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Linear $*$-Derivations on C^*-Algebras*

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Abstract

It is shown that every almost linear almost $*$-derivation $h : A \to A$ on a unital C^*-algebra, JC^*-algebra, or Lie C^*-algebra A is a linear $*$-derivation when $h(rx) = rh(x)$ $(r > 1)$ for all $x \in A$.

We moreover prove the Cauchy–Rassias stability of linear $*$-derivations on unital C^*-algebras, on unital JC^*-algebras, or on unital Lie C^*-algebras.

Keywords and Phrases: *JC^*-algebra, Lie C^*-algebra, Linear $*$-derivation, Stability, Linear functional equation.*

1. Introduction

Let X and Y be Banach spaces with norms $|| \cdot ||$ and $\| \cdot \|$, respectively. Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(||x||^p + ||y||^p)$$

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for all $x, y \in X$. Rassias [8] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - 2^p}||x||^p$$

for all $x \in X$. Găvruta [1] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x + y) - f(x) - f(y)\| \leq \varphi(x, y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{1}{2}\widetilde{\varphi}(x, x)$$

for all $x \in G$. Park [5] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Jun and Lee [2] proved the Cauchy–Rassias stability of Jensen's equation. C. Park and W. Park [7] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra.

Recently, Trif [9] proved the stability of a functional equation deriving from an inequality of Popoviciu for convex functions. And Park [6] applied the Trif's result to the Trif functional equation in Banach modules over a C^*-algebra.

Throughout this paper, let A be a unital C^*-algebra with norm $\| \cdot \|$, and $U(A)$ be the unitary group of A. Let l and d be integers with $2 \leq l \leq d - 1$, and r a real number greater than 1.

In this paper, we prove that every almost linear almost $*$-derivation $h : A \to A$ on a unital C^*-algebra, JC^*-algebra, or Lie C^*-algebra A is a linear $*$-derivation when $h(rx) = rh(x)$ $(r > 1)$ for all $x \in A$, and prove the Cauchy–Rassias stability of linear $*$-derivations on unital C^*-algebras, on unital JC^*-algebras, or on unital Lie C^*-algebras.

2. Linear $*$-Derivations on C^*-Algebras

Throughout this section, assume that $h(rx) = rh(x)$ for all $x \in A$.

We are going to investigate linear $*$-derivations on C^*-algebras associated with the Cauchy functional equation.

Theorem 2.1 *Let $h : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} r^{-j} \varphi(r^j x, r^j y) < \infty, \tag{2.i}$$

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \le \varphi(x, y), \tag{2.ii}$$

$$\|h(r^n u^*) - h(r^n u)^*\| \le \varphi(r^n u, r^n u), \tag{2.iii}$$

$$\|h(r^n uy) - h(r^n u)y - r^n u h(y)\| \le \varphi(r^n u, y) \tag{2.iv}$$

for all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid \ |\lambda| = 1\}$, all $u \in U(A)$, $n = 0, 1, \cdots$, and all $x, y \in A$. Then the mapping $h : A \to A$ is a $\mathbb{C}$-linear $*$-derivation.

Proof. Since $h(0) = rh(0)$, $h(0) = 0$. Put $\mu = 1 \in \mathbb{T}^1$ in (2.ii). By (2.ii) and the assumption that $h(rx) = rh(x)$ for all $x \in A$,

$$\begin{aligned} \|h(x + y) - h(x) - h(y)\| &= \frac{1}{r^n} \|h(r^n x + r^n y) - h(r^n x) - h(r^n y)\| \\ &\le \frac{1}{r^n} \varphi(r^n x, r^n y) \end{aligned}$$

which tends to zero as $n \to \infty$ by (2.i). So

$$h(x + y) = h(x) + h(y) \tag{2.1}$$

for all $x, y \in A$.

Put $y = 0$ in (2.ii). By (2.ii) and the assumption that $h(rx) = rh(x)$ for all $x \in A$,

$$\|h(\mu x) - \mu h(x)\| = \frac{1}{r^n} \|h(r^n \mu x) - \mu h(r^n x)\| \le \frac{1}{r^n} \varphi(r^n x, 0),$$

which tends to zero as $n \to \infty$ by (2.i). So

$$h(\mu x) = \mu h(x) \tag{2.2}$$

for all $x \in A$.

Now let $\lambda \in \mathbb{C}$ ($\lambda \neq 0$) and M an integer greater than $4|\lambda|$. Then $|\frac{\lambda}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [3, Theorem 1], there exist three elements $\mu_1, \mu_2, \mu_3 \in \mathbb{T}^1$ such that $3\frac{\lambda}{M} = \mu_1 + \mu_2 + \mu_3$. Thus by (2.1) and (2.2)

$$
\begin{aligned}
h(\lambda x) &= h(\frac{M}{3} \cdot 3\frac{\lambda}{M}x) = M \cdot h(\frac{1}{3} \cdot 3\frac{\lambda}{M}x) = \frac{M}{3}h(3\frac{\lambda}{M}x) \\
&= \frac{M}{3}h(\mu_1 x + \mu_2 x + \mu_3 x) = \frac{M}{3}(h(\mu_1 x) + h(\mu_2 x) + h(\mu_3 x)) \\
&= \frac{M}{3}(\mu_1 + \mu_2 + \mu_3)h(x) = \frac{M}{3} \cdot 3\frac{\lambda}{M}h(x) \\
&= \lambda h(x)
\end{aligned}
$$

for all $x \in A$. Hence

$$h(\zeta x + \eta y) = h(\zeta x) + h(\eta y) = \zeta h(x) + \eta h(y)$$

for all $\zeta, \eta \in \mathbb{C}(\zeta, \eta \neq 0)$ and all $x, y \in A$. And $h(0x) = 0 = 0h(x)$ for all $x \in A$. So the mapping $h : A \to A$ is a $\mathbb{C}$-linear mapping.

By (2.iii) and the assumption that $h(rx) = rh(x)$ for all $x \in A$,

$$\|h(u^*) - h(u)^*\| = \frac{1}{r^n}\|h(r^n u^*) - h(r^n u)^*\| \leq \frac{1}{r^n}\varphi(r^n u, r^n u),$$

which tends to zero as $n \to \infty$ by (2.i). So

$$h(u^*) = h(u)^*$$

for all $u \in U(A)$. Since $h : A \to A$ is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements (see [4, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^m \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in U(A)$),

$$
\begin{aligned}
h(x^*) &= h(\sum_{j=1}^m \overline{\lambda_j} u_j^*) = \sum_{j=1}^m \overline{\lambda_j} h(u_j^*) = \sum_{j=1}^m \overline{\lambda_j} h(u_j)^* = (\sum_{j=1}^m \lambda_j h(u_j))^* \\
&= h(\sum_{j=1}^m \lambda_j u_j)^* = h(x)^*
\end{aligned}
$$

for all $x \in A$.

By (2.iv) and the assumption that $h(rx) = rh(x)$ for all $x \in A$,

$$\begin{aligned}\|h(uy) - h(u)y - uh(y)\| &= \frac{1}{r^{2n}}\|h(r^n u \cdot r^n y) - h(r^n u)r^n y - r^n uh(r^n y)\| \\ &\le \frac{1}{r^{2n}}\varphi(r^n u, r^n y) \le \frac{1}{r^n}\varphi(r^n u, r^n y),\end{aligned}$$

which tends to zero as $n \to \infty$ by (2.i). So

$$h(uy) = h(u)y + uh(y)$$

for all $u \in U(A)$ and all $y \in A$. Since $h : A \to A$ is $\mathbb{C}$-linear and each $x \in A$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{m} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in U(A)$),

$$\begin{aligned}h(xy) &= h(\sum_{j=1}^{m} \lambda_j u_j y) = \sum_{j=1}^{m} \lambda_j h(u_j y) = \sum_{j=1}^{m} \lambda_j (h(u_j)y + u_j h(y)) \\ &= h(\sum_{j=1}^{m} \lambda_j u_j)y + (\sum_{j=1}^{m} \lambda_j u_j)h(y) = h(x)y + xh(y)\end{aligned}$$

for all $x, y \in A$. Hence the mapping $h : A \to A$ is a $\mathbb{C}$-linear $*$-derivation, as desired. □

Corollary 2.2. *Let $h : A \to A$ be a mapping for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned}\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| &\le \theta(\|x\|^p + \|y\|^p), \\ \|h(r^n u^*) - h(r^n u)^*\| &\le 2r^{np}\theta, \\ \|h(r^n uy) - h(r^n u)y - r^n uh(y)\| &\le \theta(r^{np} + \|y\|^p)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, \cdots$, and all $x, y \in A$. Then the mapping $h : A \to A$ is a $\mathbb{C}$-linear $*$-derivation.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 2.1. □

Theorem 2.3. *Let $h : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying* (2.i), (2.iii), *and* (2.iv) *such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \le \varphi(x, y) \tag{2.v}$$

for $\mu = 1, i$, and all $x, y \in A$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $h : A \to A$ is a $\mathbb{C}$-linear $*$-derivation.

Proof. Put $\mu = 1$ in (2.v). By the same reasoning as in the proof of Theorem 2.1, the mapping $h : A \to A$ is additive. By the same reasoning as in the proof of [8, Theorem], the additive mapping $h : A \to A$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (2.v). By the same method as in the proof of Theorem 2.1, one can obtain that

$$h(ix) = ih(x)$$

for all $x \in A$. For each element $\lambda \in \mathbb{C}$, $\lambda = s + it$, where $s, t \in \mathbb{R}$. So

$$\begin{aligned} h(\lambda x) &= h(sx + itx) = sh(x) + th(ix) = sh(x) + ith(x) = (s + it)h(x) \\ &= \lambda h(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in A$. So

$$h(\zeta x + \eta y) = h(\zeta x) + h(\eta y) = \zeta h(x) + \eta h(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in A$. Hence the mapping $h : A \to A$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 2.1. □

Now we are going to investigate linear $*$-derivations on C^*-algebras associated with the Jensen functional equation.

Theorem 2.4. *Let $h : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying (2.i), (2.iii), and (2.iv) such that*

$$\|2h(\frac{\mu x + \mu y}{2}) - \mu h(x) - \mu h(y)\| \leq \varphi(x, y), \tag{2.vi}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Then the mapping $h : A \to A$ is a $\mathbb{C}$-linear $*$-derivation.

Proof. Put $\mu = 1 \in \mathbb{T}^1$ in (2.vi). By (2.vi) and the assumption that $h(rx) = rh(x)$ for all $x \in A$,

$$\begin{aligned} \|2h(\frac{x + y}{2}) - h(x) - h(y)\| &= \frac{1}{r^n}\|2h(\frac{r^n x + r^n y}{2}) - h(r^n x) - h(r^n y)\| \\ &\leq \frac{1}{r^n}\varphi(r^n x, r^n y), \end{aligned}$$

which tends to zero as $n \to \infty$ by (2.i). So

$$2h(\frac{x+y}{2}) = h(x) + h(y) \tag{2.3}$$

for all $x, y \in A$. Let $y = 0$ in (2.3). Then $2h(\frac{x}{2}) = h(x)$ for all $x \in A$. Thus

$$h(x+y) = 2h(\frac{x+y}{2}) = h(x) + h(y)$$

for all $x, y \in A$.

The rest of the proof is the same as in the proof of Theorem 2.1. $\square$

One can obtain similar results to Corollary 2.2 and Theorem 2.3 for the Jensen functional equation.

We are going to investigate linear $*$-derivations on C^*-algebras associated with the Trif functional equation.

Theorem 2.5. *Let $h : A \to A$ be a mapping for which there exists a function $\varphi : A^d \to [0, \infty)$ such that*

$$\sum_{j=0}^{\infty} r^{-j} \varphi(r^j x_1, \cdots, r^j x_d) < \infty, \tag{2.vii}$$

$$\begin{aligned} &\| d\ {}_{d-2}C_{l-2} h(\frac{\mu x_1 + \cdots + \mu x_d}{d}) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j) \\ &\quad - l \sum_{1 \le j_1 < \cdots < j_l \le d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l}) \| \le \varphi(x_1, \cdots, x_d), \qquad (2.viii) \\ &\qquad \| h(r^n u^*) - h(r^n u)^* \| \le \varphi(\underbrace{r^n u, \cdots, r^n u}_{d \text{ times}}), \qquad (2.ix) \\ &\| h(r^n u y) - h(r^n u) y - r^n u h(y) \| \le \varphi(r^n u, \underbrace{y, \cdots, y}_{d-1 \text{ times}}) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, \cdots,$ and all $x_1, \cdots, x_d \in A$. Then the mapping $h : A \to A$ is a $\mathbb{C}$-linear $*$-derivation.

Proof. The proof is similar to the proofs of Theorems 2.1 and 2.4. $\square$

One can obtain similar results to Corollary 2.2 and Theorem 2.3 for the Trif functional equation.

3. Linear $*$-Derivations on JC^*-Algebras and on Lie C^*-Algebras

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [?]). Let H be a complex Hilbert space, regarded as the "state space" of a quantum mechanical system. Let $L(H)$ be the real vector space of all bounded self-adjoint linear operators on H, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $L(H)$ is a (nonassociative) algebra via the *anti-commutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra* if $x^2 \circ (x \circ y) = x \circ (x^2 \circ y)$ holds.

A complex Jordan algebra C with product $x \circ y$ and involution $x \mapsto x^*$ is called a JB^*-algebra if C carries a Banach space norm $\|\cdot\|$ satisfying $\|x \circ y\| \le \|x\| \cdot \|y\|$ and $\|\{xx^*x\}\| = \|x\|^3$. Here $\{xy^*z\} := x \circ (y^* \circ z) - y^* \circ (z \circ x) + z \circ (x \circ y^*)$ denotes the *Jordan triple product* of $x, y, z \in C$. A Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a *JC^*-algebra*.

Throughout this section, assume that $h(rx) = rh(x)$ for all $x \in A$.

We are going to investigate linear $*$-derivations on JC^*-algebras associated with the Cauchy functional equation.

Theorem 3.1. *Let A be a unital JC^*-algebra. Let $h : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying* (2.i), (2.ii), *and* (2.iii) *such that*

$$\|h(r^n u \circ y) - h(r^n u) \circ y - r^n u \circ h(y)\| \le \varphi(r^n u, y)$$

for all $u \in U(A)$, $n = 0, 1, \cdots$, and all $y \in A$. Then the mapping $h : A \to A$ is a $\mathbb{C}$-linear $*$-derivation.

Proof. The proof is similar to the proof of Theorem 2.1.

Corollary 3.2. *Let A be a unital JC^*-algebra. Let $h : A \to A$ be a mapping for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \le \theta(\|x\|^p + \|y\|^p),$$
$$\|h(r^n u^*) - h(r^n u)^*\| \le 2r^{np}\theta,$$
$$\|h(r^n u \circ y) - h(r^n u) \circ y - r^n u \circ h(y)\| \le \theta(r^{np} + \|y\|^p)$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in U(A)$, $n = 0, 1, \cdots$, *and all* $x, y \in A$. *Then the mapping* $h : A \to A$ *is a* $\mathbb{C}$*-linear* $*$*-derivation.*

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 3.1.

On JC^*-algebras, one can obtain similar results to Theorems 2.3, 2.4, and 2.5.

A unital C^*-algebra A, endowed with the Lie product $[x, y] = xy - yx$ on A, is called a *Lie* C^**-algebra*. A $\mathbb{C}$-linear mapping D on a Lie C^*-algebra A is called a *Lie derivation* if $D([x, y]) = [D(x), y] + [x, D(y)]$ holds for all $x, y \in A$.

Theorem 3.3. *Let* A *be a unital Lie* C^**-algebra. Let* $h : A \to A$ *be a mapping for which there exists a function* $\varphi : A^2 \to [0, \infty)$ *satisfying* (2.i), (2.ii), *and* (2.iii) *such that*

$$\|h([r^n u, y]) - [h(r^n u), y] - [r^n u, h(y)]\| \leq \varphi(r^n u, y)$$

for all $u \in U(A)$, $n = 0, 1, \cdots$, *and all* $y \in A$. *Then the mapping* $h : A \to A$ *is a* $\mathbb{C}$*-linear* $*$*-derivation.*

Proof. The proof is similar to the proof of Theorem 2.1. □

Corollary 3.4. *Let* A *be a unital Lie* C^**-algebra. Let* $h : A \to A$ *be a mapping for which there exist constants* $\theta \geq 0$ *and* $p \in [0, 1)$ *such that*

$$\begin{aligned} \|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| &\leq \theta(\|x\|^p + \|y\|^p), \\ \|h(r^n u^*) - h(r^n u)^*\| &\leq 2r^{np}\theta, \\ \|h([r^n u, y]) - [h(r^n u), y] - [r^n u, h(y)]\| &\leq \theta(r^{np} + \|y\|^p) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, *all* $u \in U(A)$, $n = 0, 1, \cdots$, *and all* $x, y \in A$. *Then the mapping* $h : A \to A$ *is a* $\mathbb{C}$*-linear* $*$*-derivation.*

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 3.3. □

On Lie C^*-algebras, one can obtain similar results to Theorems 2.3, 2.4, and 2.5.

4. Stability of Linear $*$-Derivations on C^*-Algebras

We are going to show the Cauchy–Rassias stability of linear $*$-derivations on C^*-algebras associated with the Cauchy functional equation.

Theorem 4.1. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x, y) := \sum_{j=0}^{\infty} 2^{-j} \varphi(2^j x, 2^j y) < \infty, \tag{4.i}$$

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \varphi(x, y), \tag{4.ii}$$

$$\|h(2^n u^*) - h(2^n u)^*\| \leq \varphi(2^n u, 2^n u), \tag{4.iii}$$

$$\|h(2^n uy) - h(2^n u)y - 2^n uh(y)\| \leq \varphi(2^n u, y) \tag{4.iv}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, \cdots$, and all $x, y \in A$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \leq \frac{1}{2} \widetilde{\varphi}(x, x) \tag{4.v}$$

for all $x \in A$.

Proof. Put $\mu = 1 \in \mathbb{T}^1$ in (4.ii). It follows from Găvruta Theorem [**?**] that there exists a unique additive mapping $D : A \to A$ satisfying the inequality (4.v). The additive mapping $D : A \to A$ is given by

$$D(x) = \lim_{n \to \infty} \frac{1}{2^n} h(2^n x) \tag{4.1}$$

for all $x \in A$.

Put $y = 0$ in (4.ii). It follows from (4.ii) that

$$\frac{1}{2^n} \|h(2^n \mu x) - \mu h(2^n x)\| \leq \frac{1}{2^n} \varphi(2^n x, 0),$$

which tends to zero as $n \to \infty$ by (4.i) for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Hence

$$D(\mu x) = \lim_{n \to \infty} \frac{h(2^n \mu x)}{2^n} = \lim_{n \to \infty} \frac{\mu h(2^n x)}{2^n} = \mu D(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$.

By the same method as in the proof of Theorem 2.1, one can show that the mapping $D : A \to A$ is $\mathbb{C}$-linear.

By (4.i) and (4.iii), we get

$$D(u^*) = \lim_{n \to \infty} \frac{h(2^n u^*)}{2^n} = \lim_{n \to \infty} \frac{h(2^n u)^*}{2^n} = (\lim_{n \to \infty} \frac{h(2^n u)}{2^n})^* = D(u)^*$$

for all $u \in U(A)$. By the same method as the proof of Theorem 2.1, one can show that

$$D(x^*) = D(x)^*$$

for all $x \in A$.

It follows from (4.1) that

$$D(x) = \lim_{n\to\infty} \frac{h(2^{2n}x)}{2^{2n}} \tag{4.2}$$

for all $x \in A$. By (4.iv),

$$\begin{aligned}\frac{1}{2^{2n}}\|h(2^n u \cdot 2^n y) - h(2^n u)2^n y - 2^n u h(2^n y)\| &\le \frac{1}{2^{2n}}\varphi(2^n u, 2^n y) \\ &\le \frac{1}{2^n}\varphi(2^n u, 2^n y) \end{aligned} \tag{4.3}$$

for all $x, y \in A$. By (4.i), (4.2), and (4.3),

$$\begin{aligned} D(uy) &= \lim_{n\to\infty} \frac{h(2^{2n}uy)}{2^{2n}} = \lim_{n\to\infty} \frac{h(2^n u \cdot 2^n y)}{2^{2n}} = \lim_{n\to\infty} (\frac{1}{2^n}h(2^n u)y + u\frac{1}{2^n}h(2^n y)) \\ &= D(u)y + uD(y) \end{aligned}$$

for all $u \in U(A)$ and all $y \in A$.

The rest of the proof is similar to the proof of Theorem 2.1. Hence the mapping $D : A \to A$ is a $\mathbb{C}$-linear $*$-derivation satisfying the inequality (4.v), as desired. □

Corollary 4.2. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| &\le \theta(\|x\|^p + \|y\|^p), \\ \|h(2^n u^*) - h(2^n u)^*\| &\le 2 \cdot 2^{np}\theta, \\ \|h(2^n uy) - h(2^n u)y - 2^n u h(y)\| &\le \theta(2^{np} + \|y\|^p) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, \cdots$, and all $x, y \in A$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in A$.

Proof. Define $\varphi(x,y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 4.1. □

One can obtain a similar result to Theorem 2.3 for the Cauchy functional equation.

Now we are going to show the Cauchy–Rassias stability of linear $*$-derivations on C^*-algebras associated with the Jensen functional equation.

Theorem 4.3. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^2 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} 3^{-j}\varphi(3^j x, 3^j y) <\infty, \tag{4.vi}$$

$$\|2h(\frac{\mu x+\mu y}{2}) - \mu h(x) - \mu h(y)\| \leq \varphi(x,y), \tag{4.vii}$$

$$\|h(3^n u^*) - h(3^n u)^*\| \leq \varphi(3^n u, 3^n u), \tag{4.viii}$$

$$\|h(3^n uy) - h(3^n u)y - 3^n u h(y)\| \leq \varphi(3^n u, y) \tag{4.ix}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, \cdots$, and all $x, y \in A \setminus \{0\}$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \leq \frac{1}{3}(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x)) \tag{4.x}$$

for all $x \in A \setminus \{0\}$.

Proof. Put $\mu = 1 \in \mathbb{T}^1$ in (4.vii). It follows from Jun and Lee Theorem [2, Theorem 1] that there exists a unique additive mapping $D : A \to A$ satisfying the inequality (4.x). The additive mapping $D : A \to A$ is given by

$$D(x) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n x)$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 2.1 and 4.1. □

Corollary 4.4. *Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \leq \theta(\|x\|^p + \|y\|^p),$$
$$\|h(3^n u^*) - h(3^n u)^*\| \leq 2 \cdot 3^{np}\theta,$$
$$\|h(3^n uy) - h(3^n u)y - 3^n u h(y)\| \leq \theta(3^{np} + \|y\|^p)$$

for all $\mu \in \mathbb{T}^1$*, all* $u \in U(A)$*,* $n = 0, 1, \cdots,$ *and all* $x, y \in A$*. Then there exists a unique* $\mathbb{C}$*-linear* $*$*-derivation* $D : A \to A$ *such that*

$$\|h(x) - D(x)\| \leq \frac{3 + 3^p}{3 - 3^p} \theta \|x\|^p$$

for all $x \in A \setminus \{0\}$*.*

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 4.3. □

One can obtain a similar result to Theorem 2.3 for the Jensen functional equation.

Now we are going to show the Cauchy–Rassias stability of linear $*$-derivations on C^*-algebras associated with the Trif functional equation.

Theorem 4.5. *Let* $q = \frac{l(d-1)}{d-l}$ *and* $q' = -\frac{l}{d-l}$*. Let* $h : A \to A$ *be a mapping with* $h(0) = 0$ *for which there exists a function* $\varphi : A^d \to [0, \infty)$ *such that*

$$\widetilde{\varphi}(x_1, \cdots, x_d) := \sum_{j=0}^{\infty} q^{-j} \varphi(q^j x_1, \cdots, q^j x_d) < \infty, \tag{4.xi}$$

$$\begin{aligned} &\|d \ {}_{d-2}C_{l-2} h\Big(\frac{\mu x_1 + \cdots + \mu x_d}{d}\Big) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j) \\ &\qquad - l \sum_{1 \leq j_1 < \cdots < j_l \leq d} \mu h\Big(\frac{x_{j_1} + \cdots + x_{j_l}}{l}\Big)\| \leq \varphi(x_1, \cdots, x_d), \end{aligned} \tag{4.xii}$$

$$\|h(q^n u^*) - h(q^n u)^*\| \leq \varphi(\underbrace{q^n u, \cdots, q^n u}_{d \text{ times}}), \tag{4.xiii}$$

$$\|h(q^n u y) - h(q^n u) y - q^n u h(y)\| \leq \varphi(q^n u, \underbrace{y, \cdots, y}_{d-1 \text{ times}}) \tag{4.xiv}$$

for all $\mu \in \mathbb{T}^1$*, all* $u \in U(A)$*,* $n = 0, 1, \cdots,$ *and all* $x_1, \cdots, x_d \in A$*. Then there exists a unique* $\mathbb{C}$*-linear* $*$*-derivation* $D : A \to A$ *such that*

$$\|h(x) - D(x)\| \leq \frac{1}{l \cdot {}_{d-1}C_{l-1}} \widetilde{\varphi}(qx, \underbrace{q'x, \cdots, q'x}_{d-1 \text{ times}}) \tag{4.xv}$$

for all $x \in A$*.*

Proof. Put $\mu = 1 \in \mathbb{T}^1$ in (4.xii). It follows from Trif Theorem [9, Theorem 3.1] that there exists a unique additive mapping $D : A \to A$ satisfying the inequality (4.xv). The additive mapping $D : A \to A$ is given by

$$D(x) = \lim_{n\to\infty} \frac{1}{q^n} h(q^n x)$$

for all $x \in A$.

The rest of the proof is similar to the proofs of Theorems 2.1 and 4.1. $\square$

Corollary 4.6. *Let $q = \frac{l(d-1)}{d-l}$. Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{gathered}
\|d \ {}_{d-2}C_{l-2} h(\frac{\mu x_1 + \cdots + \mu x_d}{d}) + {}_{d-2}C_{l-1} \sum_{j=1}^{d} \mu h(x_j) \\
-l \sum_{1\leq j_1 < \cdots < j_l \leq d} \mu h(\frac{x_{j_1} + \cdots + x_{j_l}}{l})\| \leq \theta(\sum_{j=1}^{d} \|x_j\|^p), \\
\|h(q^n u^*) - h(q^n u)^*\| \leq d q^{np}\theta, \\
\|h(q^n uy) - h(q^n u)y - q^n u h(y)\| \leq \theta(q^{np} + (d-1)\|y\|^p)
\end{gathered}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, \cdots$, and all $x_1, \cdots, x_d \in A$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \leq \frac{q^{1-p}(q^p + (d-1)r^p)\theta}{l \ {}_{d-1}C_{l-1}(q^{1-p} - 1)} \|x\|^p$$

for all $x \in A$.

Proof. Define $\varphi(x_1, \cdots, x_d) = \theta(\sum_{j=1}^{d} \|x_j\|^p)$, and apply Theorem 4.5. $\square$

One can obtain a similar result to Theorem 2.3 for the Trif functional equation.

5. Stability of Linear $*$-Derivations on JC^*-Algebras and On Lie C^*-Algebras

We are going to show the Cauchy–Rassias stability of linear $*$-derivations on JC^*-algebras associated with the Cauchy functional equation.

Theorem 5.1. *Let A be a unital JC^*-algebra. Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying* (4.i), (4.ii), *and* (4.iii) *such that*

$$\|h(2^n u \circ y) - h(2^n u) \circ y - 2^n u \circ h(y)\| \le \varphi(2^n u, y)$$

for all $u \in U(A)$, $n = 0, 1, \cdots$, and all $y \in A$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : A \to A$ satisfying the inequality (4.v).

Proof. The proof is similar to the proofs of Theorems 2.1 and 4.1. □

Corollary 5.2. *Let A be a unital JC^*-algebra. Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \ge 0$ and $p \in [0, 1)$ such that*

$$\|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| \le \theta(\|x\|^p + \|y\|^p),$$
$$\|h(2^n u^*) - h(2^n u)^*\| \le 2 \cdot 2^{np}\theta,$$
$$\|h(2^n u \circ y) - h(2^n u) \circ y - 2^n u \circ h(y)\| \le \theta(2^{np} + \|y\|^p)$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, \cdots$, and all $x, y \in A$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \le \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in A$.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 5.1. □

On JC^*-algebras, one can obtain similar results to Theorems 4.3 and 4.5, and Corollaries 4.4 and 4.6.

Now we are going to show the Cauchy–Rassias stability of linear $*$-derivations on Lie C^*-algebras associated with the Cauchy functional equation.

Theorem 5.3. *Let A be a unital Lie C^*-algebra. Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying* (4.i), (4.ii), *and* (4.iii) *such that*

$$\|h([2^n u, y]) - [h(2^n u), y] - [2^n u, h(y)]\| \le \varphi(2^n u, y)$$

for all $u \in U(A)$, $n = 0, 1, \cdots$, and all $y \in A$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : A \to A$ satisfying the inequality (4.v).

Proof. The proof is similar to the proofs of Theorems 2.1 and 4.1. $\square$

Corollary 5.4. *Let A be a unital Lie C^*-algebra. Let $h : A \to A$ be a mapping with $h(0) = 0$ for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} \|h(\mu x + \mu y) - \mu h(x) - \mu h(y)\| &\leq \theta(\|x\|^p + \|y\|^p), \\ \|h(2^n u^*) - h(2^n u)^*\| &\leq 2 \cdot 2^{np}\theta, \\ \|h([2^n u, y]) - [h(2^n u), y] - [2^n u, h(y)]\| &\leq \theta(2^{np} + \|y\|^p) \end{aligned}$$

for all $\mu \in \mathbb{T}^1$, all $u \in U(A)$, $n = 0, 1, \cdots$, and all $x, y \in A$. Then there exists a unique $\mathbb{C}$-linear $*$-derivation $D : A \to A$ such that

$$\|h(x) - D(x)\| \leq \frac{2\theta}{2 - 2^p}\|x\|^p$$

for all $x \in A$.

Proof. Define $\varphi(x, y) = \theta(\|x\|^p + \|y\|^p)$, and apply Theorem 5.3. $\square$

On Lie C^*-algebras, one can obtain similar results to Theorems 4.3 and 4.5, and Corollaries 4.4 and 4.6.

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