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Water Supply Network in a Fuzzy Environment ; Maximum-Entropy Approach

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Abstract

The maximum entropy principle initiated by Jaynes [5] is a powerful optimization technique of determining the distribution of random system in the case of partial or incomplete information or data available about the system [7,10,12,13]. In our real world problem input data or parameters are often fuzzy or imprecise because of incomplete or non-obtainable information. The system is pipe line network , Which delivers known demands from source, Which are fuzzy number, to consumers. Traditional mathematical programming is unable to solve this type of problem. The paper deals with the estimation of optimal distribution of water in a water supply network by modifying the maximum entropy principle after ranking of fuzzy number.

Keywords and Phrases: *Triangular fuzzy number, Ranking of fuzzy numbers, Maximum entropy principle, Shanon entropy.*

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1. Introduction

In recent years, fuzzy set theory has been given place in the realm of logistics. Zimmermann [15] first applied the fuzzy set theory concept with some suitable membership functions to solve linear programming problem with several objective functions. He showed that solutions obtained by fuzzy linear programming are always efficient.

In this paper we introduce looped water supply network. Network has a single source from where water is distributed to different demand which are known and fuzzy number. We estimate the flow rate. In our practical life demand , source are not always crisp. First we have constructed an equivalent crisp model by ranking fuzzy numbers with respect to their total integral value. In order to calculate the flow rate (in m^3/hr) in the pipes of a looped system accurately we need additional information to enable us to calculate the head losses around the loops and so determine the flow rates. The conventional computer oriented pipe network analysis method [1,10] are useless in this situation as they require data on pipe lengths, diameters, and friction coefficients which we do not have. We can then infer some sort of "most likely" flow rate based only upon the incomplete information that we possess.

2. Preliminary Concepts

The problem considered in this study arises when a decision maker has vague or imprecise data in hand. At this point the coefficients in the problem can be defined by fuzzy numbers [4, 6]. Before developing optimization model, some basic concepts of fuzzy set theory and ranking fuzzy number functions are described below.

2.1 Fuzzy Sets and Fuzzy Numbers

Definition 2.1. *Define a universe of discourse , X, as a collection of objects all having the same characteristics. A fuzzy set $\widetilde{A}$ of X is defined by membership function $\mu_{\widetilde{A}} : X \longrightarrow [0,1]$. $\mu_{\widetilde{A}}(x)$ is the degree of membership of x in $\widetilde{A}$. The closer the value of $\mu_{\widetilde{A}}(x)$ is to unity, higher the grade of x in $\widetilde{A}$. Therefore, $\widetilde{A}$ is completely characterized by the set of ordered pairs :* $\widetilde{A} = \Big\{(x, \mu_{\widetilde{A}}(x))|\ x \in X\Big\}$.

Definition 2.2. *Let $F(\Re)$ be a set of all triangular fuzzy numbers in real line $\Re$. A triangular fuzzy number $\widetilde{A} \in F(\Re)$ is a fuzzy number with the membership function $\mu_{\widetilde{A}} : \Re \longrightarrow [0,1]$ parameterized by a triplet (a_L, a, a_R) where a_L and a_R denote the lower and upper limits of the support of a fuzzy number $\widetilde{A}$ with mode a :*

$$\mu_{\widetilde{A}}(x) = \begin{cases} \mu^L_{\widetilde{A}}(x) = \dfrac{x - a_L}{a - a_L}, & if a_L \le x \le a \\ 1 \quad , & x = a \\ \mu^R_{\widetilde{A}}(x) = \dfrac{x - a_R}{a - a_R}, & if a \le x \le a_R \end{cases} \tag{1}$$

where $a_L \le a \le a_R$ are real numbers, and $\mu^L_{\widetilde{A}}(x)$ and $\mu^R_{\widetilde{A}}(x)$ are the left membership function and right membership function of the fuzzy number $\widetilde{A}$.
$\mu^L_{\widetilde{A}}(x) : [a_L, a] \longrightarrow [0,1]$ is continuous and strictly increasing and $\mu^R_{\widetilde{A}}(x) : [a, a_R] \longrightarrow [0,1]$ is continuous and strictly decreasing function.

Figure 1: Triangular Fuzzy Number

In the triangular fuzzy number $\widetilde{A}$ denoted by $\widetilde{A} = (a_L, a, a_R)$, the parameter a gives the maximal grade of $\mu_{\widetilde{A}}(a)$, i.e. $\mu_{\widetilde{A}}(a)$=1; it is the most possible value of the evaluation data. a_L and a_R are the lower and upper bounds of the available areas of the evaluation data. For a non fuzzy number A, this can be expressed as $\widetilde{A} = (a, a, a)$. A triangular fuzzy number is therefore a generalization of a non-fuzzy number.

By the extension principle Zadeh [14], an arithmetics operation $\widetilde{W} = \widetilde{U} * \widetilde{V}$ of two triangular fuzzy numbers $\widetilde{U} = (u_L, u, u_R)$ and $\widetilde{V} = (v_L, v, v_R)$, is a triangular fuzzy number whose membership function $\mu_{\widetilde{W}}(x)$ is defined by Dubois and Prade [2] and Kaufmann and Gupta [6] as :

$$\mu_{\widetilde{W}}(z) = \mu_{\tilde{U} * \tilde{V}}(z) = \sup_{z = x * y} min\{\mu_{\tilde{U}}(x), \mu_{\tilde{V}}(y)\}$$

where $*$ denotes the one of the arithmetic operations $+$, $-$ or $\times$. Then, the result of fuzzy addition $\widetilde{U} \oplus \widetilde{V} = (u_L + v_L, u + v, u_R + v_R)$ is also a triangular fuzzy number. The operation of fuzzy subtraction $\widetilde{U} - \widetilde{V} = (u_L - v_R, u - v, u_R - v_L)$ and scaler multiplication

$$\begin{aligned} k \times \widetilde{v} &= (kv_L, kv, kv_R), \quad k \ge 0 \\ &= (kv_R, kv, kv_L), \quad k < 0 \end{aligned}$$

are also fuzzy number.

2.2 Ranking Fuzzy Numbers with Respect to their Total Integral Value

Before making a decision, decision-makers have to assess the alternatives with fuzzy numbers and rank these fuzzy numbers correspondingly Liou and Wang [8]. It can be seen that ranking of fuzzy numbers is an important procedure in solving the fuzzy programming problem. Many methods have been proposed for ranking of fuzzy numbers. A relatively simple computation and easily understood method proposed by Liou and Wang [8] is considered in this study.

Definition 2.3. *Let $\widetilde{A}$ be a triangular fuzzy number with membership function (1). The left integral value of$\widetilde{A}$ is defined as*

$$I_L(\widetilde{A}) = \int_0^1 (\mu^L_{\widetilde{A}})^{-1}(y)\,\mathrm{d}y \tag{2}$$

and the right integral value of $\widetilde{A}$ is defined as

$$I_R(\widetilde{A}) = \int_0^1 (\mu^R_{\widetilde{A}})^{-1}(y)\,\mathrm{d}y \tag{3}$$

Where $(\mu^R_{\widetilde{A}})^{-1}(y)$ and $(\mu^L_{\widetilde{A}})^{-1}$
(y)$aretheinversefunctionsof\mu^L_{\widetilde{A}}(x)$ and $\mu^R_{\widetilde{A}}(x)$ respctively.

For TFN $\widetilde{A} = (a_L, a, a_R)$

$(\mu^L_{\widetilde{A}})^{-1}(y) = a_L + (a - a_L)$y and $(\mu^R_{\widetilde{A}})^{-1}(y) = a_R + (a - a_R)$y , y $\in [0,1]$

Thus, we have

$$I_L(\widetilde{A}) = \frac{1}{2}(a_L + a) \tag{4}$$

and

$$I_R(\widetilde{A}) = \frac{1}{2}(a + a_R), \tag{5}$$

The left integral value $I_L(\widetilde{A})$ and the right integral value $I_R(\widetilde{A})$ of a TFN $\widetilde{A}$ are geometrically interpreted as the areas of trapezoids OLPQ and ORPQ respectively (Figure 1)

Definition 2.4. *Let $\widetilde{A}$ be a triangular fuzzy number with membership function (1), then the total λ -integral value of $\widetilde{A}$ with index of optimism $\lambda \in [0,1]$ is defined as*

$$\begin{aligned} I_T^{\lambda}(\widetilde{A}) &= \lambda I_R(\widetilde{A}) + (1-\lambda) I_L(\widetilde{A}) \\ &= \frac{1}{2}[\lambda a_R + a + (1-\lambda) a_L] \end{aligned} \tag{6}$$

where $I_R(\widetilde{A})$ and $I_R(\widetilde{A})$ are the right and left integral values of $\widetilde{A}$ respectively.

The left integral value is used to reflect the pessimistic viewpoint and the right integral value is used to reflect the optimistic viewpoint of the decision maker. The total λ-integral value is a convex combination of right and left integral values through an index of optimism.

Remarks. *When $\widetilde{A}$ is a crisp number, denoted by (a,a,a), then*

$$\begin{aligned} I_T^{\lambda}(\widetilde{A}) &= \lambda I_R(\widetilde{A}) + (1-\lambda) I_L(\widetilde{A}) \\ &= \frac{1}{2}[\lambda a + a + (1-\lambda) a] = a \end{aligned}$$

It is observed that the total integral value of a non-fuzzy number $\widetilde{A}$ $[= (a,a,a)]$ is a for all values of λ.

Definition 2.5. *Let $S = \{\widetilde{A}_1, \widetilde{A}_2, \cdots, \widetilde{A}_n\}$ be a set of convex fuzzy numbers, and that ranking function R $R : S \longrightarrow \Re$ is a mapping from S to the real line $\Re$. For any distinct $\widetilde{A}_i, \widetilde{A}_j \in S$, the ranking function has the following properties :*

$$\begin{array}{llcll} (1) & R(\widetilde{A}_i) & < & R(\widetilde{A}_j) & \textit{implies} \quad \widetilde{A}_i < \widetilde{A}_j, \\ (2) & R(\widetilde{A}_i) & = & R(\widetilde{A}_j) & \textit{implies} \quad \widetilde{A}_i = \widetilde{A}_j, \\ (3) & R(\widetilde{A}_i) & > & R(\widetilde{A}_j) & \textit{implies} \quad \widetilde{A}_i > \widetilde{A}_j \end{array}$$

In this study, total λ-integral value of fuzzy numbers is used as the ranking function. Under a given level of optimism $\lambda \in [0,1]$, fuzzy numbers can be ordered by comparing their total integral value with their λ- value. That is, for any distinct $\widetilde{A}_i, \widetilde{A}_j \in F(\Re) \subset S$, we use the following criteria to rank a fuzzy number based on Definition 2.5:

$$\begin{array}{llll}
(1) & I_T^\lambda(\widetilde{A}_i) < I_T^\lambda(\widetilde{A}_j) & \textit{implies} & \widetilde{A}_i < \widetilde{A}_j \quad (\widetilde{A}_i \text{ is smaller than } \widetilde{A}_j), \\
(2) & I_T^\lambda(\widetilde{A}_i) = I_T^\lambda(\widetilde{A}_j) & \textit{implies} & \widetilde{A}_i = \widetilde{A}_j \quad (\widetilde{A}_i \text{ is equal to } \widetilde{A}_j), \\
(3) & I_T^\lambda(\widetilde{A}_i) > I_T^\lambda(\widetilde{A}_j) & \textit{implies} & \widetilde{A}_i > \widetilde{A}_j \quad (\widetilde{A}_i \text{ is greater than } \widetilde{A}_j).
\end{array}$$

Obviously, the ranking function for fuzzy numbers is affected by the decision maker's degree of optimism, which is represented by the parameters $\lambda \in [0,1]$. A larger λ specifies a higher degree of optimism. For instance, when $\lambda = 1$, the total integral value $I_T^1(\widetilde{A}) = I_R(\widetilde{A})$ represents an optimistic decision maker's point of view. On the other hand, when $\lambda = 0$, the total integral value $I_T^0(\widetilde{A}) = I_L(\widetilde{A})$ indicates pessimistic decision maker's point of view.

$$\text{When} \quad \lambda = 0.5, \quad I_T^{0.5}(\widetilde{A}) = \frac{1}{2}(I_R(\widetilde{A}) + I_L(\widetilde{A})) = \frac{(a_L + 2a + a_R)}{4}$$

which is the same as ordinary representation of $\widetilde{A}$ Kaufmann and Gupta , [6]. It indicates a moderately optimistic decision maker's point of view and is well qualified to be a defuzzification of the fuzzy number $\widetilde{A}$.

3. The Maximum Entropy Principle Probabilistic Distribution

Consider a random process which can be described by discrete random variable X with n possible outcomes $\{ x_1, x_2, \cdots, x_N\}$. Define p_i , $i = 1, 2, \cdots, N$, to be the probability that X has the values x_i , $i = 1, 2, \cdots, N$. i.e. $P(X = x_i) = p_i$. The probabilities are not known. Some information is available about the random process in the form of M expectation function

$$\sum_{j=1}^{N} p_j a_{ij}(x) = E[a_i], \quad i = 1, 2, \cdots, M \tag{7}$$

$$\sum_{j=1}^{N} p_j = 1 \tag{8}$$

Our problem is that of finding a probability assignment which avoids bias while agreeing with whatever information is given. The distribution random process possesses great deal disorder or chaos. This measure of disorder was given by Shanon [9] as:

$$S(p_1, p_2, \cdots, p_N) = -K \sum_{j=1}^{N} p_j \ ln p_j \tag{9}$$

where K is a positive constant called the Boltzmann constant, depends upon the unit measurement of entropy. Jaynes's Maximum Entropy Principle casts the problem of determining the discrete probabilities p into the form of an optimization problem. Modified form of Maximum Entropy Principle [3] is used to generate solution to wider , more general problems where the available information is not complete.

Problem

$$\text{Maximize} \quad S \;=\; -\sum_{j=1}^{N} p_j \, ln(p_j), \quad (K = 1) \tag{10}$$

Subject to linear constraints :

$$\sum_{j=1}^{N} p_j \;=\; 1 \tag{11}$$

$$\sum_{j=1}^{N} p_j \, b_{ij} \;=\; c_i \,, \quad i = 1, 2, \cdots, M \tag{12}$$

The solution of the problem I is

$$p_j \;=\; \exp\Big\{ -\mu - \sum_{i=1}^{M} \beta_i b_{ij} \Big\}, \; j = 1, 2, \cdots, N \tag{13}$$

where β_i , $i = 1, 2, \cdots, M$ and μ are Lagrange multipliers associated with (12) and (11) respectively. To determine the probabilities p_j , $j = 1, 2, \cdots, N$

from the equation (13) it is therefore necessary to know the values of Lagrange multipliers. This can be found solving the equations

$$\sum_{j=1}^{N} \exp\Big\{-\sum_{i=1}^{M} \beta_i b_{ij}\Big\} = \exp(\mu) \tag{14}$$

$$\sum_{j=1}^{N} b_{ij} \exp\Big\{-\mu-\sum_{i=1}^{M} \beta_i b_{ij}\Big\} = c_i\,, \quad i=1,2,\cdots,M \tag{15}$$

Solving the $(M+1)$ non-linear equations we can find $(M+1)$ Lagrange multipliers, which is awkward and tedious and required laborious numerical calculation. We modify this process in the following way :

Assume that the matrix $B=[b^*_{ij}]_{(M+1)\times N}$ where $b^*_{ij}=b_{ij}$, $i=1,2,\cdots,M$

$$= 1\ ,\ i=M+1\ ,\ j=1,2,\cdots,N$$

has the rank $(M+1)$ [$M+1<N$, otherwise equations (14) and (15) would themselves be sufficient to determine the unknown probabilities uniquely and would consist of complete rather than partial information.]

Then we can write

$$\begin{array}{rcl} p_1 & = & \psi_1(p_{M+2},p_{M+3},\cdots,p_N) \\ p_2 & = & \psi_2(p_{M+2},p_{M+3},\cdots,p_N) \\ \cdots & \cdots & \cdots\cdots\cdots\cdots \\ p_{M+1} & = & \psi_{M+1}(p_{M+2},p_{M+3},\cdots,p_N) \end{array} \tag{16}$$

Then original problem is then reduced to unconstrained optimization problem. For maximization $\frac{\partial S}{\partial p_j}=0$. Then

$$p_j \prod_{i=1}^{M+1} \psi_i^{(\frac{\partial \psi_i}{\partial p_j})} = \exp\Big[-\Big\{1+\sum_{i=1}^{M+1}\frac{\partial \psi_i}{\partial p_j}\Big\}\Big], \quad j=M+2,M+3,\cdots,N. \tag{17}$$

Hence $N-(M+1)$ equations and $N-(M+1)$ unknowns , which can be solved to find the values $(p_{M+2},p_{M+3},\cdots,p_N)$ and then from the relations (16) we can find the whole set $(p_1,p_2,\cdots,p_N)$

4. Water Supply Network Analysis

Water distribution system connect consumers to sources of water, using hydraulic components, such as pipes, valves and reservoirs. The engineer faced with the design of such a system or of additions to an existing system, has to select sizes of its components. Also he has to consider the way in which the operational components will used to supply the required demands with adequate pressures. A problem is to estimate of least biased pipe flow rate through different known direction of a looped water network from known demand, which is usually made that all the demands occur at the nodes. In order to calculate the flow rates in the pipes of a looped system accurately we need additional information to enable us to calculate the head losses around the two loops and so determine the flow rates. The conventional computer oriented pipe network analysis [1,9,10, 11] are useless in this situation as they required data on pipe lengths, diameters, friction coefficients etc. which we do not have.

Again in our real world problem input data or parameters are often fuzzy or imprecise because of incomplete or non-obtainable information. Traditional mathematical programming techniques, obviously, can not solve this type of problem. Here the demands, sources are characterized as triangular fuzzy number (TFN). First these model have been converted into deterministic ones via the ranking function of the fuzzy numbers with respect to their total integral values and maximum entropy technique developed in the previous section can successfully used in determining the flow rate.

5. Small Looped Water Network Problem

Consider a looped water supply network **Fig-2** with 7 nodes and 8 pipes whose source and demand are triangular fuzzy number. Note **Fig-2** shows the connectivity of the system and the flow direction in each pipe but does not give any data for the individual pipes. If the network were branched we would have no difficulty in calculating the flow rates in the pipes of the branched system. We could simply work backwards along each branch accumulating demand quantities as flow rates until we reach the single source. Let Q_i be the assumed flow rate($m^3/hour$) through i th pipe ($i = 1, 2, \cdots, 8$). Then we have equations

$$
\begin{aligned}
Q_1 &= (1060, 1120, 1200) \\
Q_2 + Q_3 &= (940, 1020, 1110) \\
Q_2 - Q_7 &= (90, 100, 110) \\
Q_8 + Q_6 &= (190, 200, 210) \\
Q_5 - Q_6 &= (320, 330, 340) \\
Q_3 - Q_4 - Q_5 &= (110, 120, 130) \\
Q_4 + Q_7 - Q_8 &= (260, 270, 290)
\end{aligned}
\tag{18}
$$

First convert this model into crisp one by (6). Which gives

$$
\begin{aligned}
Q_1 &= 1090 + 70\lambda \\
Q_2 + Q_3 &= 980 + 85\lambda \\
Q_2 - Q_7 &= 95 + 10\lambda \\
Q_8 + Q_6 &= 195 + 10\lambda \\
Q_5 - Q_6 &= 325 + 10\lambda \\
Q_3 - Q_4 - Q_5 &= 115 + 10\lambda \\
Q_4 + Q_7 - Q_8 &= 265 + 15\lambda
\end{aligned}
\tag{19}
$$

All input and output flow rates have also been scaled by appropriate factor of $(980 + 85\lambda)$ and using (17) we find the least biased estimation of the pipe flow rate for different values of λ

λ	Flow rate			
0	$Q_1 = 1090$	$Q_2 = 355.210$	$Q_3 = 639.895$	$Q_4 = 144.562$
	$Q_5 = 379.323$	$Q_6 = 55.827$	$Q_7 = 260.153$	$Q_8 = 141.030$
0.3	$Q_1 = 1111$	$Q_2 = 362.123$	$Q_3 = 649.287$	$Q_4 = 147.134$
	$Q_5 = 382.564$	$Q_6 = 56.289$	$Q_7 = 263.753$	$Q_8 = 142.158$
0.5	$Q_1 = 1125$	$Q_2 = 366.145$	$Q_3 = 656.054$	$Q_4 = 149.566$
	$Q_5 = 385.134$	$Q_6 = 56.916$	$Q_7 = 265.573$	$Q_8 = 14.612$
0.7	$Q_1 = 1139$	$Q_2 = 370.923$	$Q_3 = 662.254$	$Q_4 = 150.857$
	$Q_5 = 388.028$	$Q_6 = 57.734$	$Q_7 = 268.873$	$Q_8 = 146.331$
1	$Q_1 = 1160$	$Q_2 = 377.834$	$Q_3 = 672.012$	$Q_4 = 153.237$
	$Q_5 = 391.985$	$Q_6 = 58.152$	$Q_7 = 271.904$	$Q_8 = 147.876$

Table : 1 / Flow rate for different values of λ

6. Conclusion

For large engineering systems such as pipe line networks, traffic networks etc, complete and accurate data is rarely available and often the calculated performance depends upon'engineering judgements' in supplying guesses at values for data items in order to be able to use existing computer programs. Again in our real world problem input data or parameters are often fuzzy or imprecise because of incomplete or non-obtainable information. Traditional mathematical programming techniques, obviously, can not solve this type of problem. Here the demands, sources are characterized as triangular fuzzy number (TFN).

The above the process shown how the entropy measure of uncertainty and the Maximum Entropy Principle can be used to infer least biased results using incomplete data. In the network described we have specified initially the direction of flow and demand, source which are TFN. If the direction of flow is not given we shall consider all possible kinds of direction of the network and estimate the corresponding maximum entropy for each case. We shall select the one which yields the greatest value. Initially the problem has been converted into crisp one by ranking of fuzzy numbers with respect to their integral values and then maximum entropy principle can successfully used in determining the flow rate.

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