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On Certain Subclass of λ-Bazilevič Functions of Type $\alpha + i\mu^*$

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Abstract

In the present paper, the authors introduce a new subclass $B_n(\lambda, \alpha, \mu, A, B, z)$ of $\lambda-$Bazilevič functions of type $\alpha + i\mu$. The subordination relations and inequality properties are discussed by making use of differential subordination method. The results presented here generalize and improve some known results, and some other new results are obtained.

Keywords and Phrases: *λ-Bazilevič functions of type $\alpha + i\mu$, Differential subordination.*

1. Introduction and Definitions

Let $\mathcal{A}_n$ denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \quad (n \in N = \{1, 2, 3, \ldots\}), \tag{1.1}$$

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which are analytic in the unit disk $\mathcal{U} = \{z : \ z < 1\}$. Also let $\mathcal{S}_n^*(\beta)$ denote the usual class of starlike functions of order β, $0 \leq \beta < 1$.

Let $f(z)$ and $F(z)$ be analytic in $\mathcal{U}$. Then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathcal{U}$, if there exists an analytic function $\omega(z)$ in $\mathcal{U}$ such that $\omega(z) \leq z$ and $f(z) = F(\omega(z))$, denoted by $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathcal{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathcal{U}) \subset F(\mathcal{U})$ (see [1]).

The following class of analytic functions were studied by various authors (see [3]).

Definition 1. Let $B_n(\alpha, \mu, \beta, g(z))$ denote the class of functions in $\mathcal{A}_n$ satisfying the inequality

$$\Re\left\{\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha+i\mu}\right\} > \beta \quad (z \in \mathcal{U}), \tag{1.2}$$

where $\alpha \geq 0, \ \mu \in R, \ 0 \leq \beta < 1$ and $g(z) \in \mathcal{S}_n^*(\beta)$. The function $f(z)$ in this class is said to be Bazilevič function of type $\alpha + i\mu$ and of order β.

In the present paper, we define the following class of analytic functions.

Definition 2. Let $B_n(\lambda, \alpha, \mu, A, B, g(z))$ denote the class of functions in $\mathcal{A}_n$ satisfying the inequality

$$(1-\lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{g(z)}\right)^{\alpha+i\mu} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\left(\frac{f'(z)}{g'(z)}\right)^{\alpha+i\mu} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}), \tag{1.3}$$

where $0 \leq \lambda \leq 1$, $\alpha \geq 0$, $\mu \in R$, $-1 \leq B \leq 1$, $A \neq B$, $A \in R$ and $g(z) \in \mathcal{S}_n^*(\beta)$. All the powers in (1.3) are principal values, below we apply this agreement. The function $f(z)$ in this class is said to be $\lambda-$Bazilevič function of type $\alpha+i\mu$.

If $\alpha = 1, \ \mu = 0, \ A = 1-2\beta$ and $B = -1$, then the class $B_n(\lambda, \alpha, \mu, A, B, g(z))$ reduces to the class of $\lambda-$close-to-convex functions of order β, $0 \leq \beta < 1$. If $\alpha = 0, \ \mu = 0, \ A = 1$ and $B = -1$, then the class $B_n(\lambda, \alpha, \mu, A, B, g(z))$ reduces to the class of $\lambda-$convex functions [6]. If $\alpha = 0, \ \mu = 0, \ A = 1 - 2\beta$ and $B = -1$, then the class $B_n(\lambda, \alpha, \mu, A, B, g(z))$ reduces to the class of $\lambda-$convex functions of order β, $0 \leq \beta < 1$. If $A = 1 - 2\beta$ and $B = -1$, then the class $B_n(\lambda, \alpha, \mu, A, B, g(z))$ reduces to the class of $\lambda-$Bazilevič functions of type $\alpha + i\mu$ and of order β, $0 \leq \beta < 1$.

Li [3], Owa [4], Owa and Nunokawa [5] discussed the related properties of

the classes $B_n(\lambda,\alpha,\mu,1-2\beta,-1,z)$, $B_n(0,\alpha,0,1-2\beta,-1,z)$ and $B_n(\lambda,1,0,1-2\beta,-1,z)$, respectively. In the present paper, we will discuss the subordination relations and inequality properties of the class $B_n(\lambda,\alpha,\mu,A,B,z)$. The results presented here generalize and improve some known results, and some other new results are obtained.

2. Preliminaries Results

In order to establish our main results, we shall require the following lemmas.

Lemma 1 ([7]). *Let* $F(z)=1+b_nz^n+b_{n+1}z^{n+1}+\cdots$ *be analytic in* $\mathcal{U}$, $h(z)$ *be analytic and convex in* $\mathcal{U}$, $h(0)=1$. *If*

$$F(z)+\frac{1}{c}zF'(z)\prec h(z), \tag{2.1}$$

where $c\neq 0$ *and* $\Re\, c\geq 0$, *then*

$$F(z)\prec \frac{c}{n}z^{-\frac{c}{n}}\int_0^z t^{\frac{c}{n}-1}h(t)dt\prec h(z),$$

and $\frac{c}{n}z^{-\frac{c}{n}}\int_0^z t^{\frac{c}{n}-1}h(t)dt$ *is the best dominant for (2.1).*

Lemma 2 ([8]). *Let* $f(z)=\sum_{k=1}^{\infty}a_kz^k$ *be analytic in* $\mathcal{U}$, $g(z)=\sum_{k=1}^{\infty}b_kz^k$ *be analytic and convex in* $\mathcal{U}$. *If* $f(z)\prec g(z)$, *then* $a_k\leq b_1$, *for* $k=1,2,\ldots$.

Lemma 3. *Let* $0\leq\lambda\leq 1$, $\alpha\geq 0$, $\mu\in R$, $\alpha+i\mu\neq 0$, $-1\leq B\leq 1$, $A\neq B$ *and* $A\in R$. *Then* $f(z)\in B_n(\lambda,\alpha,\mu,A,B,z)$ *if and only if*

$$q(z)+\frac{1}{\alpha+i\mu}zq'(z)\prec\frac{1+Az}{1+Bz}, \tag{2.2}$$

where $q(z)=(1-\lambda)\,(f(z)/z)^{\alpha+i\mu}+\lambda(f'(z))^{\alpha+i\mu}$.

Proof. Let

$$\left(\frac{f(z)}{z}\right)^{\alpha+i\mu}=m(z). \tag{2.3}$$

Then, by taking the derivatives in the both sides of (2.3), we have

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} = m(z) + \frac{z}{\alpha+i\mu}m'(z),$$

that is,

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} = \left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \frac{z}{\alpha+i\mu}\left(\left(\frac{f(z)}{z}\right)^{\alpha+i\mu}\right)'. \tag{2.4}$$

Substituting $f(z)$ by $zf'(z)$ in (2.4), we have

$$\left(1+\frac{zf''(z)}{f'(z)}\right)(f'(z))^{\alpha+i\mu} = (f'(z))^{\alpha+i\mu} + \frac{z}{\alpha+i\mu}\left((f'(z))^{\alpha+i\mu}\right)'. \tag{2.5}$$

From equalities (2.4) and (2.5), we get

$$\begin{aligned} & (1-\lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)(f'(z))^{\alpha+i\mu} \\ = & \left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \lambda\,(f'(z))^{\alpha+i\mu}\right] \\ & + \frac{z}{\alpha+i\mu}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \lambda\,(f'(z))^{\alpha+i\mu}\right]'. \end{aligned} \tag{2.6}$$

Now, suppose that $f(z) \in B_n(\lambda, \alpha, \mu, A, B, z)$, and let

$$q(z) = (1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \lambda\,(f'(z))^{\alpha+i\mu}.$$

Thus, from the definition of $B_n(\lambda, \alpha, \mu, A, B, z)$ and equality (2.6), we can get (2.2).

On the other hand, this deductive process can be converse. Therefore, the proof of Lemma 3 is complete.

3. Main Results and Their Proofs

Theorem 1. *Let* $0 \leq \lambda \leq 1,\ \alpha \geq 0,\ \mu \in R,\ \alpha + i\mu \neq 0,\ -1 \leq B \leq 1,\ A \neq B$ *and* $A \in R$. *If* $f(z) \in B_n(\lambda, \alpha, \mu, A, B, z)$, *then*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \lambda\left(f'(z)\right)^{\alpha+i\mu} \prec \frac{\alpha+i\mu}{n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha+i\mu}{n}-1}du \prec \frac{1+Az}{1+Bz}.$$

Proof. First let $q(z) = (1-\lambda)\left(f(z)/z\right)^{\alpha+i\mu} + \lambda\left(f'(z)\right)^{\alpha+i\mu}$, then $q(z) = 1 + b_n z^n + b_{n+1}z^{n+1} + \cdots$ is analytic in $\mathcal{U}$. Now, suppose that $f(z) \in B_n(\lambda, \alpha, \mu, A, B, z)$, by Lemma 3, we know

$$q(z) + \frac{1}{\alpha+i\mu}zq'(z) \prec \frac{1+Az}{1+Bz}.$$

It is obvious that $h(z) = (1+Az)/(1+Bz)$ is analytic and convex in $\mathcal{U}$, $h(0) = 1$. Since $\alpha + i\mu \neq 0$ and $\alpha \geq 0$, therefore it follows from Lemma 1 that

$$\begin{aligned}(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \lambda\left(f'(z)\right)^{\alpha+i\mu} &\prec \frac{\alpha+i\mu}{n}z^{-\frac{\alpha+i\mu}{n}}\int_0^z t^{\frac{\alpha+i\mu}{n}-1}h(t)dt \\ &= \frac{\alpha+i\mu}{n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha+i\mu}{n}-1}du \prec \frac{1+Az}{1+Bz}.\end{aligned}$$

Corollary 1. *Let* $0 \leq \lambda \leq 1,\ \alpha \geq 0,\ \mu \in R,\ \alpha + i\mu \neq 0$ *and* $\beta \neq 1$. *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$(1-\lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\left(f'(z)\right)^{\alpha+i\mu} \prec \frac{1+(1-2\beta)z}{1-z}\ \ (z \in \mathcal{U}),$$

then

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \lambda\left(f'(z)\right)^{\alpha+i\mu} \prec \frac{\alpha+i\mu}{n}\int_0^1 \frac{1+(1-2\beta)zu}{1-zu}u^{\frac{\alpha+i\mu}{n}-1}du\ \ (z \in \mathcal{U}), \tag{3.1}$$

and (3.1) is equivalent to

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu} + \lambda\left(f'(z)\right)^{\alpha+i\mu} \prec \beta + \frac{(1-\beta)(\alpha+i\mu)}{n}\int_0^1 \frac{1+zu}{1-zu}u^{\frac{\alpha+i\mu}{n}-1}du\ \ (z \in \mathcal{U}).$$

Theorem 2. *Let* $0 \leq \lambda \leq 1,\ \alpha \geq 0,\ \mu \in R,\ \alpha + i\mu \neq 0,\ -1 \leq B \leq 1,\ A \neq B$ *and* $A \in R$. *If* $f(z) \in B_n(\lambda, \alpha, \mu, A, B, z)$, *then*

$$\begin{aligned}\inf_{z\in\mathcal{U}} \Re\left\{\frac{\alpha+i\mu}{n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha+i\mu}{n}-1}du\right\} &< \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu}+\lambda\left(f'(z)\right)^{\alpha+i\mu}\right\}\\ &< \sup_{z\in\mathcal{U}} \Re\left\{\frac{\alpha+i\mu}{n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha+i\mu}{n}-1}du\right\}.\end{aligned}$$

Proof. Suppose that $f(z) \in B_n(\lambda, \alpha, \mu, A, B, z)$, from Theorem 1 we know

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu}+\lambda\left(f'(z)\right)^{\alpha+i\mu} \prec \frac{\alpha+i\mu}{n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha+i\mu}{n}-1}du.$$

Therefore it follows from the definition of the subordination that

$$\Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu}+\lambda\left(f'(z)\right)^{\alpha+i\mu}\right\} > \inf_{z\in\mathcal{U}} \Re\left\{\frac{\alpha+i\mu}{n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha+i\mu}{n}-1}du\right\},$$

and

$$\Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu}+\lambda\left(f'(z)\right)^{\alpha+i\mu}\right\} < \sup_{z\in\mathcal{U}} \Re\left\{\frac{\alpha+i\mu}{n}\int_0^1 \frac{1+Azu}{1+Bzu}u^{\frac{\alpha+i\mu}{n}-1}du\right\}.$$

Corollary 2. *Let* $0 \leq \lambda \leq 1,\ \alpha \geq 0,\ \mu \in R,\ \alpha + i\mu \neq 0$ *and* $\beta < 1$. *If* $f(z) \in B_n(\lambda, \alpha, \mu, 1-2\beta, -1, z)$, *then*

$$\begin{aligned}&\beta + (1-\beta)\inf_{z\in\mathcal{U}} \Re\left\{\frac{\alpha+i\mu}{n}\int_0^1 \frac{1+zu}{1-zu}u^{\frac{\alpha+i\mu}{n}-1}du\right\}\\ <\ &\Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu}+\lambda(f'(z))^{\alpha+i\mu}\right\}\\ <\ &\beta + (1-\beta)\sup_{z\in\mathcal{U}} \Re\left\{\frac{\alpha+i\mu}{n}\int_0^1 \frac{1+zu}{1-zu}u^{\frac{\alpha+i\mu}{n}-1}du\right\}.\end{aligned}$$

Corollary 3. *Let* $0 \leq \lambda \leq 1,\ \alpha \geq 0,\ \mu \in R,\ \alpha + i\mu \neq 0$ *and* $\beta > 1$. *If* $f(z) \in \mathcal{A}_n$ *satisfies*

$$\Re\left\{(1-\lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha+i\mu}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)(f'(z))^{\alpha+i\mu}\right\} < \beta\ \ (z\in\mathcal{U}),$$

then

$$\begin{aligned} & \beta+(1-\beta)\sup_{z\in\mathcal{U}}\Re\left\{\frac{\alpha+i\mu}{n}\int_0^1\frac{1+zu}{1-zu}u^{\frac{\alpha+i\mu}{n}-1}du\right\} \\ < \ & \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha+i\mu}+\lambda(f'(z))^{\alpha+i\mu}\right\} \\ < \ & \beta+(1-\beta)\inf_{z\in\mathcal{U}}\Re\left\{\frac{\alpha+i\mu}{n}\int_0^1\frac{1+zu}{1-zu}u^{\frac{\alpha+i\mu}{n}-1}du\right\}. \end{aligned}$$

Theorem 3. *Let* $0\leq\lambda\leq 1$, $\alpha\geq 0$ *and* $-1\leq B<A\leq 1$. *If* $f(z)\in B_n(\lambda,\alpha,0,A,B,z)$, *then*

$$\begin{aligned} \frac{\alpha}{n}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\alpha}{n}-1}du \ & < \ \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right\} \\ & < \ \frac{\alpha}{n}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\alpha}{n}-1}du \ \ (z\in\mathcal{U}), \end{aligned} \tag{3.2}$$

and inequality (3.2) is sharp, with the extremal function defined by

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}=\frac{\alpha}{n}\int_0^1\frac{1+Auz^n}{1+Buz^n}u^{\frac{\alpha}{n}-1}du. \tag{3.3}$$

Proof. Suppose that $f(z)\in B_n(\lambda,\alpha,0,A,B,z)$, from Theorem 1 we know

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\prec\frac{\alpha}{n}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{n}-1}du.$$

Therefore it follows from the definition of the subordination and $A>B$ that

$$\begin{aligned} \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right\} \ & < \ \sup_{z\in\mathcal{U}}\Re\left\{\frac{\alpha}{n}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{n}-1}du\right\} \\ & \leq \ \frac{\alpha}{n}\int_0^1\sup_{z\in\mathcal{U}}\Re\left\{\frac{1+Azu}{1+Bzu}\right\}u^{\frac{\alpha}{n}-1}du \\ & < \ \frac{\alpha}{n}\int_0^1\frac{1+Au}{1+Bu}u^{\frac{\alpha}{n}-1}du, \end{aligned}$$

and

$$\begin{aligned}\Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right\} &> \inf_{z\in\mathcal{U}}\Re\left\{\frac{\alpha}{n}\int_0^1\frac{1+Azu}{1+Bzu}u^{\frac{\alpha}{n}-1}du\right\}\\ &\geq \frac{\alpha}{n}\int_0^1\inf_{z\in\mathcal{U}}\Re\left\{\frac{1+Azu}{1+Bzu}\right\}u^{\frac{\alpha}{n}-1}du\\ &> \frac{\alpha}{n}\int_0^1\frac{1-Au}{1-Bu}u^{\frac{\alpha}{n}-1}du.\end{aligned}$$

It is obvious that inequality (3.2) is sharp, with the extremal function defined by equality (3.3).

Corollary 4. *Let $0\leq\lambda\leq 1$, $\alpha\geq 0$ and $\beta<1$. If $f(z)\in B_n(\lambda,\alpha,0,1-2\beta,-1,z)$, then*

$$\begin{aligned}\frac{\alpha}{n}\int_0^1\frac{1-(1-2\beta)u}{1+u}u^{\frac{\alpha}{n}-1}du &< \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right\}\\ &< \frac{\alpha}{n}\int_0^1\frac{1+(1-2\beta)u}{1-u}u^{\frac{\alpha}{n}-1}du\quad(z\in\mathcal{U}),\end{aligned}\tag{3.4}$$

and inequality (3.4) is equivalent to

$$\begin{aligned}\beta+\frac{(1-\beta)\alpha}{n}\int_0^1\frac{1-u}{1+u}u^{\frac{\alpha}{n}-1}du &< \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right\}\\ &< \beta+\frac{(1-\beta)\alpha}{n}\int_0^1\frac{1+u}{1-u}u^{\frac{\alpha}{n}-1}du\quad(z\in\mathcal{U}).\end{aligned}$$

Corollary 5. *Let $\alpha\geq 0$ and $\beta<1$. If $f(z)\in\mathcal{A}_n$ satisfies*

$$\Re\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)\left(f'(z)\right)^{\alpha}\right\}>\beta\quad(z\in\mathcal{U}),$$

then

$$\begin{aligned}&\frac{\alpha}{n}\int_0^1\frac{1-(1-2\beta)u}{1+u}u^{\frac{\alpha}{n}-1}du<\Re\left\{\left(f'(z)\right)^{\alpha}\right\}\\ &<\frac{\alpha}{n}\int_0^1\frac{1+(1-2\beta)u}{1-u}u^{\frac{\alpha}{n}-1}du\quad(z\in\mathcal{U}),\end{aligned}\tag{3.5}$$

and inequality (3.5) is equivalent to

$$\beta + \frac{(1-\beta)\alpha}{n}\int_0^1 \frac{1-u}{1+u}u^{\frac{\alpha}{n}-1}du < \Re\left\{(f'(z))^{\alpha}\right\}$$

$$< \beta + \frac{(1-\beta)\alpha}{n}\int_0^1 \frac{1+u}{1-u}u^{\frac{\alpha}{n}-1}du \quad (z \in \mathcal{U}),$$

inequality (3.5) is sharp, with the extremal function defined by

$$f_{\alpha,\beta}(z) = \int_0^z \left(\frac{\alpha}{n}\int_0^1 \frac{1+(1-2\beta)t^n u}{1-t^n u}u^{\frac{\alpha}{n}-1}du\right)^{\frac{1}{\alpha}} dt. \tag{3.6}$$

Corollary 6. *Let $\lambda > 0$ and $0 \leq \beta < 1$. If $f(z) \in B_n(\lambda, 1, 0, 1-2\beta, -1, z)$, then for $z = r < 1$, we have*

$$\Re\{f'(z)\} > \frac{1}{\lambda}\int_0^1 t^{\frac{1}{\lambda}-1}\frac{1-(1-2\beta)t}{1+t}dt.$$

Remark 1. *Corollary 6 is the corresponding result obtained by Owa and Nunokawa in [5].*

By applying the similar method as in Theorem 3, we have

Theorem 4. *Let $0 \leq \lambda \leq 1$, $\alpha \geq 0$ and $-1 \leq A < B \leq 1$. If $f(z) \in B_n(\lambda, \alpha, 0, A, B, z)$, then*

$$\begin{aligned}\frac{\alpha}{n}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{n}-1}du &< \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left(f'(z)\right)^{\alpha}\right\} \\ &< \frac{\alpha}{n}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{n}-1}du \quad (z \in \mathcal{U}),\end{aligned} \tag{3.7}$$

and inequality (3.7) is sharp, with the extremal function defined by equality (3.3).

Corollary 7. *Let $0 \leq \lambda \leq 1$, $\alpha \geq 0$ and $\beta > 1$. If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left\{(1-\lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\left(f'(z)\right)^{\alpha}\right\} < \beta \quad (z \in \mathcal{U}),$$

then

$$\begin{aligned}\frac{\alpha}{n}\int_0^1 \frac{1+(1-2\beta)u}{1-u}u^{\frac{\alpha}{n}-1}du &< \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right\}\\ &< \frac{\alpha}{n}\int_0^1 \frac{1-(1-2\beta)u}{1+u}u^{\frac{\alpha}{n}-1}du \quad (z\in\mathcal{U}), \quad (3.8)\end{aligned}$$

and inequality (3.8) is equivalent to

$$\begin{aligned}\beta+\frac{(1-\beta)\alpha}{n}\int_0^1 \frac{1+u}{1-u}u^{\frac{\alpha}{n}-1}du &< \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right\}\\ &< \beta+\frac{(1-\beta)\alpha}{n}\int_0^1 \frac{1-u}{1+u}u^{\frac{\alpha}{n}-1}du \quad (z\in\mathcal{U}).\end{aligned}$$

Corollary 8. *Let $\alpha \geq 0$ and $\beta > 1$. If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Re\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)\left(f'(z)\right)^{\alpha}\right\} < \beta \quad (z\in\mathcal{U}),$$

then

$$\begin{aligned}&\frac{\alpha}{n}\int_0^1 \frac{1+(1-2\beta)u}{1-u}u^{\frac{\alpha}{n}-1}du < \Re\left\{\left(f'(z)\right)^{\alpha}\right\}\\ &< \frac{\alpha}{n}\int_0^1 \frac{1-(1-2\beta)u}{1+u}u^{\frac{\alpha}{n}-1}du \quad (z\in\mathcal{U}), \quad (3.9)\end{aligned}$$

and inequality (3.9) is equivalent to

$$\begin{aligned}&\beta+\frac{(1-\beta)\alpha}{n}\int_0^1 \frac{1+u}{1-u}u^{\frac{\alpha}{n}-1}du < \Re\left\{\left(f'(z)\right)^{\alpha}\right\}\\ &< \beta+\frac{(1-\beta)\alpha}{n}\int_0^1 \frac{1-u}{1+u}u^{\frac{\alpha}{n}-1}du \quad (z\in\mathcal{U}),\end{aligned}$$

inequality (3.9) is sharp, with the extremal function defined by equality (3.6).

Remark 2. *Corollary 7-8 improve the corresponding results of Corollary 6-7 in [3], respectively.*

If $\Re\omega \geq 0$, then $(\Re\omega)^{\frac{1}{2}} \leq \Re\omega^{\frac{1}{2}} \leq \omega(z)^{\frac{1}{2}}$ (see [2, 9]). So we have

Theorem 5. *Let $0 \leq \lambda \leq 1$, $\alpha \geq 0$ and $-1 \leq B < A \leq 1$. If $f(z) \in B_n(\lambda, \alpha, 0, A, B, z)$, then*

$$\left(\frac{\alpha}{n}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{n}-1}du\right)^{\frac{1}{2}} < \Re\left\{\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right]^{\frac{1}{2}}\right\} < \left(\frac{\alpha}{n}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{n}-1}du\right)^{\frac{1}{2}} \quad (z\in\mathcal{U}), \tag{3.10}$$

and inequality (3.10) is sharp, with the extremal function defined by equality (3.3).

Proof. From Theorem 1 we know

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha} \prec \frac{1+Az}{1+Bz}.$$

Since $-1 \leq B < A \leq 1$, we have

$$0 \leq \frac{1-A}{1-B} < \Re\left\{(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right\} < \frac{1+A}{1+B}.$$

Thus, from inequality (3.2), we can get inequality (3.10). It is obvious that inequality (3.10) is sharp, with the extremal function defined by equality (3.3).

By applying the similar method as in Theorem 5, we have

Theorem 6. *Let $0 \leq \lambda \leq 1$, $\alpha \geq 0$ and $-1 \leq A < B \leq 1$. If $f(z) \in B_n(\lambda, \alpha, 0, A, B, z)$, then*

$$\left(\frac{\alpha}{n}\int_0^1 \frac{1+Au}{1+Bu}u^{\frac{\alpha}{n}-1}du\right)^{\frac{1}{2}} < \Re\left\{\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\alpha}+\lambda\left(f'(z)\right)^{\alpha}\right]^{\frac{1}{2}}\right\} < \left(\frac{\alpha}{n}\int_0^1 \frac{1-Au}{1-Bu}u^{\frac{\alpha}{n}-1}du\right)^{\frac{1}{2}} \quad (z\in\mathcal{U}), \tag{3.11}$$

and inequality (3.11) is sharp, with the extremal function defined by equality (3.3).

Remark 3. *From Theorem 5-6 we also can obtain the corresponding results about some other special classes of analytic functions, here we don't give unnecessary details any more.*

Theorem 7. *Let* $0 \leq \lambda \leq 1,\ \alpha \geq 0,\ -1 \leq B \leq 1,\ A \neq B$ *and* $A \in R$. *If* $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in B_n(\lambda, \alpha, 0, A, B, z)$, *then*

$$a_{n+1} \leq \frac{A - B}{(n + \alpha)(\lambda n + 1)}, \tag{3.12}$$

and inequality (3.12) is sharp, with the extremal function defined by equality (3.3).

Proof. Suppose that $f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \in B_n(\lambda, \alpha, 0, A, B, z)$, then we have

$$\begin{aligned} & (1 - \lambda)\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)(f'(z))^{\alpha} \\ = \ & 1 + (n + \alpha)(\lambda n + 1)a_{n+1}z^n + \cdots \prec \frac{1 + Az}{1 + Bz}. \end{aligned}$$

It follows from Lemma 2 that

$$(n + \alpha)(\lambda n + 1)a_{n+1} \leq A - B. \tag{3.13}$$

Thus, from (3.13), we can get (3.12). Notice that

$$f(z) = z + \frac{A - B}{(n + \alpha)(\lambda n + 1)}z^{n+1} + \cdots \in B_n(\lambda, \alpha, 0, A, B, z),$$

we obtain that inequality (3.12) is sharp.

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