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A Continuous Deterministic Inventory System for Deteriorating Items with Inventory-Level-Dependent Time Varying Demand Rate

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Abstract

A continuous deterministic inventory model for deteriorating items is developed under instantaneous replenishment and stock-dependent time - varying demand rate. A numerical example is taken to determine the optimal average cost. A sensitivity analysis is carried out to demonstrate the effects of changing parameter values on the optimal solution of the system.

Keywords and Phrases: *Inventory model, Deterioration, Demand, Replenishment.*

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1. Introduction

The classical approach in deterministic inventory modelling is to assume a uniform demand rate. Many inventory management experts have observed that for some items, the demand rate is directly related to the amount of inventory displayed. According to Levin *et al.* (1972), "at times, the presence of inventory has a motivational effect on the people around it. It is a common belief that large piles of goods displayed in a supermarket will lead the customer to buy more". Silver and Peterson (1979) have noted that the sales at the retail level tend to be proportional to the inventory displayed. These observations have attracted many marketing researchers and practitioners to investigate the situation where the demand rate is dependent on the level of the on-hand inventory. Gupta and Vrat (1986) presented an inventory model for stock - dependent consumption rate. However, their expressions, for average total system cost was based on the demand rate which was based on initial stock-level (or order quantity) rather than instantaneous inventory level. Mondal and Phaujder (1989) suggested corrections to their model. The idea that the demand rate would decline along with stock-level throughout the cycle was reflected first in the model developed by Baker and Urban (1988). Pal, Goswami and Chaudhuri (1993) extended the model of Baker and Urban (1988) to the case of deteriorating items. Datta and Pal (1990) modified the concept of Baker and Urban (1988) by assuming that demand rate would decline along with stock-level down to a certain level, and then it would become constant for the rest of the cycle. One of the terminal conditions used in this model was that the cycle would end with zero stock. Later on, Urban (1992) relaxed the terminal condition of zero-ending inventory and suggested that, in an inventory system with inventory - level - dependent demand rate, it was more profitable to utilize higher inventory levels resulting in greater demand. Recently Roy and Samanta (2004) developed one type of continuous order-level inventory model for deteriorating items under instantaneous replenishment and stock-dependent time-varying demand.

In this paper, an order - level inventory model has been developed for deteriorating items. The demand rate is stock-dependent as well as time dependent also. It is proved that the results reduce to the corresponding results for the standard EOQ model when demand is constant and deterioration approaches zero. Sensitivity of the decision variables is studied to see how far the output of the model is affected by changes or errors in its input parameters based on an numerical example.

2. Notations and Assumptions

The following notations and assumptions are used for developing the model.

(i) $f(t) = (a + be^{-t})[Q(t)]^{\beta}$ is the demand rate at time t defined in the interval $[0, T], Q(t)$ is the on-hand inventory at time $t(0 \leq t \leq T)$ and $(a+b)[Q(0)]^{\beta}$ is the rate of demand at time zero. Here $a(> 0), b(> 0)$ are scale parameters and $\beta(0 < \beta < 1)$ denotes the shape parameter and is the measure of responsiveness of the demand rate to changes in the level of the on-hand inventory. The variable $Q(t)$ is assumed to be continuous in time and a, b, β are taken as constants.

(ii) h is the holding cost per unit per unit time.

(iii) A is the replenishment cost per cycle.

(iv) C_1 is the unit cost of the item.

(v) Replenishment is instantaneous and lead time is zero.

(vi) No shortage in inventory is allowed.

(vii) T is the length of a cycle.

(viii) A Constant fraction $\theta(0 < \theta << 1)$ of the on-hand inventory deteriorates per unit time.

3. Formulation of the problem and its analysis

At the begining of each cycle, the inventory level decreases rapidly because the quantity of demand is greater at a higher level of inventory. As the inventory is depleted, the rate of decrease of inventory level slows down. Ultimately the inventory reaches the zero level at the end of the cycle time T. The instantaneous state of $Q(t)$ over the cycle time T is given by the following first order nonlinear differential equation

$$\frac{dQ(t)}{dt} + \theta Q(t) = -(a + be^{-t})[Q(t)]^{\beta};\ 0 \leq t \leq T,\ a > 0,\ b > 0,\ 0 < \beta < 1 \quad (3.1)$$

where $Q(0) = q$, initial inventory, and $Q(T) = 0$. Hence

$$\begin{aligned}[][Q(t)]^{1-\beta} &= q^{(1-\beta)} \exp\left(-\theta(1-\beta)t\right) \\ &\quad -(1-\beta)\exp\left(-\theta(1-\beta)t\right)\int_0^t (a+be^{-t})\exp\left(\theta(1-\beta)t\right)dt \end{aligned} \tag{3.2}$$

using $Q(T) = 0$, we have

$$q^{\gamma} = \frac{a}{\theta}\left\{\exp(\theta\gamma T) - 1\right\} + \frac{b\gamma}{(\theta\gamma-1)}\left\{\exp\left((\theta\gamma-1)T\right) - 1\right\} \tag{3.3}$$

where

$$\gamma = 1 - \beta \tag{3.4}$$

From (3.2) and (3.3) we get

$$\begin{aligned}[][Q(t)]^{\gamma} &= a\gamma T\left[1 - \frac{t}{T} - \frac{\theta\gamma}{2T}(T^2-t^2) + \frac{\theta^2\gamma^2}{6T}(T^3-t^3)\right. \\ &\quad -\frac{\gamma\theta b}{aT}\left\{(1+\theta\gamma)(T-t) - \frac{\theta\gamma}{2}(T^2-t^2)\right\} \\ &\quad \left.-\frac{b}{aT}(1+\theta\gamma+\theta^2\gamma^2)(e^{-T}-e^{-t})\right] \end{aligned} \tag{3.5}$$

neglecting θ^3 and higher powers of θ since $0 < \theta << 1$. Hence

$$\begin{aligned} Q(t) &= (a\gamma)^{\frac{1}{\gamma}} T^{(\frac{1}{\gamma}-2)}\left[T^2 - \frac{tT}{\gamma} - \frac{\theta T}{2}(T^2-t^2) + \frac{\theta^2\gamma T}{6}(T^3-t^3)\right. \\ &\quad -\frac{b\theta}{a}T\left\{(1+\theta\gamma)(T-t) - \frac{\theta\gamma}{2}(T^2-t^2)\right\} \\ &\quad -\frac{bT}{a\gamma}(1+\theta\gamma+\theta^2\gamma^2)(e^{-T}-e^{-t}) \\ &\quad +\frac{\beta}{2\gamma^2}\left\{t^2 + \frac{\theta^2\gamma^2}{4}(T^2-t^2)^2 + \frac{\gamma^2 b^2\theta^2}{a^2}(T-t)^2\right. \\ &\quad +\frac{b^2}{a^2}(e^{-T}-e^{-t})^2(1+2\theta\gamma+3\theta^2\gamma^2) + \theta\gamma t(T^2-t^2) \end{aligned}$$

$$
\begin{aligned}
&-\frac{\theta^2\gamma^2}{3}t(T^3-t^3)+\frac{2\theta\gamma b}{a}\left\{(1+\theta\gamma)(Tt-t^2)-\frac{\theta\gamma}{2}(T^2t-t^3)\right\}\\
&+\frac{2bt}{a}(1+\theta\gamma+\theta^2\gamma^2)(e^{-T}-e^{-t})\\
&+\frac{b\theta^2\gamma^2}{a}(T^2-t^2)(T-t)+\frac{b\theta\gamma}{a}(1+\theta\gamma)(e^{-T}-e^{-t})(T^2-t^2)\\
&-\frac{b\theta^2\gamma^2}{3a}(e^{-T}-e^{-t})(T^3-t^3)\\
&+\frac{2b^2\theta\gamma}{a^2}(e^{-T}-e^{-t})(T-t)\left\{1+2\theta\gamma-\frac{\theta\gamma}{2}(T+t)\right\}\Bigg\}\Bigg]
\end{aligned}
\tag{3.6}
$$

neglecting θ^3 and higher powers of θ since $0<\theta<<1$.

The inventory I_T in a cycle is given by

$$
\begin{aligned}
I_T \;=\; & \int_0^T Q(t)dt=(a\gamma)^{\frac{1}{\gamma}}T^{(\frac{1}{\gamma}-2)}\Bigg[T^3\left(1-\frac{1}{2\gamma}\right)-\frac{\theta}{3}T^4+\frac{\theta^2\gamma T^5}{8}\\
&-\frac{b\theta}{a}\left\{(1+\theta\gamma)\frac{T^3}{2}-\frac{\theta\gamma T^4}{3}\right\}-\frac{b}{a\gamma}(1+\theta\gamma+\theta^2\gamma^2)\\
&\times\big(e^{-T}(T-1)+1\big)T+\frac{\beta}{2\gamma^2}\Bigg\{\frac{T^3}{3}+\frac{2\theta^2\gamma^2}{15}T^5+\frac{\gamma^2b^2\theta^2}{3a^2}T^3\\
&+\frac{b^2}{2a^2}(1+2\theta\gamma+3\theta^2\gamma^2)\left(e^{-2T}(2T+3)-4e^{-T}+1\right)+\frac{\theta\gamma}{4}T^4\\
&-\frac{\theta^2\gamma^2}{10}T^5+\frac{\theta\gamma b}{a}\left\{(1+\theta\gamma)\frac{T^3}{3}-\frac{\theta\gamma}{4}T^4\right\}\\
&+\frac{b}{a}\left(1+\theta\gamma+\theta^2\gamma^2\right)\left(e^{-T}(T^2+2T+2)-2\right)+\frac{5b\theta^2\gamma^2}{12a}T^4\\
&+\frac{b\theta\gamma}{3a}(1+\theta\gamma)\left\{2e^{-T}(T^3-3T-3)+6-3T^2\right\}\\
&-\frac{b\theta^2\gamma^2}{12a}\left\{3e^{-T}(T^4-4T^2-8T-8)-4T^3+24\right\}\\
&+\frac{b^2\theta\gamma}{a^2}\Big\{(1+2\theta\gamma)\left(e^{-T}(T^2-2)-2T+2\right)\\
&-\frac{\theta\gamma}{3}\left(2e^{-T}(T^3-3T-3)-3T^2+6\right)\Big\}\Bigg\}\Bigg]
\end{aligned}
\tag{3.7}
$$

Total deterioration in a cycle

$$\begin{aligned} D &= q - \text{ Total demand}, \quad (q = \text{ initial inventory }) \\ &= q - \int_0^T (a+be^{-t})[Q(t)]^\beta dt \\ &= q - a\int_0^T [Q(t)]^\beta dt - b\int_0^T e^{-t}[Q(t)]^\beta dt \end{aligned} \tag{3.8}$$

After some calculations neglecting θ^3 and higher powers of θ, we have

$$\begin{aligned} \int_0^T [Q(t)]^\beta dt &= (a\gamma)^{\frac{\beta}{\gamma}} T^{\left(\frac{\beta}{\gamma}-2\right)} \Bigg[T^3\left(1-\frac{\beta}{2\gamma}\right) - \frac{\theta\beta}{3}T^4 + \frac{\theta^2\beta\gamma}{8}T^5 \\ &\quad - \frac{b\beta\theta}{a}\left\{(1+\theta\gamma)\frac{T^3}{2} - \frac{\theta\gamma}{3}T^4\right\} \\ &\quad - \frac{b\beta}{a\gamma}(1+\theta\gamma+\theta^2\gamma^2)\left(e^{-T}(T-1)+1\right)T \\ &\quad + \frac{\beta(2\beta-1)}{2\gamma^2}\Bigg\{\frac{T^3}{3} + \frac{2\theta^2\gamma^2}{15}T^5 + \frac{\gamma^2b^2\theta^2}{3a^2}T^3 \\ &\quad + \frac{b^2}{2a^2}(1+2\theta\gamma+3\theta^2\gamma^2)\left(e^{-2T}(2T+3)-4e^{-T}+1\right) \\ &\quad + \frac{\theta\gamma}{4}T^4 - \frac{\theta^2\gamma^2}{10}T^5 + \frac{\theta\gamma b}{a}\left\{(1+\theta\gamma)\frac{T^3}{3} - \frac{\theta\gamma}{4}T^4\right\} \\ &\quad + \frac{b}{a}(1+\theta\gamma+\theta^2\gamma^2)\left(e^{-T}(T^2+2T+2)-2\right) + \frac{5b\theta^2\gamma^2}{12a}T^4 \\ &\quad + \frac{b\theta\gamma}{3a}(1+\theta\gamma)\left\{2e^{-T}(T^3-3T-3)+6-3T^2\right\} \\ &\quad - \frac{b\theta^2\gamma^2}{12a}\left\{3e^{-T}(T^4-4T^2-8T-8)-4T^3+24\right\} \\ &\quad + \frac{b^2\theta\gamma}{a^2}\Big\{(1+2\theta\gamma)\left(e^{-T}(T^2-2)-2T+2\right) \\ &\quad - \frac{\theta\gamma}{3}\left(2e^{-T}(T^3-3T-3)-3T^2+6\right)\Big\}\Bigg\}\Bigg] \end{aligned} \tag{3.9}$$

and

$$
\begin{aligned}
&\int_0^T e^{-t}\,[Q(t)]^{\beta}\,dt \\
= \quad & (a\gamma)^{\frac{\beta}{\gamma}} T^{\left(\frac{\beta}{\gamma}-2\right)} \Bigg[T^2(1-e^{-T}) - \frac{\beta T}{\gamma}\left\{1-e^{-T}(T+1)\right\} \\
& -\frac{\theta\beta T}{2}\left\{2e^{-T}(T+1)+T^2-2\right\} \\
& +\frac{\theta^2\beta\gamma}{6}T\left\{3e^{-T}(T^2+2T+2)+T^3-6\right\} \\
& -\frac{b\beta\theta}{a}T\left\{(1+\theta\gamma)(e^{-T}+T-1)-\frac{\theta\gamma}{2}\left(2e^{-T}(T+1)+T^2-2\right)\right\} \\
& -\frac{b\beta T}{2a\gamma}(1+\theta\gamma+\theta^2\gamma^2)(e^{-T}-e^{-2T}-1) \\
& +\frac{\beta(2\beta-1)}{2\gamma^2}\left\{2-e^{-T}(T^2+2T+2)\right. \\
& +\frac{\theta^2\gamma^2}{4}\left(T^4-4T^2+24-8e^{-T}(T^2+3T+3)\right) \\
& +\frac{\gamma^2 b^2\theta^2}{a^2}\left(T^2-2T+2-2e^{-T}\right) \\
& +\frac{b^2}{3a^2}(1+2\theta\gamma+3\theta^2\gamma^2)(3e^{-2T}-3e^{-T}-e^{-3T}+1) \\
& +\theta\gamma\left(T^2-6+2e^{-T}(T^2+3T+3)\right) \\
& -\frac{\theta^2\gamma^2}{3}\left(T^3-24+3e^{-T}(T^3+4T^2+8T+8)\right) \\
& +\frac{2\theta\gamma b}{a}\left\{T-2+e^{-T}(T+2)+\theta\gamma\left(T-\frac{T^2}{2}+1-e^{-T}(T+1)^2\right)\right\} \\
& +\frac{b}{2a}(1+\theta\gamma+\theta^2\gamma^2)\left(4e^{-T}-1-e^{-2T}(2T+3)\right) \\
& +\frac{b\theta^2\gamma^2}{a}\left\{T^3-T^2-2T+6-2e^{-T}(2T+3)\right\} \\
& +\frac{b\theta\gamma}{a}(1+\theta\gamma)\left\{e^{-T}(T^2-1)-\frac{T^2}{2}+\frac{1}{4}-e^{-2T}\left(T^2-\frac{T}{2}-\frac{3}{4}\right)\right\}
\end{aligned}
$$

$$-\frac{b\theta^2\gamma^2}{3a}\left\{e^{-T}(T^3-6)+\frac{3}{8}e^{-2T}\left(6T^2+14T+15\right)-\frac{T^3}{2}+\frac{3}{8}\right\}$$
$$+\frac{b^2\theta\gamma}{4a^2}\left\{6e^{-2T}+8e^{-T}(T-1)-4T+2\right.$$
$$\left.+\theta\gamma\left(e^{-2T}(4T^2-2T+9)-4e^{-T}(T^2-4T+3)+2T^2-8T+3\right)\right\}\Big\}\Big] \tag{3.10}$$

The average system cost

$$C(T)=\frac{1}{T}\left[A+C_1D+hI_T\right] \tag{3.11}$$

where T is the length of a cycle, A is the replenishment cost per cycle, C_1 is the unit cost of the item, D is the total deterioration in a cycle given by (3.8), h is the holding cost per unit per unit time and I_T is the total inventory in a cycle given by (3.7).

Our problem is to determine T^* which minimizess $C(T)$ of the inventory system. The necessary condition for $C(T)$ to be minimum is

$$\frac{d}{dT}C(T)=0$$

which gives

$$-\frac{1}{T^2}(A+C_1D+hI_T)+\frac{1}{T}\left(C_1\,\frac{dD}{dT}+h\frac{dI_T}{dT}\right)=0 \tag{3.12}$$

provided that T^* will satisfy the condition

$$\frac{d^2C(T)}{dT^2}>0. \tag{3.13}$$

In the limit as $\theta\rightarrow 0$, $\beta\rightarrow 0, b\rightarrow 0$ then

$$T^*=\left(\frac{2A}{ah}\right)^{\frac{1}{2}}$$

which is standard EOQ cycle time. The EOQ is

$$q^*\rightarrow aT^*=\left(\frac{2aA}{h}\right)^{\frac{1}{2}}$$

which is standard Wilson (1934) square - root formula for the EOQ.

Again as $\theta \to 0, \beta \to 0, b \to 0$

$$C(T^*) \to \frac{ahT^*}{2} + \frac{A}{T^*}$$

which is well known result for the Wilson (1934) EOQ model. It is quite impossible to find out an explicit expression for T by solving (3.12). However solving (3.12) numerically, one can easily find T^* and then the optimal average total inventory cost $C(T^*)$ can be found from (3.11).

4. Numerical Example

Let $A = 220$, $C_1 = 12$, $a = 3.8$, $b = 0.16$, $h = 0.4$, $\theta = 0.02$ and $\beta = 0.2$ in appropriate units. Based on these input data, the computer outputs are

$$T^* = 9.4958, \; C(T^*) = 38.7615, \; q^* = 74.0506.$$

It is checked that these results satisfy the sufficient condition (3.13).

Sensitivity Analysis

The sensitivity analysis is performed by changing the value of each of the parameters by -50%, -20%, 20% and 50%, taking one parameter at a time and keeping the remaining parameters unchanged. We now study sensitivity of the optimal solution to changes in the values of the different parameters associated with the system based on the above example. The results are shown in the following table.

A careful study of the table reveals the following points :

(i) T^* is insensitive to changes in the value of the parameter b and it is moderately sensitive to changes in A, C_1, a, h, θ and β.

(ii) $C(T^*)$ is insensitive to changes in the values of the parameters C_1, b and moderately sensitive to changes in A, a, h, θ and β.

(iii) Table 1 shows that q^* is insensitive to changes in the values of the parameters b, β and moderately sensitive to changes in A, C_1, a, h and θ.

Here we have assumed that insensitive, moderately sensitive and highly sensitive imply that % changes are -10 to $+10$, -50 to $+50$ and more respectively.

Table 1. Sensitivity Analyisis

Parameter	% change	% change in T^*	% change in $C(T^*)$	% change in q^*
A	-50	-26.596	-34.416	-33.654
	-20	-9.517	-12.549	-12.512
	20	8.548	11.463	11.674
	50	20.061	27.149	28.027
C_1	-50	15.056	-8.302	20.836
	-20	5.112	-3.040	6.932
	20	-4.299	2.749	-5.717
	50	-9.655	6.430	-12.690
a	-50	48.010	-30.042	-27.943
	-20	13.390	-10.688	-10.266
	20	-9.716	9.496	9.453
	50	-20.283	22.073	22.507
b	-50	-0.019	-0.054	-0.285
	-20	-0.007	-0.021	-0.114
	20	0.007	0.021	0.114
	50	0.019	0.054	0.285
h	-50	12.724	-13.764	17.525
	-20	4.597	-5.290	6.227
	20	-4.070	5.041	-5.414
	50	-9.373	12.196	-12.327

Parameter	% change	% change in T^*	% change in $C(T^*)$	% change in q^*
θ	-50	12.512	-12.486	11.020
	-20	4.661	-4.887	4.180
	20	-4.233	4.745	-3.857
	50	-9.867	11.609	-9.060
β	-50	22.870	-9.830	-8.270
	-20	8.578	-4.161	-3.570
	20	-7.935	4.692	3.871
	50	-18.890	13.481	9.905

5. Conclusion

In the present paper we have studied a continuous deterministic inventory model for deteriorating items with stock-dependent time - varying demand rate. If the demand rate is dependent on inventory level as well as on time, a retailer may display each of his items in large quantities to generate greater demand. Then there must arise problems of space allocation for each item and investment requirements resulting from the increased inventory levels.

However, success depends on the correctness of the estimation of the input parameters. In reality, however management is most likely to be uncertain of the true values of these parameters, moreover, their values may be changed over time due to their complex structures. Therefore it is more reasonable to assume that these parameters are known only within some given ranges.

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