_2^2 = c_{31}c_{42} \tag{26}$$

The right eigen vector $\tilde{X} = \left[x_1, x_2, x_3, x_4\right]^T$ corresponding to the eigenvalue λ can be written as

$$\tilde{X} = \left[\left(c_{42} - \lambda^2\right), -\lambda c_{43},\ \lambda\left(c_{42} - \lambda^2\right), -\lambda^2 c_{43}\right]^T \tag{27}$$

From (27) we can easily calculate the eigenvector $\tilde{X}_i$ corresponding to the eigenvalue $\lambda = \lambda_i$, i = 1, 2, 3, 4. For our reference we shall use the following notations :

$$\tilde{X}_1 = \left[\tilde{X}\right]_{\lambda=\lambda_1}, \quad \tilde{X}_2 = \left[\tilde{X}\right]_{\lambda=-\lambda_1}, \quad \tilde{X}_3 = \left[\tilde{X}\right]_{\lambda=\lambda_2}, \quad \tilde{X}_4 = \left[\tilde{X}\right]_{\lambda=-\lambda_2} \tag{28}$$

The left eigenvector $\tilde{Y} = \left[y_1, y_2, y_3, y_4\right]$ corresponding to the eigenvalue λ can be calculated as:

$$\tilde{Y} = \left[\left(\lambda^2 - c_{42}\right), \ \lambda c_{34}\ c_{42}, \frac{\lambda\left(\lambda^2 - c_{42}\right)}{c_{31}}, \ \frac{\lambda^2}{c_{31}} \right] \tag{29}$$

For simplicity, henceforth, we shall denote them as

$$\tilde{Y}_1 = \left|\tilde{Y}\right|_{\lambda=\lambda_1}, \quad \tilde{Y}_2 = \left|\tilde{Y}\right|_{\lambda=-\lambda_1}, \quad \tilde{Y}_3 = \left|\tilde{Y}\right|_{\lambda=\lambda_2}, \quad \tilde{Y}_4 = \left|\tilde{Y}\right|_{\lambda=-\lambda_2} \tag{30}$$

4.1 THERMAL SHOCK SEMI–SPACE PROBLEM

We consider a semi–space homogeneous viscoelastic medium without heat source occupying in the region $x \geq 0$.

For this case the solution of the equation (23) is of the form

$$\tilde{V}(x,p) = C_1 \tilde{X}_2 e^{-\lambda_1 x} + C_2 \tilde{X}_4 e^{-\lambda_2 x} \tag{31}$$

where the terms containing exponentials of growing nature in the space variables have been discarded. The constants C_1 and C_2 are to be determined from the boundary conditions.

$$\theta(0,\ t) = \theta_o H(t) \qquad \sigma(0,\ t) = 0 \tag{32}$$

where $H(t)$ is a Heaviside unit step function.

Taking Laplace Transform of (32) we get

$$\bar{\theta}(0,p) = \frac{\theta_o}{p} \ \text{and} \ \bar{\sigma}(0,p) = 0 \tag{33}$$

From (31) we can find

$$\tilde{u}(x,p) = c_{43}\left[C_1 \lambda_1 e^{-\lambda_1 x} - C_2 \lambda_2 e^{-\lambda_2 x}\right] \tag{34}$$

$$\bar{\theta}(x,p) = \left(c_{42} - \lambda_1^2\right)C_1 e^{-\lambda_1 x} + \left(c_{42} - \lambda_2^2\right)C_2 e^{-\lambda_2 x} \tag{35}$$

Using (33) in (34) and (35) we get,

$$\bar{u}(x,p) = -\frac{\alpha\theta_o}{p\left(\lambda_1^2 - \lambda_2^2\right)}\left[\lambda_1 e^{-\lambda_1 x} - \lambda_2 e^{-\lambda_2 x}\right] \tag{36}$$

$$\bar{\theta}(x,p) = \frac{\theta_o}{p\left(\lambda_1^2 - \lambda_2^2\right)}\left[\left(\lambda_1^2 - \alpha p^2\right)e^{-\lambda_1 x} - \left(\lambda_2^2 - \alpha p^2\right)e^{-\lambda_2 x}\right] \tag{37}$$

From (36), (37) and (19) we get

$$\bar{\sigma}(x,p) = \frac{\alpha p\theta_o}{\lambda_1^2 - \lambda_2^2}\left[e^{-\lambda_1 x} - e^{-\lambda_2 x}\right] \tag{38}$$

4.2 PROBLEM FOR A LAYER MEDIA

Now we consider a viscoelastic medium occupying in the region $0 \le x \le X$ with adiabatic thermal boundary $x = X$.

For this case the solution of the equation (23) without heat source is of the form

$$\tilde{V}(x,p) = C_1\tilde{X}_1 e^{\lambda_1 x} + C_2\tilde{X}_2 e^{-\lambda_1 x} + C_3\tilde{X}_3 e^{\lambda_2 x} + C_4\tilde{X}_4 e^{-\lambda_2 x} \tag{39}$$

where the constants C_i, $i = 1, 2, 3, 4$ are to be determined from the following conditions.

The surface $x = 0$ is taken as traction free. Hence

$$\sigma(0, t) = 0, \quad \text{or,} \quad \bar{\sigma}(0,p) = 0 \tag{40}$$

and is subjected to a thermal shock

$$\theta(0,\ t)\ =\ \theta_o H(t), \quad \text{or}, \quad \overline{\theta}(0,p) = \frac{\theta_o}{p} \tag{41}$$

At the rigid base $x\ =\ X$

$$u(X,\ t)\ =\ 0, \quad \text{or}, \quad \overline{u}(X,p) = 0 \tag{42}$$

and

$$q(X,\ t)\ =\ 0, \quad \text{or}, \ \overline{q}(X,p) = 0 \tag{43}$$

where q denotes the component of heat flux vector perpendicular to the surface of the layer. Using Fourier's law of heat conduction which is valid for the theory of thermoelasticity of one relaxation time parameter, equation (42) reduces to

$$\overline{\theta}^{/}(X,p) = 0 \tag{44}$$

From (39) using (40) – (44) we get,

$$C_1 = -\frac{\theta_o e^{-\lambda_1 X}}{2p\left(\lambda_1^2 - \lambda_2^2\right)\cosh\left(\lambda_1 X\right)}$$

$$C_2 = -\frac{\theta_o e^{\lambda_1 X}}{2p\left(\lambda_1^2 - \lambda_2^2\right)\cosh\left(\lambda_1 X\right)}$$

$$C_3 = \frac{\theta_o e^{-\lambda_2 X}}{2p\left(\lambda_1^2 - \lambda_2^2\right)\cosh\left(\lambda_2 X\right)}$$

$$C_4 = \frac{\theta_o e^{\lambda_2 X}}{2p\left(\lambda_1^2 - \lambda_2^2\right)\cosh\left(\lambda_2 X\right)} \tag{45}$$

Thus the displacement and temperature field can be written form (30) as

$$\overline{u}(x,p) = \frac{-\alpha\theta_o}{p\left(\lambda_1^2 - \lambda_2^2\right)}\left[\lambda_1 \frac{\sinh\{\lambda_1(X-x)\}}{\cosh\left(\lambda_1 X\right)} - \lambda_2 \frac{\sinh\{\lambda_2(X-x)\}}{\cosh\left(\lambda_2 X\right)}\right] \tag{46}$$

$$\bar{\theta}(x, p) = \frac{\theta_o}{p\left(\lambda_1^2 - \lambda_2^2\right)}\left[\left(\lambda_1^2 - \alpha p^2\right)\frac{\cosh\{\lambda_1(X - x)\}}{\cosh(\lambda_1 X)} - \left(\lambda_2^2 - \alpha p^2\right)\frac{\cosh\{\lambda_2(X - x)\}}{\cosh(\lambda_2 X)}\right] \tag{47}$$

From (46), (47) and (19) we get,

$$\bar{\sigma}(x, p) = \frac{\alpha p \theta_o}{\left(\lambda_1^2 - \lambda_2^2\right)}\left[\frac{\cosh\{\lambda_1(X - x)\}}{\cosh(\lambda_1 X)} - \frac{\cosh\{\lambda_2(X - x)\}}{\cosh(\lambda_2 X)}\right] \tag{48}$$

4.3 PLANE DISTRIBUTION OF HEAT SOURCES IN A VISCOELASTIC MEDIUM

In this case we assume that heat source acts on the plane $x = 0$ and is of the form

$$Q(x, t) = Q_o H(t)\delta(x) \tag{49}$$

where Q_o is the constant heat and $\delta(x)$ is Dirae's delta function. The laplace transform of (49) is

$$\bar{Q}(x, p) = Q_o \frac{\delta(x)}{p} \tag{50}$$

Assuming the regularity condition at infinity as in Das et al [22], the solution of the equation (23) can be written as

$$\tilde{V}(x, p) = a_2(x)\tilde{X}_2 e^{-\lambda_1 x} + a_4(x)\tilde{X}_4 e^{-\lambda_2 x}\ , \quad \text{for } x > 0 \tag{51}$$

where

$$a_2(x) = \frac{1}{\tilde{Y}_2\tilde{X}_2}\int_{z=-\infty}^{x}\tilde{Y}_2\tilde{B}(z, p)e^{-\lambda_1 z}dz\ , \qquad x > 0$$

Therefore

$$a_2(x)=\frac{1}{\tilde{Y}_2\tilde{X}_2}\int_{z=-\infty}^{x}\frac{\lambda_1\left(\lambda_1^2-c_{42}\right)}{c_{31}}\frac{Q_o(1+\tau_o p)}{p}\delta(z)e^{-\lambda_1 z}dz\,,\qquad x>0$$

$$=\frac{1}{\tilde{Y}_2\tilde{X}_2}\frac{\lambda_1\left(\lambda_1^2-c_{42}\right)}{c_{31}}\frac{Q_o(1+\tau_o p)}{p}$$

$$a_2(x)=\frac{Q_o}{p^2}\frac{\lambda_1\left(\lambda_1^2-c_{42}\right)}{\tilde{V}_2} \tag{52}$$

Similarly

$$a_4(x)=\frac{Q_o}{p^2}\frac{\lambda_2\left(\lambda_2^2-c_{42}\right)}{\tilde{V}_4} \tag{53}$$

where $\tilde{V}_2=\tilde{Y}_2\tilde{X}_2$ and $\tilde{V}_4=\tilde{Y}_4\tilde{X}_4$

The displacement and temperature field can be written from (51) as

$$\bar{u}(x,p)=\frac{Q_o\alpha}{p^2}\left[\frac{\lambda_1^2\left(\lambda_1^2-c_{42}\right)}{\tilde{V}_2}e^{-\lambda_1 x}+\frac{\lambda_2^2\left(\lambda_2^2-c_{42}\right)}{\tilde{V}_4}e^{-\lambda_2 x}\right] \tag{54}$$

and

$$\bar{\theta}(x,p)=-\frac{Q_o}{p^2}\left[\frac{\lambda_1\left(\lambda_1^2-c_{42}\right)^2}{\tilde{V}_2}e^{-\lambda_1 x}+\frac{\lambda_2\left(\lambda_2^2-c_{42}\right)^2}{\tilde{V}_4}e^{-\lambda_2 x}\right] \tag{55}$$

using (54) and (55) in (19) we can write stress component $\bar{\sigma}(x,p)$ as

$$\bar{\sigma}(x,p)=-\frac{Q_o}{p^2}\left[\frac{\lambda_1^3\left(\lambda_1^2-c_{42}\right)}{\tilde{V}_2}e^{-\lambda_1 x}+\frac{\lambda_2^3\left(\lambda_2^2-c_{42}\right)}{\tilde{V}_4}e^{-\lambda_2 x}\right]$$

$$+\frac{Q_o}{p^2}\left[\frac{\lambda_1\left(\lambda_1^2-c_{42}\right)^2}{\tilde{V}_2}e^{-\lambda_1 x}+\frac{\lambda_2\left(\lambda_2^2-c_{42}\right)^2}{\tilde{V}_4}e^{-\lambda_2 x}\right] \tag{56}$$

Equations (54), (55) and (56) determine completely the state of the solid for $x > 0$. The solution for the whole space (when the space $x \le 0$ is also included) is obtained from (54), (55) and (56) by taking the symmetries under consideration. Thus, considering the heat source to act at the location $x = c$ instead of $x = 0$, we may write down the field variables as follows

$$\bar{u}(x,p) = \frac{Q_o\alpha}{p^2}\left[\frac{\lambda_1^2\left(\lambda_1^2 - c_{42}\right)}{\tilde{V}_2} e^{\pm(-\lambda_1(x-c))} + \frac{\lambda_2^2\left(\lambda_2^2 - c_{42}\right)}{\tilde{V}_4} e^{\pm(-\lambda_2(x-c))}\right] \tag{57}$$

$$\bar{\theta}(x,p) = -\frac{Q_o}{p^2}\left[\frac{\lambda_1\left(\lambda_1^2 - c_{42}\right)^2}{\tilde{V}_2} e^{\pm(-\lambda_1(x-c))} + \frac{\lambda_2\left(\lambda_2^2 - c_{42}\right)^2}{\tilde{V}_4} e^{\pm(-\lambda_2(x-c))}\right] \tag{58}$$

$$\bar{\sigma}(x,p) = -\frac{Q_o}{p^2}\left[\frac{\lambda_1^3\left(\lambda_1^2 - c_{42}\right)}{\tilde{V}_2} e^{\pm(-\lambda_1(x-c))} + \frac{\lambda_2^3\left(\lambda_2^2 - c_{42}\right)}{\tilde{V}_4} e^{\pm(-\lambda_2|x-c|)}\right]$$

$$+\frac{Q_o}{p^2}\left[\frac{\lambda_1\left(\lambda_1^2 - c_{42}\right)^2}{\tilde{V}_2} e^{\pm(-\lambda_1(x-c))} + \frac{\lambda_2\left(\lambda_2^2 - c_{42}\right)^2}{\tilde{V}_4} e^{\pm(-\lambda_1(x-c))}\right] \tag{59}$$

where the upper(plus) sign denotes the solution in the region $x \le c$, while the lower (minus) sign denotes the solution in the region $x > c$.

5. Numerical Solution

The inversion of Laplace transform for temperature and stresses in the space time domain are very complex and we prefer to develop an efficient computer software for the purpose of inversion of this integral transform. For the inversion of Laplace transform we follow the method of Bellman [24] and choose seven values of the time t_i, i = 1 to 7 at which stresses and temperature are to be determined where t_i are the roots of the Legendre polynomial of degree seven.

$$\frac{4\mu}{3k} = 0.8 \quad , \quad \beta = 0.05 \quad , \quad A = 0.106 \quad , \quad \in = 0.00165$$

$$\tau_o = 0.02 \quad , \quad c = 5 \quad , \quad \alpha^* = 0.5 \quad , \quad \text{and} \quad X = 4$$

6. Concluding Remarks

In order to study the stress and temperature we have drawn several graphs (coupled and generalized; Fig.1 to Fig.8) for the different cases i) thermal shock problem, ii) problem of lawer medium and iii) plane distribution of heat sources on whole and semi space, for different values of the space variable and at times $t_1 = 0.025775$, $t_2 = 0.138382$, $t_3 = 0.352509$, $t_4 = 0.693147$, t_5 1.21376, $t_6 = 2.04612$ and $t_7 = 3.67119$. It is observed that

1) The characteristic of the stress σ and temperature θ for the material considered in [16] are almost the same in respect of wave propagation for all the problems of i), ii) and iii)

2) For fixed values of x, the amplitudes of stress σ gradually decreases with greater wave length as t increases.

3) For fixed values of time the absolute values of the stress σ and temperature θ gradually decreases as x increases.

4) For fixed values of space variable the absolute values of stress σ and temperature decreases as t increases.

5) The distribution of temperature for the problem (iii) is symmetric with respect to the position of the heat source.

Again it presents ome typical graphs (Fig.9 to Fig.16) from [23] for numerical checks to support the reliability of the present results.

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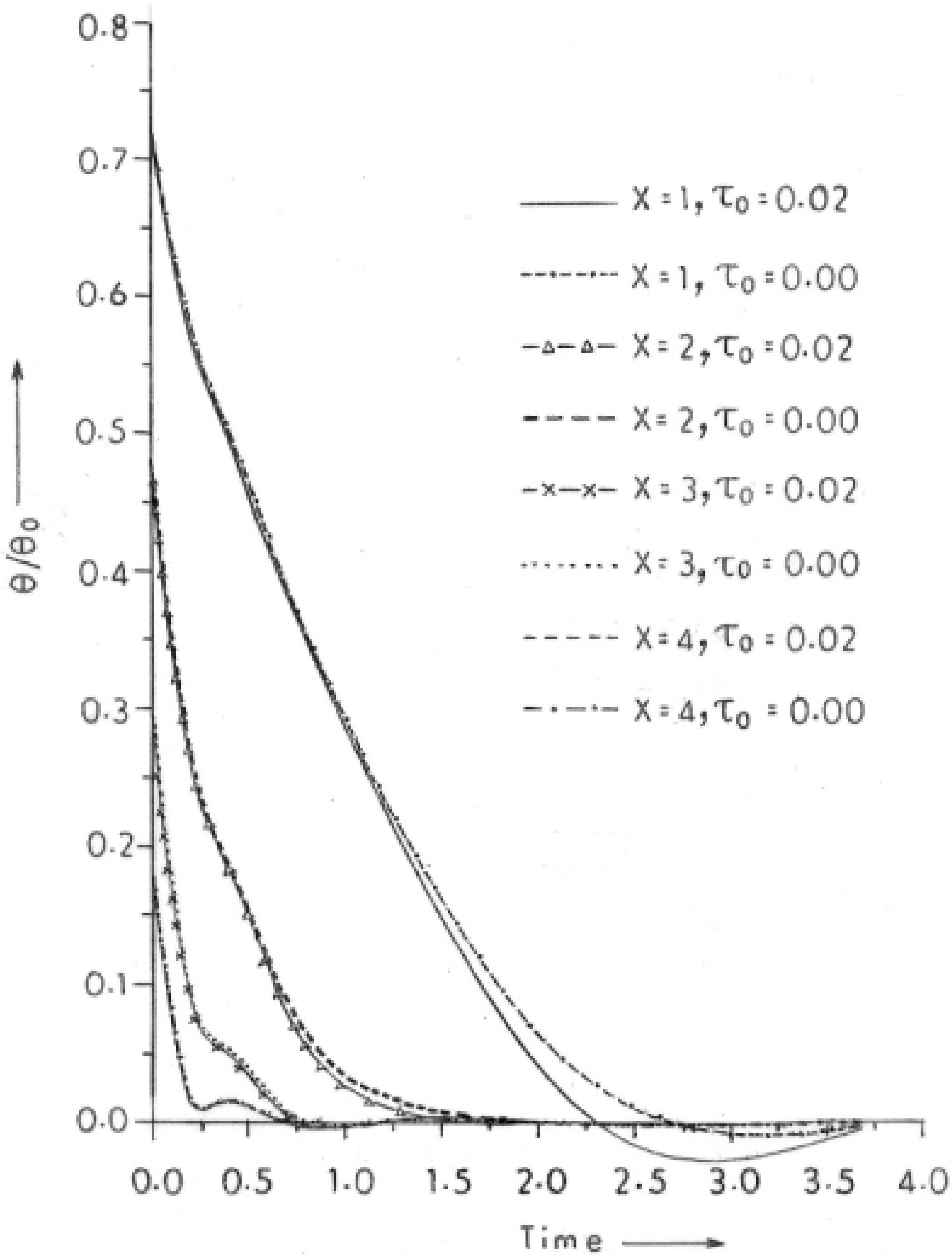

Fig.1: Distribution of Temperature for prob-1

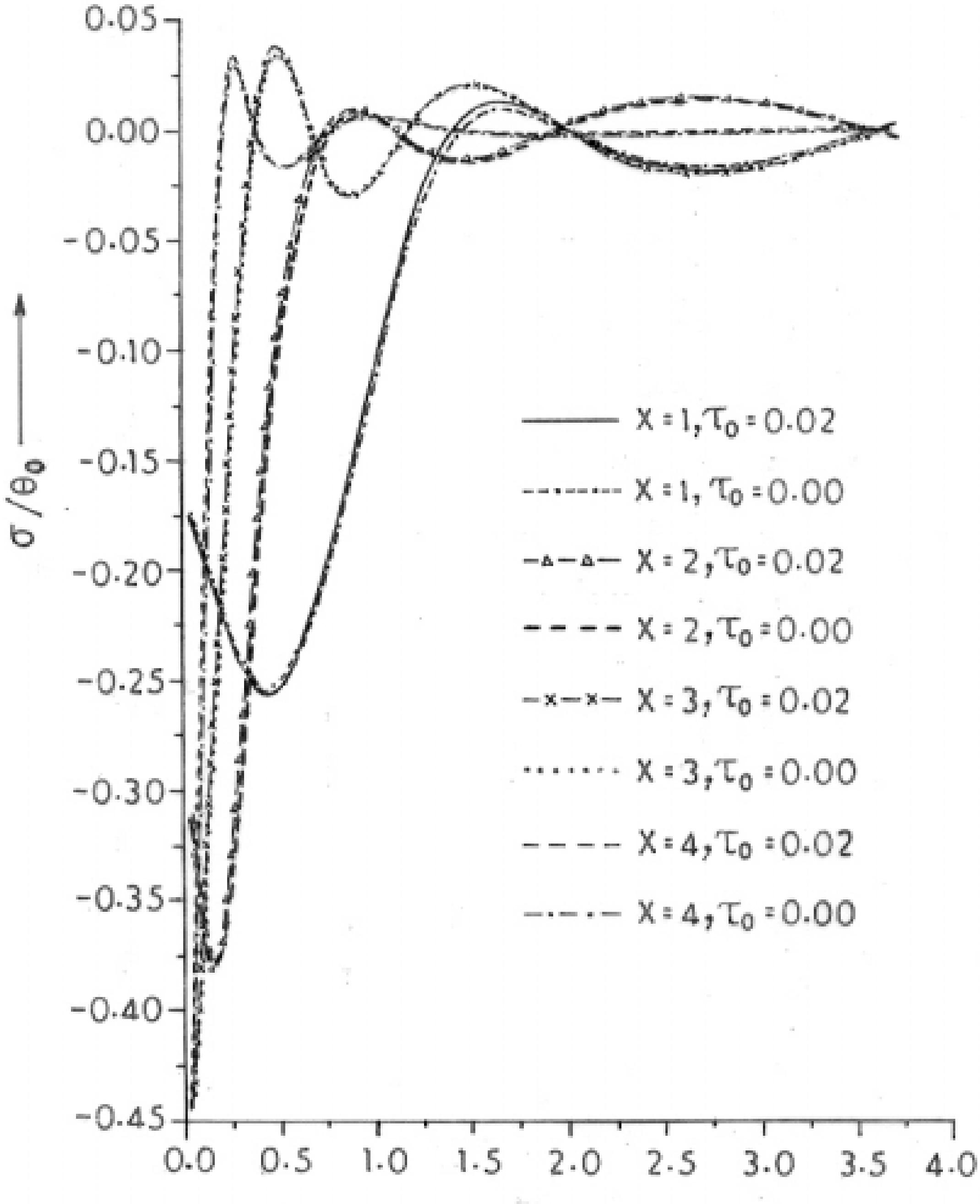

Fig.2: Ditribution of Stresses for prob. 1

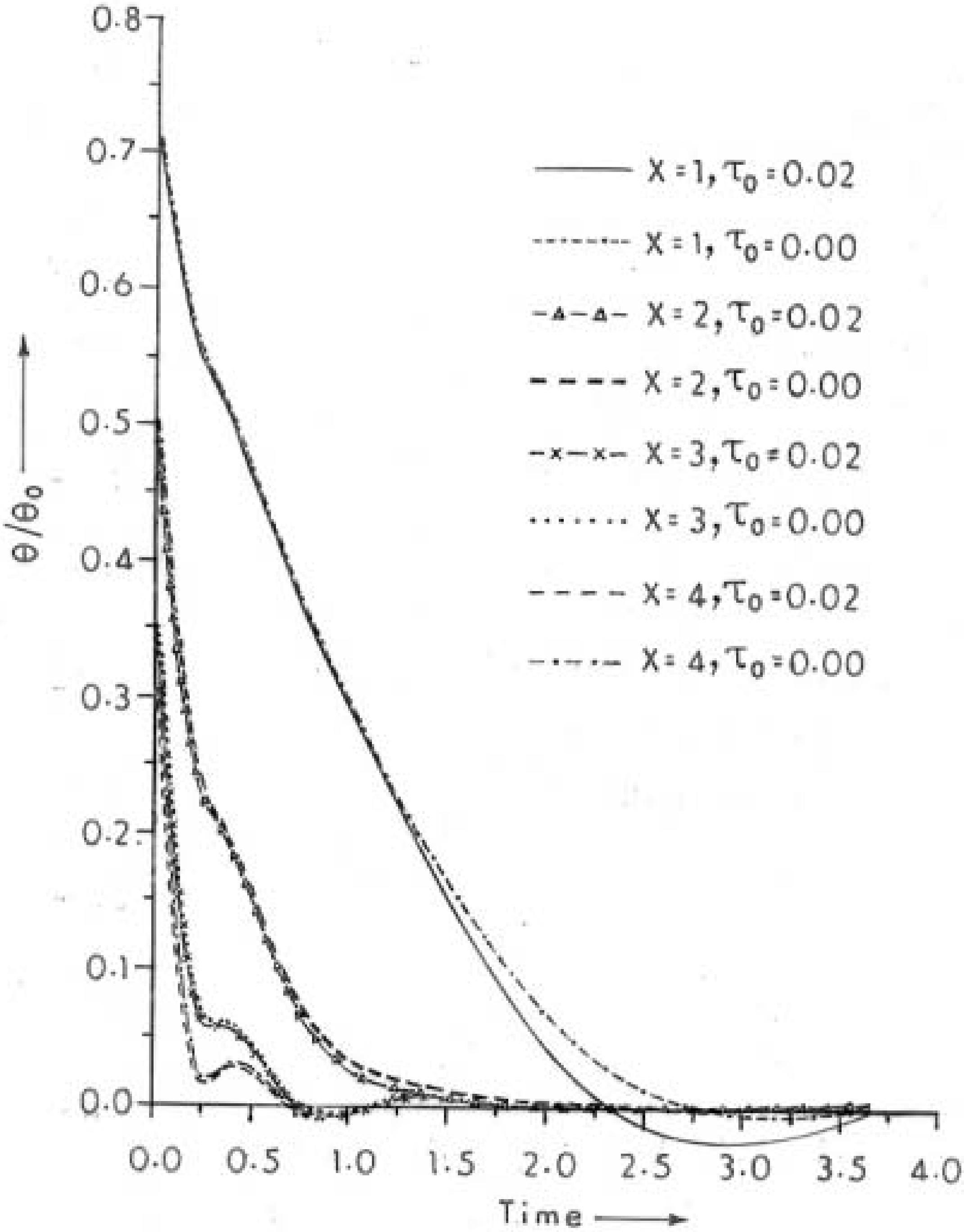

Fig. 3: Distribution of Temperature for prob. 2

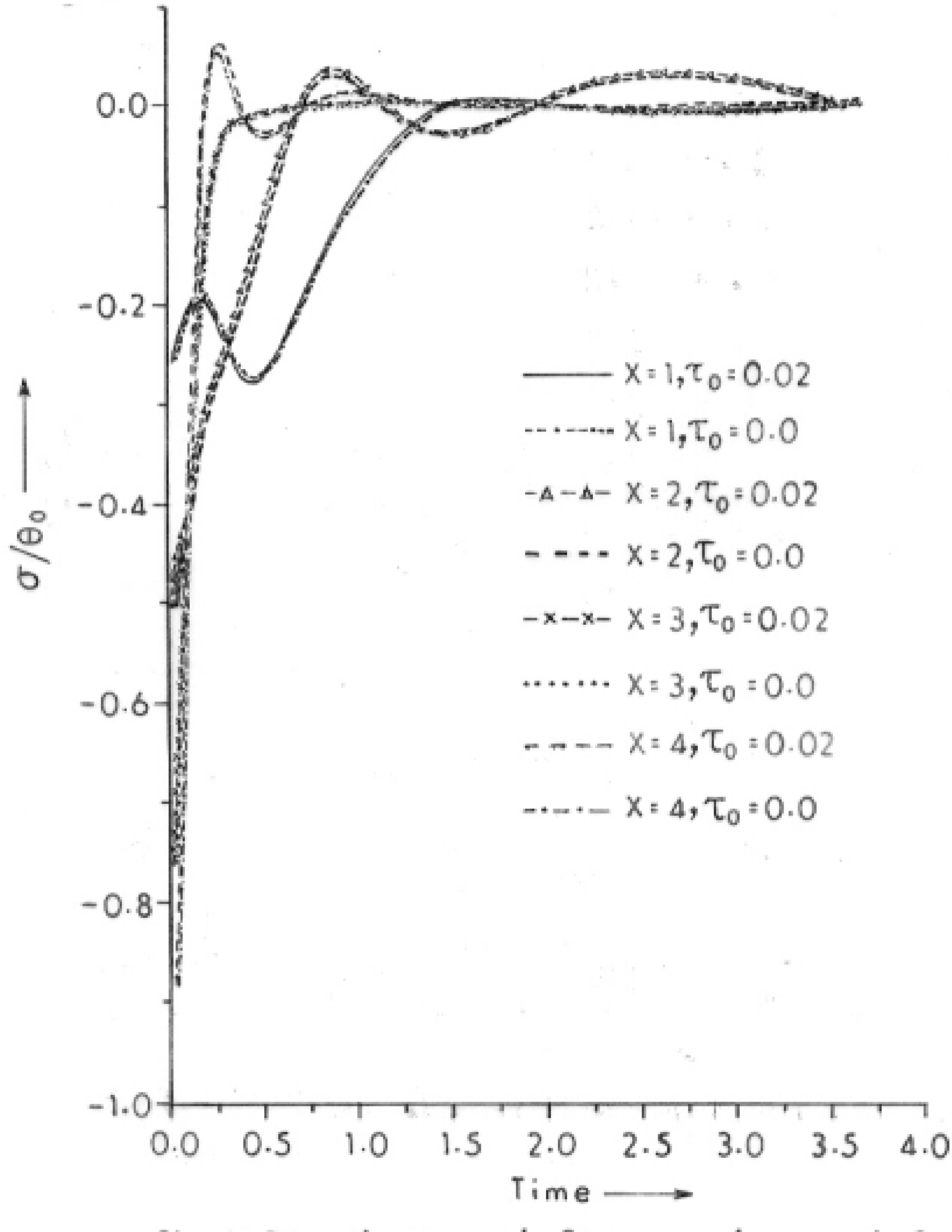

Fig.4: Distribution of Stresses for prob.2

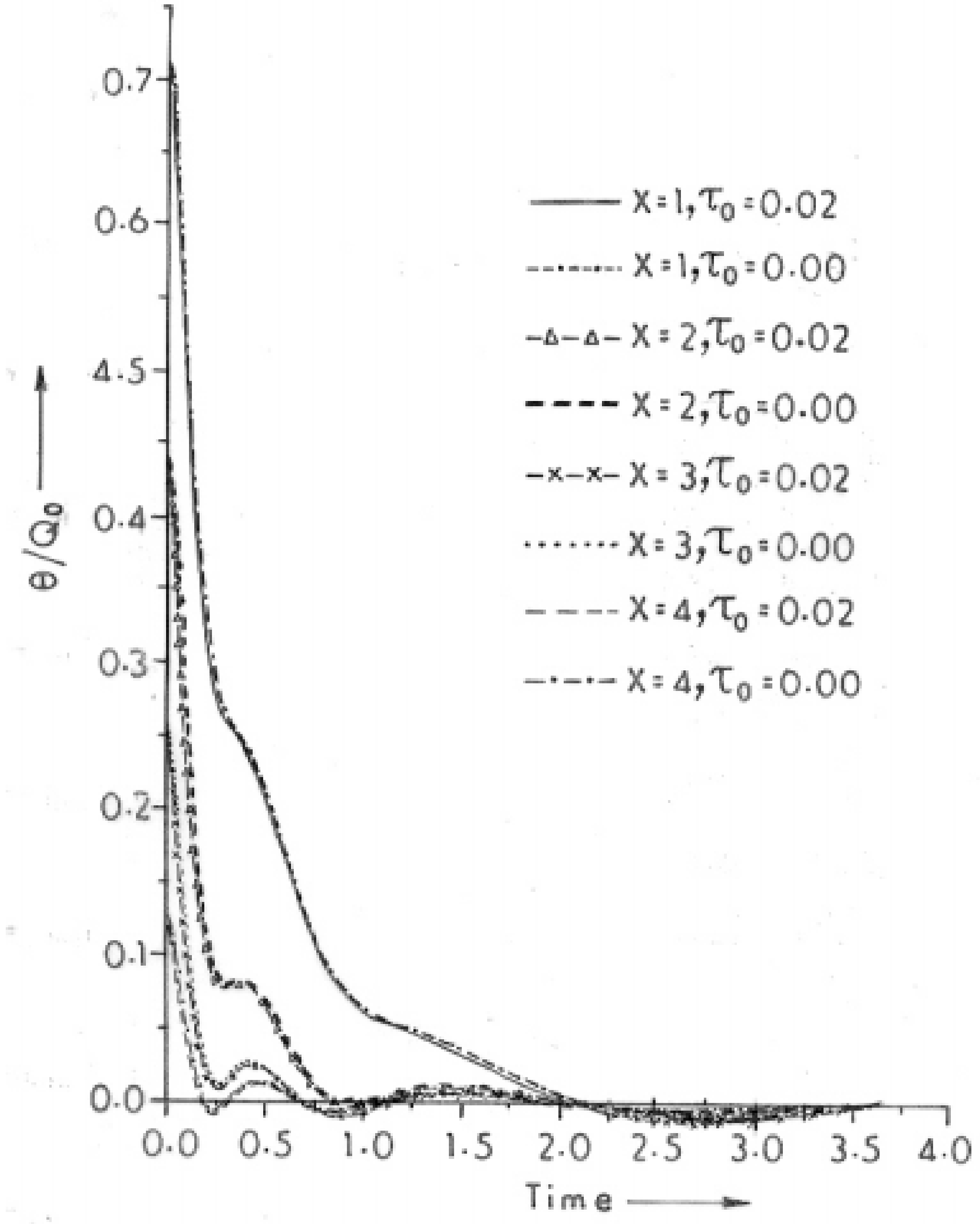

Fig. 5: Distribution of Temperature for prob. 3

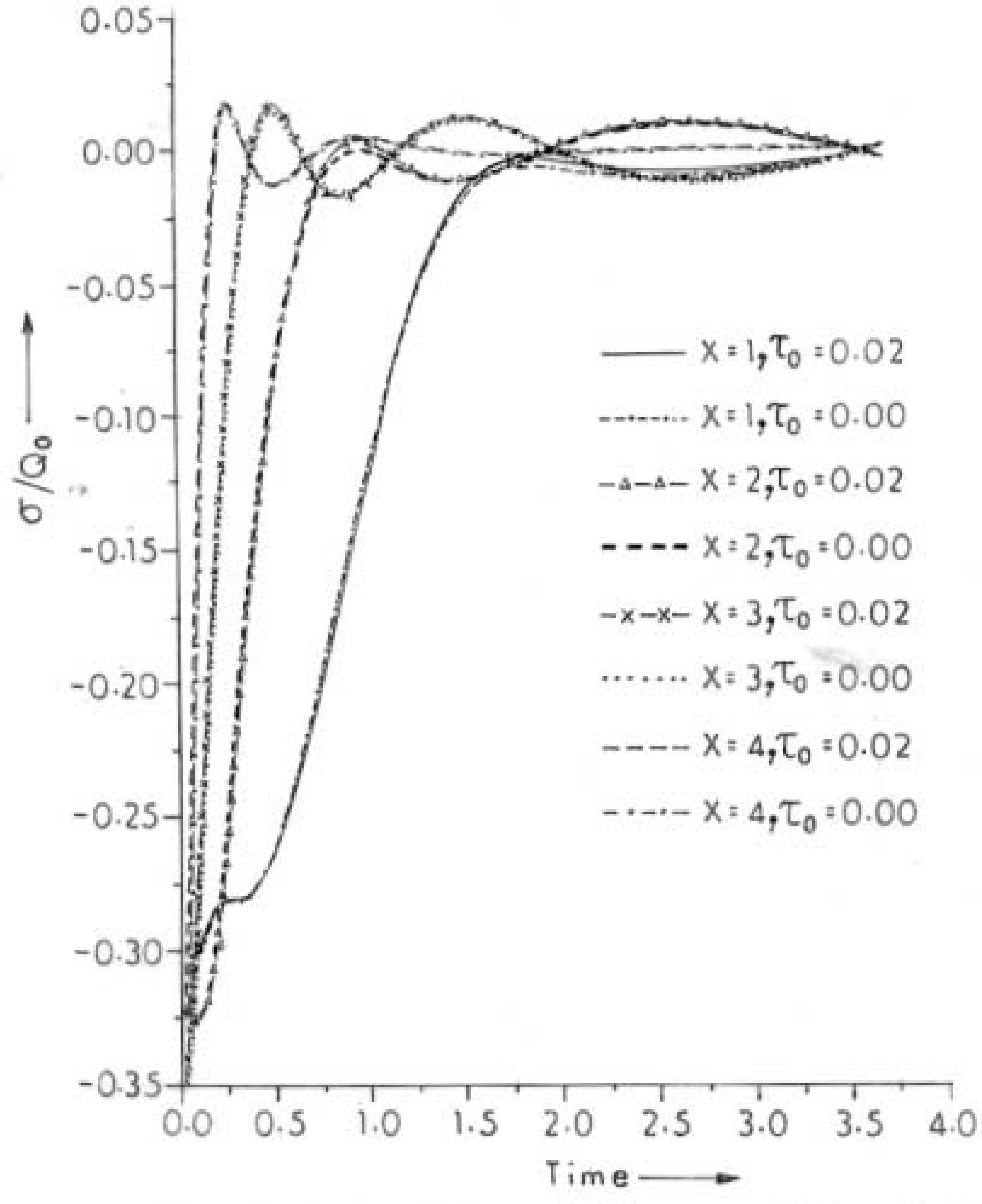

Fig. 6: Distribution of Stresses for prob. 3

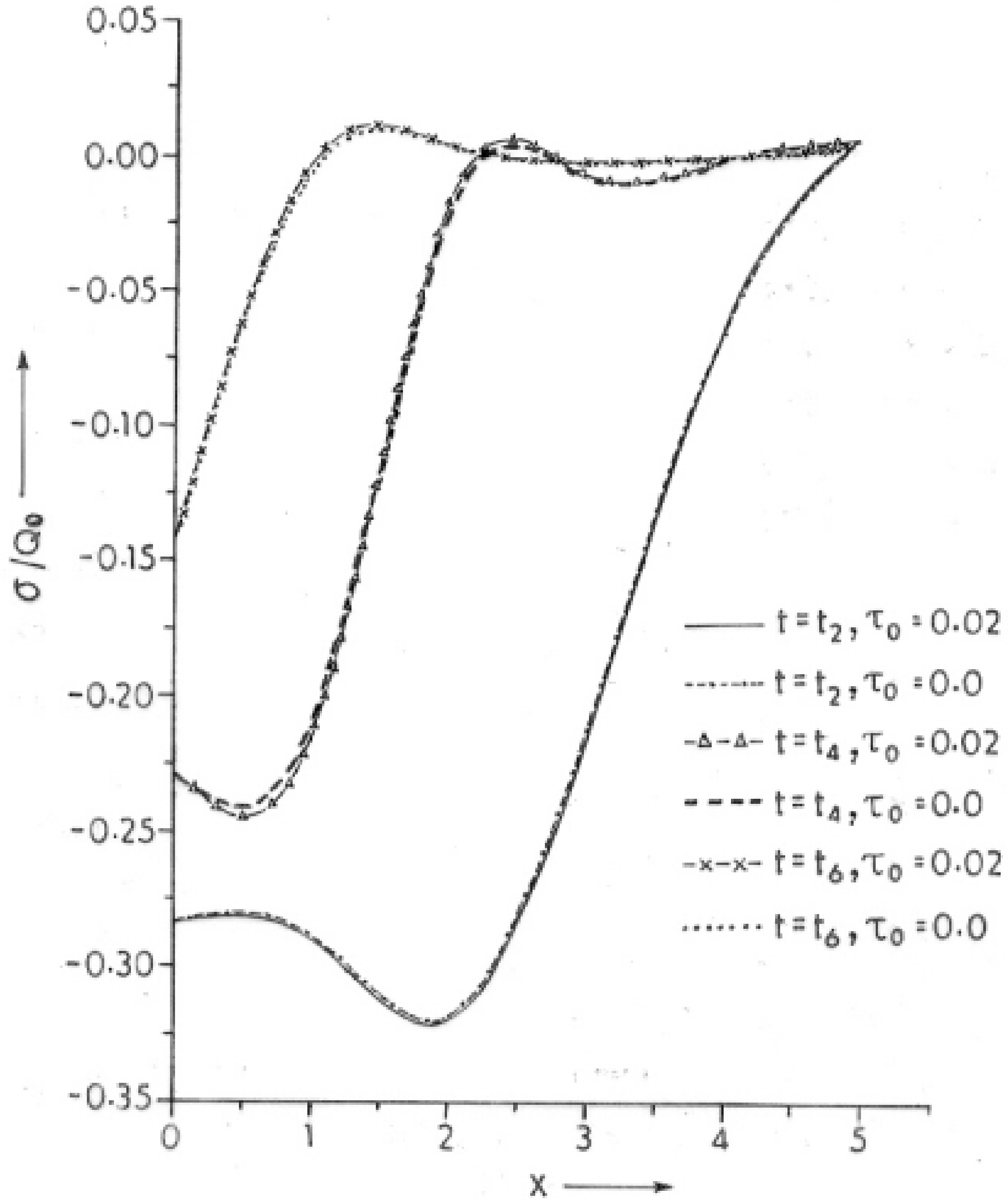

Fig. 7: Distibution of Stresses for fixed values of time (prob-3)

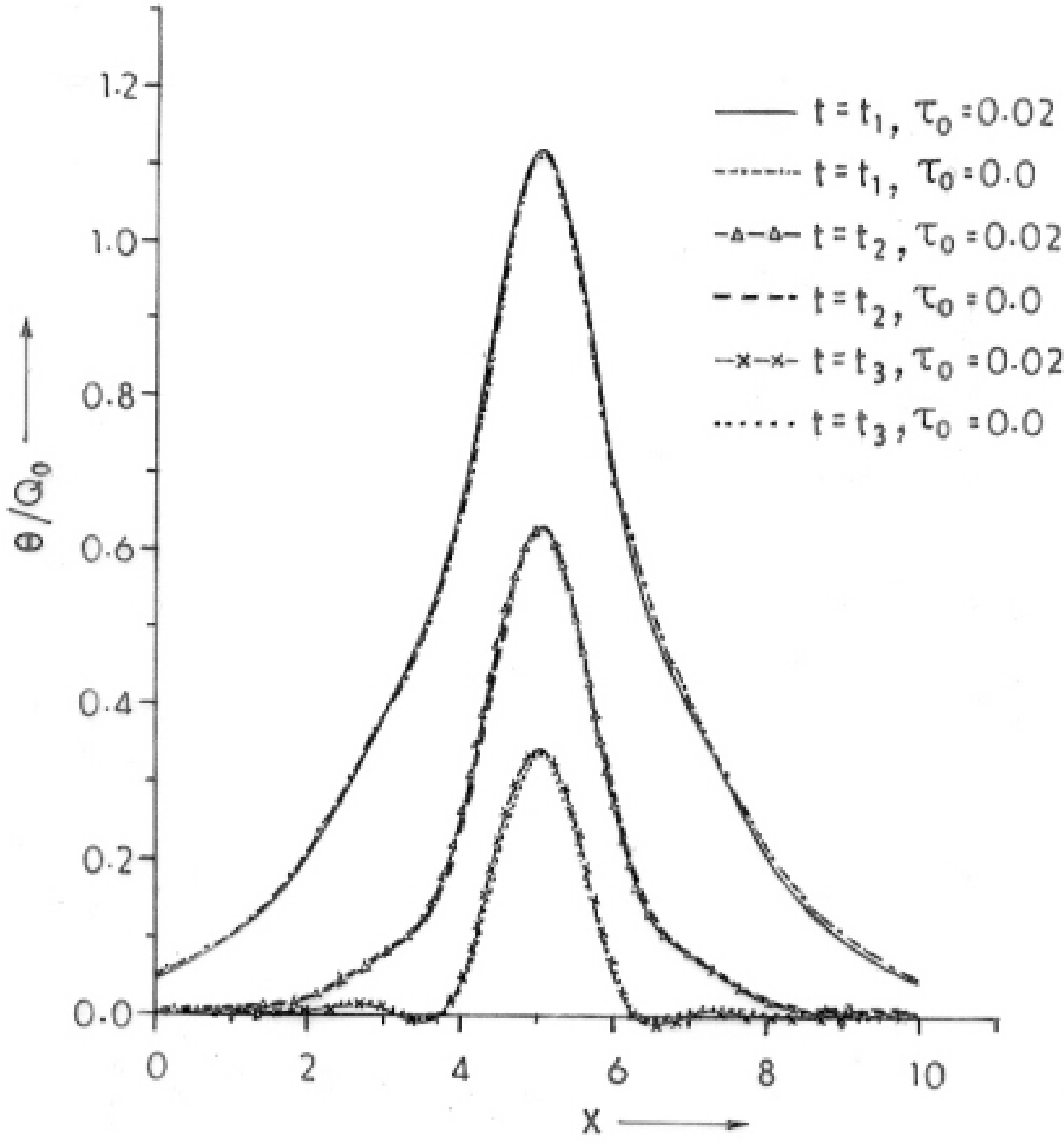

Fig. 8: Distribution of Temperature for fixed values of time (prob-3).

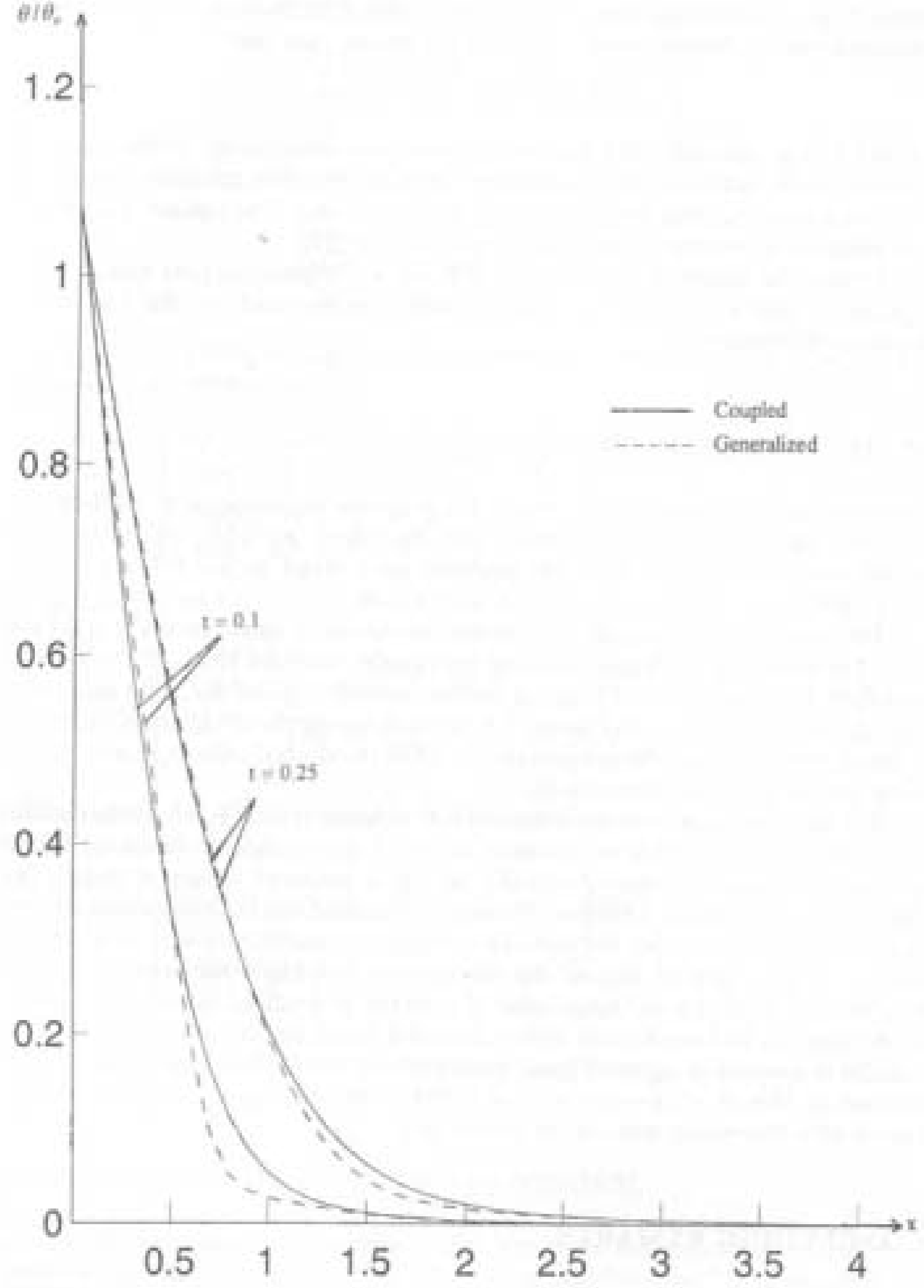

Figure 9. Temperature distribution for Problem 1.

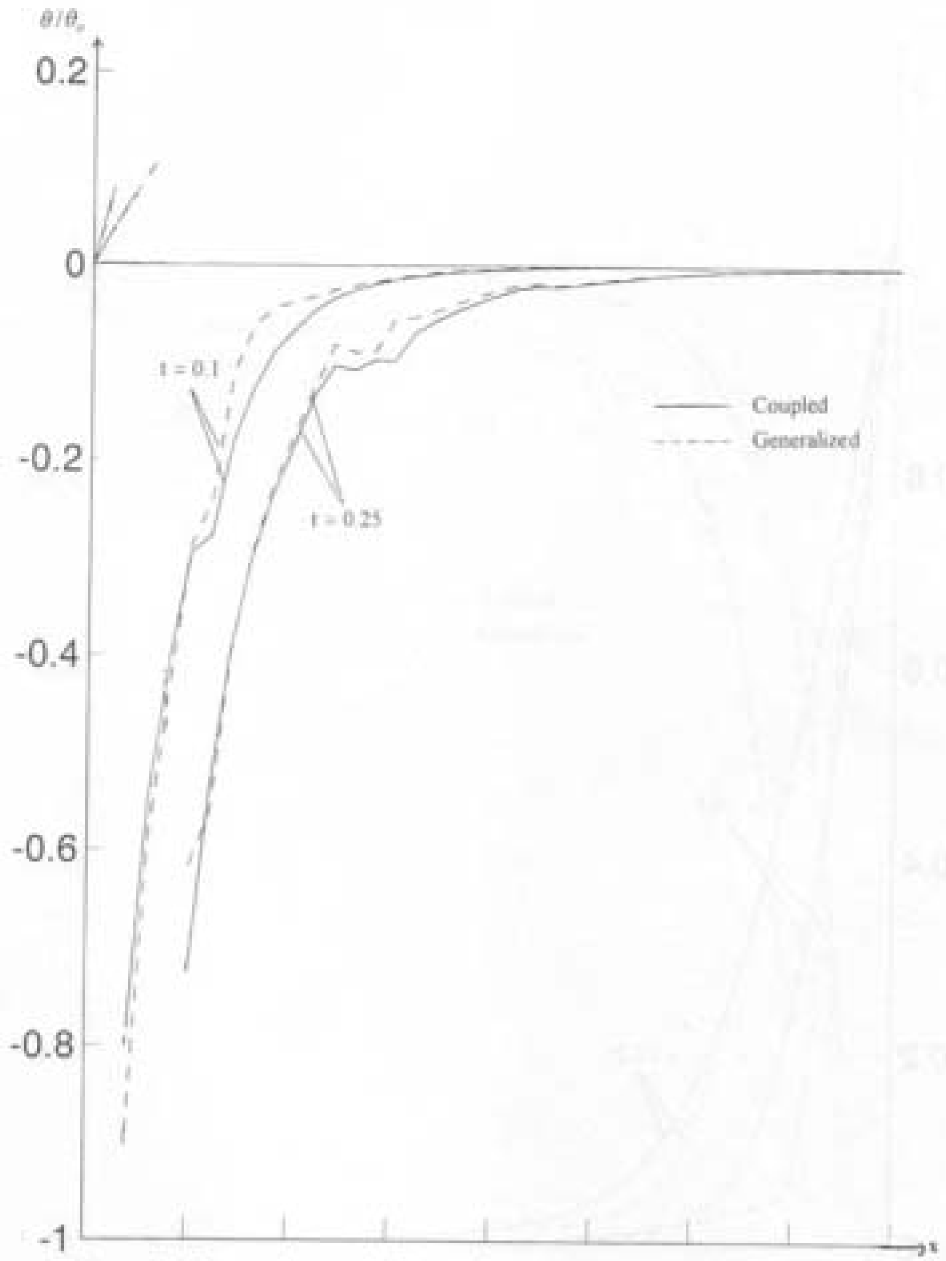

Figure 10 Stress distribution for Problem 1.

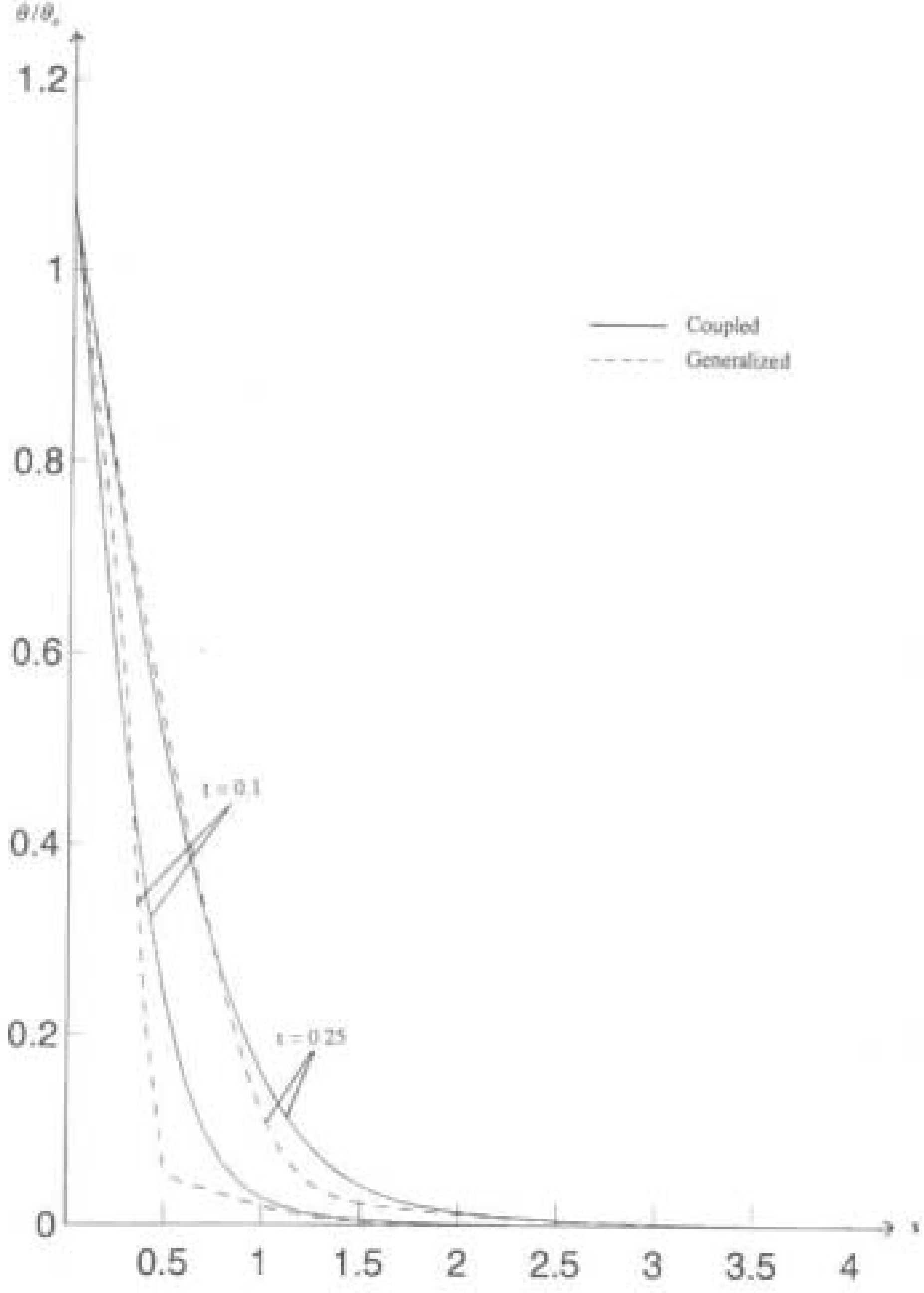

Figure 11 Temperature distribution for Problem 2.

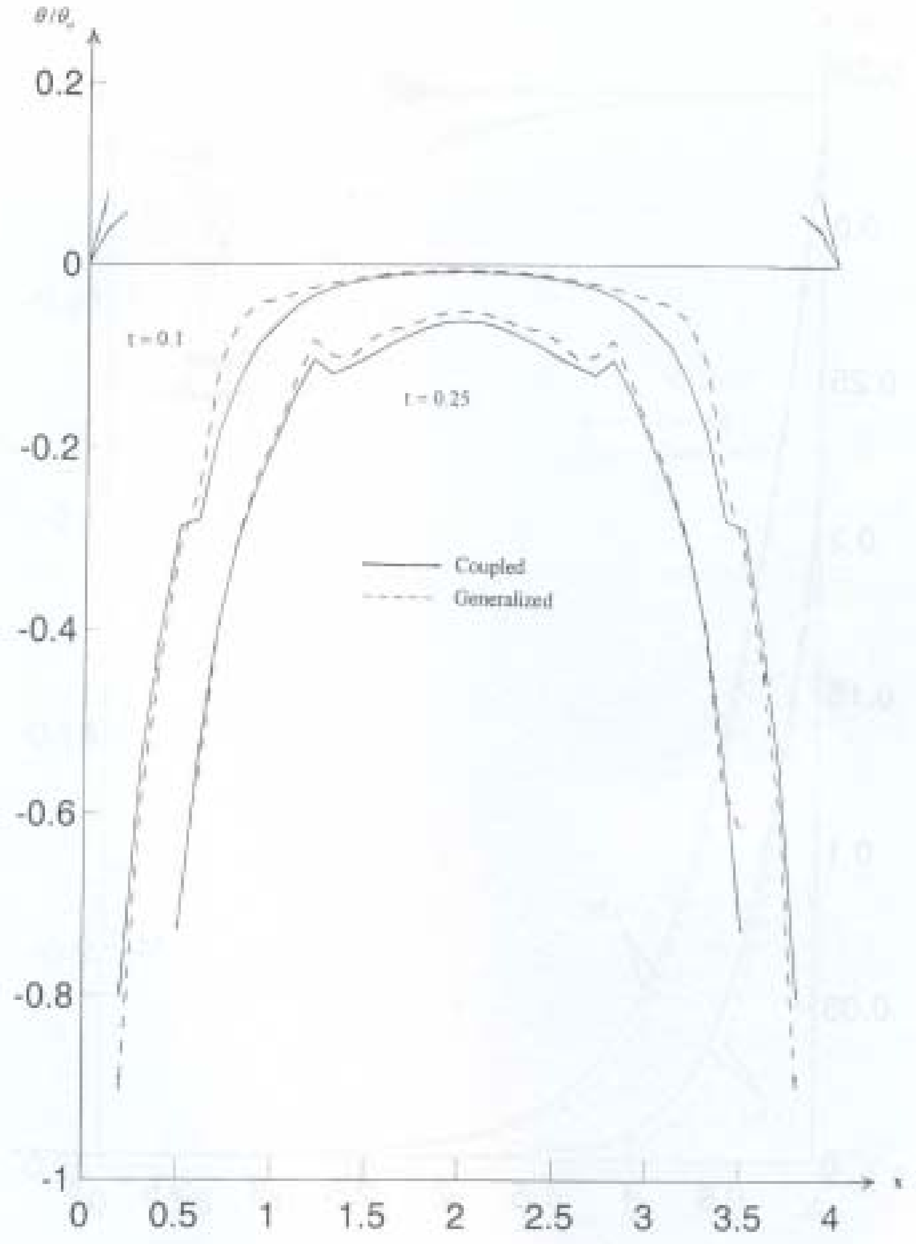

Figure 12 Stress distribution for Problem 2.

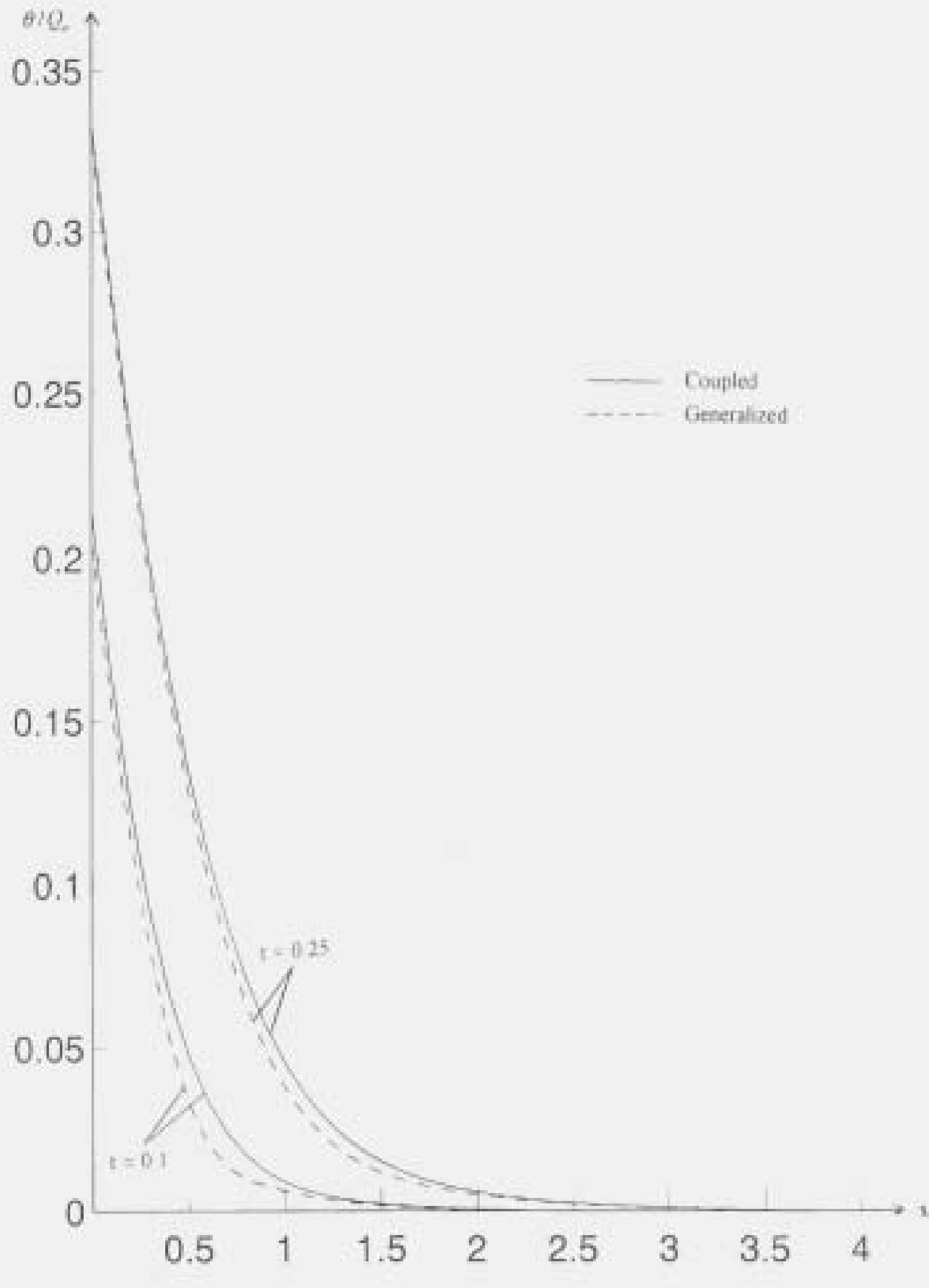

Figure 13 Temperature distribution for the whole space in Problem 3.

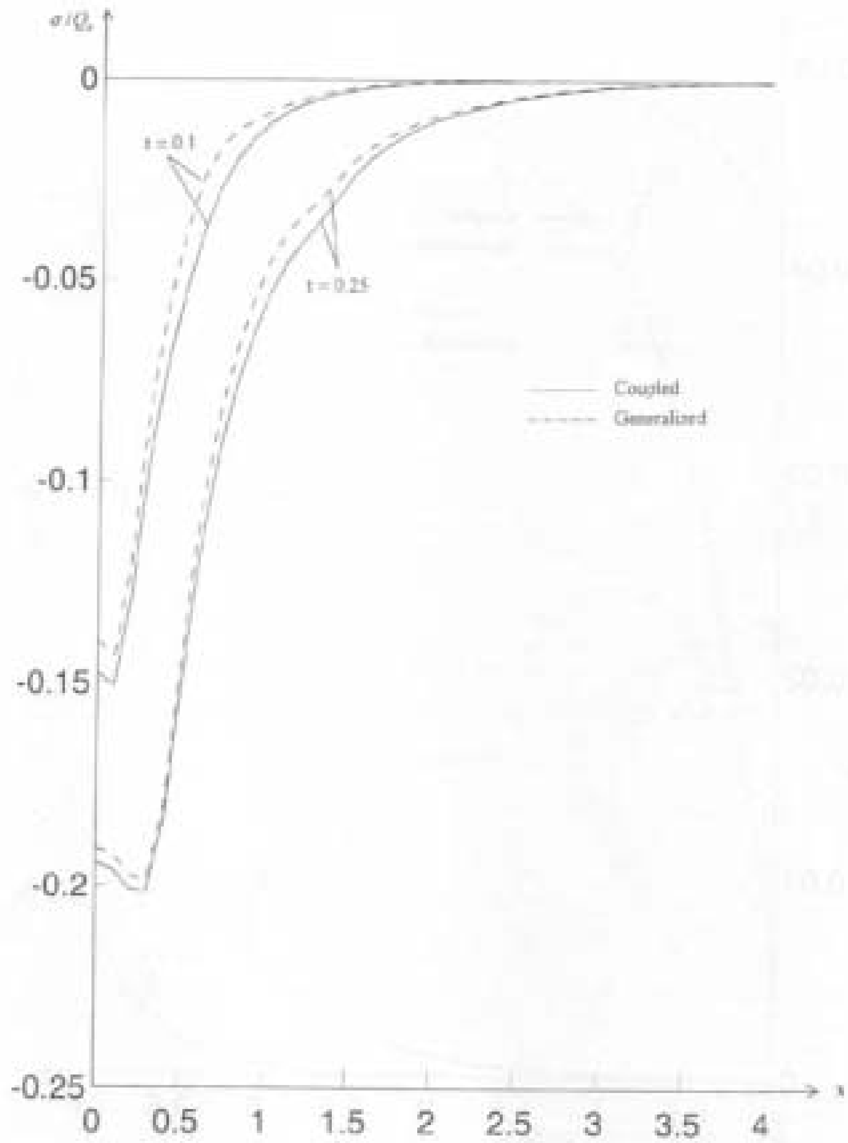

Figure 14 Stress distribution for the whole space in Problem 3.

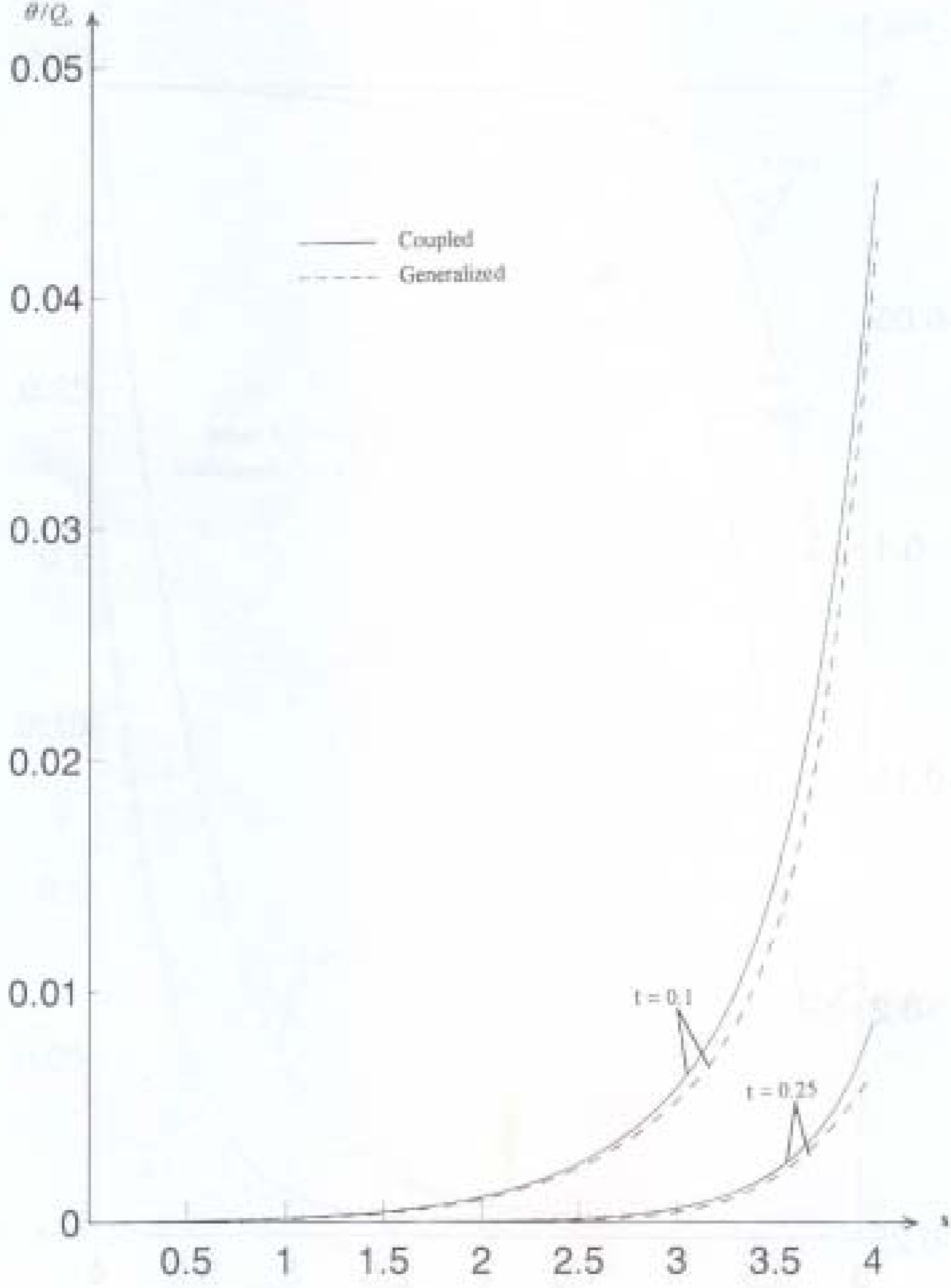

Figure 15 Temperature distribution for the semi-space in Problem 3.

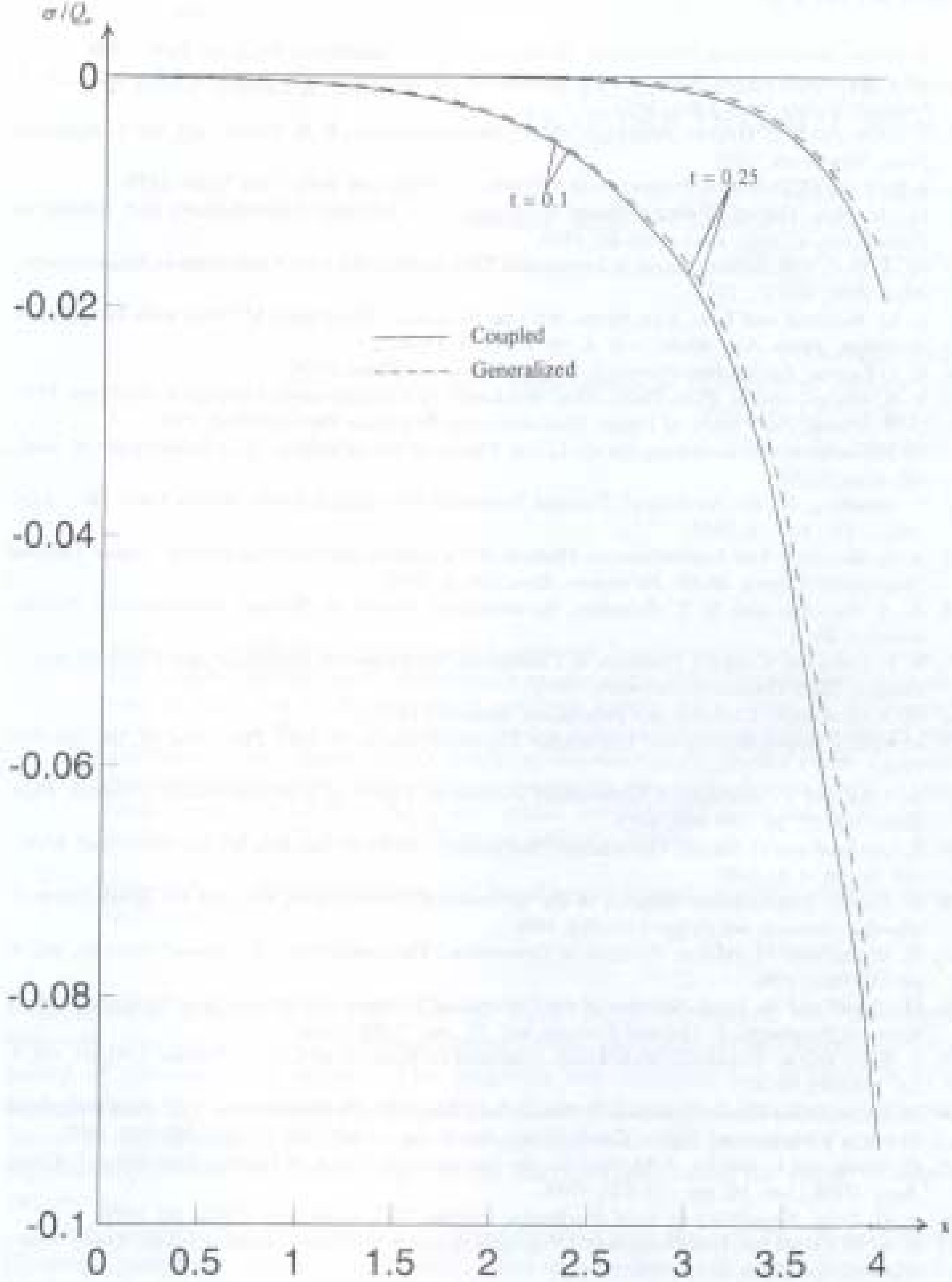

Figure 16 Stress distribution for the semi-space in Problem 3.