Tamsui Oxford Journal of Mathematical Sciences **23(1)** (2007) 105-125
Aletheia University

Ostrowski-Grüss-Čebyšev Type Inequalities Involving Several Functions*

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Received July 28, 2005, Accepted August 10, 2005.

Abstract

In this paper, we establish some Ostrowski-Grüss-Čebyšev type inequalities involving several functions whose modulus of the derivatives are convex functions.

Keywords and Phrases: *Ostrowski inequality, Grüss inequality, Čebyšev inequality, Convex functions, Log-convex functions.*

1. Introduction

Throughout, let

$$\|h'\|_\infty := \sup_{t\in(a,b)} |h'(t)|,$$

$$S(f,g) = f(x)\,g(x) - \frac{1}{2(b-a)}\left[g(x)\int_a^b f(t)\,dt + f(x)\int_a^b g(t)\,dt\right],$$

and

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right)$$

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*2000*Mathematics Subject Classification.* 26D15, 26D20.

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where $x \in [a,b]$, h' exists and is bounded on (a,b) and f, g are integrable on $[a,b]$.

The *Ostrowski's inequality* [5] states that if f' exists and is bounded on (a,b), then, for all $x \in [a,b]$, we have the inequality

$$\left|\int_a^b f(t)dt - f(x)(b-a)\right| \leq \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right] \|f'\|_\infty . \quad (1.1)$$

The *Čebyšev's inequality* [6] states that if f', $g' \in L_\infty[a,b]$, then we have the inequality

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty . \quad (1.2)$$

The *Grüss's inequality* [6], states that if $m \leq f(x) \leq M$ and $n \leq f(x) \leq N < \infty$ for all $x \in [a,b]$, then we have the inequality

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \quad (1.3)$$

where m, M, n, N are real numbers.

For some recent results which generalize, improve and extend the inequalities (1.1)-(1.3), see $[1-11]$.

Recall the following definitions of a convex function and a log-convex function:

Let $f:[a,b] \to R$ and $g:[a,b] \to (0,\infty)$. The functions f and g are called convex on $[a,b]$ and log-convex on $[a,b]$, respectively, if

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y),$$

and

$$f(tx + (1-t) y) \leq [f(x)]^\lambda [f(y)]^{1-\lambda}$$

for all $x, y \in [a,b]$ and $\lambda \in [0,1]$.

In [10], Pachpatte established the following four theorems about Ostrowski-Grüss-Čebyšev type inequalities:

Theorem A. *Let f, $g : [a,b] \to R$ be absolutely continuous functions on $[a,b]$.*

(a_1) *If $|f'|$ and $|g'|$ are convex on $[a,b]$, then*

$$|S(f,g)| \le \frac{1}{4}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)$$
$$\times\left\{|g(x)|\left[|f'(x)|+\|f'\|_\infty\right]+|f(x)|\left[|g'(x)|+\|g'\|_\infty\right]\right\}$$

for $x \in [a,b]$.

(a_2) *If $|f'|$ and $|g'|$ are log-convex on $[a,b]$, then*

$$|S(f,g)| \le \frac{1}{2(b-a)}\left\{|g(x)|\,|f'(x)|\int_a^b |x-t|\left(\frac{A-1}{\ln A}\right)dt\right.$$
$$\left.+\,|f(x)|\,|g'(x)|\int_a^b |x-t|\left(\frac{B-1}{\ln B}\right)dt\right\}$$

for $x \in [a,b]$, where

$$A=\frac{|f'(t)|}{|f'(x)|} \quad \text{and} \quad B=\frac{|g'(t)|}{|g'(x)|}\,. \tag{1.4}$$

Theorem B. *Let f and $g : [a,b] \to R$ be absolutely continuous functions on $[a,b]$.*

(b_1) *If $|f'|$ and $|g'|$ are convex on $[a,x]$ and $[x,b]$, then*

$$|S(f,g)| \le \frac{1}{2}\left\{|g(x)|\,F(x)+|f(x)|\,G(x)\right\},$$

for $x \in [a,b]$, where

$$F(x)=\frac{1}{6}\left[|f'(a)|\left(\frac{x-a}{b-a}\right)^2+|f'(b)|\left(\frac{b-x}{b-a}\right)^2\right.$$
$$\left.+\left\{1+4\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\}|f'(x)|\right](b-a)$$

and

$$G(x)=\frac{1}{6}\left[\left|g'(a)\right|\left(\frac{x-a}{b-a}\right)^2+\left|g'(b)\right|\left(\frac{b-x}{b-a}\right)^2+\left\{1+4\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right\}\left|g'(x)\right|\right](b-a)$$

for $x\in[a,b]$.

(b_2) *If* $|f'|$ *and* $|g'|$ *are log-convex on* $[a,x]$ *and* $[x,b]$, *then*

$$|S(f,g)|\le\frac{1}{2}\left[|g(x)|P(x)+|f(x)|Q(x)\right],$$

for $x\in[a,b]$, *where*

$$P(x)=(b-a)\left[\left|f'(a)\right|\left(\frac{x-a}{b-a}\right)^2\frac{A_1\ln A_1+1-A_1}{(\ln A_1)^2}+\left|f'(b)\right|\left(\frac{b-x}{b-a}\right)^2\frac{B_1\ln B_1+1-B_1}{(\ln B_1)^2}\right],$$

$$Q(x)=(b-a)\left[\left|g'(a)\right|\left(\frac{x-a}{b-a}\right)^2\frac{A_2\ln A_2+1-A_2}{(\ln A_2)^2}+\left|g'(b)\right|\left(\frac{b-x}{b-a}\right)^2\frac{B_2\ln B_2+1-B_2}{(\ln B_2)^2}\right],$$

and

$$A_1=\frac{|f'(x)|}{|f'(a)|},\quad B_1=\frac{|f'(x)|}{|f'(b)|},\tag{1.5}$$

$$A_2=\frac{|g'(x)|}{|g'(a)|},\quad B_2=\frac{|g'(x)|}{|g'(b)|},\tag{1.6}$$

for $x\in[a,b]$.

Theorem C. *Let* f *and* $g:[a,b]\to R$ *be absolutely continuous functions on* $[a,b]$.

(c_1) *If* $|f'|$ *and* $|g'|$ *are convex on* $[a,b]$, *then*

$$|T(f,g)| \leq \frac{1}{4(b-a)^2}\int_a^b \left[|g(x)|\left[|f'(x)|+\|f'\|_\infty\right]+|f'(x)|\left[|g'(x)|+\|g'\|_\infty\right]\right]E(x)\,dx,$$

where

$$E(x)=\frac{(x-a)^2+(b-x)^2}{2} \tag{1.7}$$

for $x\in[a,b]$.

(c_2) *If* $|f'|$ *and* $|g'|$ *are log-convex on* $[a,b]$, *then*

$$|T(f,g)| \leq \frac{1}{2(b-a)^2}\int_a^b \left[|g(x)|\int_a^b |x-t|\,|f'(x)|\left(\frac{A-1}{\ln A}\right)dt + |f(x)|\int_a^b |x-t|\,|g'(x)|\left(\frac{B-1}{\ln B}\right)dt\right]dx$$

where A,B *are defined as in* (1.4).

Theorem D. *Let* f *and* $g:[a,b]\to R$ *be absolutely continuous functions on* $[a,b]$.

(d_1) *If* $|f'|$ *and* $|g'|$ *are convex on* $[a,b]$, *then*

$$|T(f,g)| \leq \frac{1}{2}\int_a^b \left[\left(\frac{x-a}{b-a}\right)^2\left[|g(x)|\left\{\frac{1}{6}|f'(a)|+\frac{1}{3}|f'(x)|\right\} + |f(x)|\left\{\frac{1}{6}|g'(a)|+\frac{1}{3}|g'(x)|\right\}\right] + \left(\frac{x-a}{b-a}\right)^2|g(x)|\left\{\frac{1}{3}|f'(x)|+\frac{1}{6}|f'(b)|\right\} + |f(x)|\left\{\frac{1}{3}|g'(x)|+\frac{1}{6}|g'(b)|\right\}\right]\right]dx,$$

(d_2) *If* $|f'|$ *and* $|g'|$ *are log-convex on* $[a,x]$ *and* $[x,b]$, *then*

$$|T(f,g)| \leq \frac{1}{2}\int_a^b \left[\left(\frac{x-a}{b-a}\right)^2\left\{|g(x)|\,|f'(a)|\,\frac{A_1\ln A_1+1-A_1}{(\ln A_1)^2}\right.\right.$$

$$+\left|f(x)\right|\left|g'(a)\right|\frac{A_2\ln A_2+1-A_2}{(\ln A_2)^2}\bigg\}$$
$$+\left(\frac{x-x}{b-a}\right)^2\left\{\left|g(x)\right|\left|f'(b)\right|\frac{B_1\ln B_1+1-B_1}{(\ln B_1)^2}\right.$$
$$\left.+\left|f(x)\right|\left|g'(b)\right|\frac{B_2\ln B_2+1-B_2}{(\ln B_2)^2}\right\}\Bigg]dx$$

where A_1, B_1 *and* A_2, B_2 *are defined as in* (1.5) *and* (1.6), *respectively.*

In this paper, we establish some inequalities which generalize Theorems A-D.

2. Main Results

Throughout in this section, let $\alpha_i \in R\ (i=1,\cdots,n), \alpha=(\alpha_1,\cdots,\alpha_n)$ with $\sum_{i=1}^{n}\alpha_i=1$ and let

$$S_\alpha(f_1,\cdots,f_n) = \prod_{i=1}^{n} f_i(x) - \frac{1}{(b-a)}\sum_{i=1}^{n}\left(\prod_{j=1,j\neq i}^{n}\alpha_i f_j(x)\int_a^b f_i(t)\,dt\right),$$

and

$$T_\alpha(f_1,\cdots,f_n) = \frac{1}{b-a}\int_a^b\prod_{i=1}^{n} f_i(x)\,dx - \sum_{i=1}^{n}\left[\left(\frac{1}{b-a}\int_a^b\prod_{j=1,j\neq i}^{n}\alpha_i f_j(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b f_i(x)\,dx\right)\right]$$

where $x\in[a,b]$, and $f_i\ (i=1,\cdots,n)$ are integrable on $[a,b]$.

In order to prove our results, we need the following identities proved in [1] and [2], respectively:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,dt + \frac{1}{b-a}\int_a^b (x-t)\left[\int_0^1 f'[(1-\lambda)x+\lambda t]\,dt\right]d\lambda \tag{2.1}$$

and

$$f(x)=\frac{1}{b-a}\int_a^b f(t)\,dt+(x-a)^2\frac{1}{b-a}\int_0^1 \lambda f'[(1-\lambda)a+\lambda x]\,d\lambda$$

$$-(b-x)^2\frac{1}{b-a}\int_0^1 \lambda f'[\lambda x+(1-\lambda)b]\,d\lambda, \tag{2.2}$$

for $x\in[a,b]$ where $f:[a,b]\to R$ is an absolutely continuous function on $[a,b]$.

Theorem 1. *Let $f_i:[a,b]\to R$ $(i=1,\cdots,n)$ be absolutely continuous functions on $[a,b]$.*

(a) If $|f_i'|$ $(i=1,\cdots,n)$ are convex on $[a,b]$, then

$$|S_\alpha(f_1,\cdots,f_n)|\le\frac{b-a}{2}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]$$

$$\times\sum_{i=1}^n\left[\prod_{j=1,j\neq i}^n|\alpha_i f_j(x)|\left(|f_i'(x)|+\|f_i'\|_\infty\right)\right], \tag{2.3}$$

for $x\in[a,b]$.

(b) If $|f_i'|$ $(i=1,\cdots,n)$ are log-convex on $[a,b]$, then

$$|S_\alpha(f_1,\cdots,f_n)|$$

$$\le\sum_{i=1}^n\left\{\frac{1}{b-a}\prod_{j=1,j\neq i}^n|\alpha_i f_j(x)|\,|f_i'(x)|\int_a^b|x-t|\left[\left(\frac{D_i-1}{\ln D_i}\right)\right]dt\right\} \tag{2.4}$$

for $x\in[a,b]$, where

$$D_i=\frac{|f_i'(t)|}{|f_i'(x)|}\ \ (i=1,\cdots,n) \tag{2.5}$$

for $x,t\in[a,b]$.

Proof. Using (2.1), we have the identities

$$f_1(x)-\frac{1}{b-a}\int_a^b f_1(t)\,dt$$

$$= \frac{1}{b-a} \int_a^b (x-t) \left[\int_0^1 f_1' \left[(1-\lambda) x + \lambda t \right] d\lambda \right] dt, \tag{2.6.1}$$

$$f_2(x) - \frac{1}{b-a} \int_a^b f_2(t)\, dt$$

$$= \frac{1}{b-a} \int_a^b (x-t) \left[\int_0^1 f_2' \left[(1-\lambda) x + \lambda t \right] d\lambda \right] dt, \tag{2.6.2}$$

$$\vdots$$

$$f_n(x) - \frac{1}{b-a} \int_a^b f_n(t)\, dt$$

$$= \frac{1}{b-a} \int_a^b (x-t) \left[\int_0^1 f_n' \left[(1-\lambda) x + \lambda t \right] d\lambda \right] dt \tag{2.6.n}$$

for $x \in [a,b]$. Multiplying both sides of (2.6.i) by $\prod_{j=1, j\neq i}^{n} \alpha_i f_j(x)$ $(i = 1, \cdots, n)$ and adding the resulting identities, we get

$$S_\alpha (f_1, \cdots, f_n)$$

$$= \sum_{i=1}^{n} \left\{ \frac{1}{b-a} \prod_{j=1, j\neq i}^{n} \alpha_i f_j(x) \int_a^b (x-t) \left[\int_0^1 f_i' \left[(1-\lambda) x + \lambda t \right] d\lambda \right] dt \right\}. \tag{2.7}$$

(a) Since $|f_i'|$ $(i = 1, \cdots, n)$ are convex on $[a,b]$, from (2.7) we get that

$$|S_\alpha (f_1, \cdots, f_n)|$$

$$\leq \sum_{i=1}^{n} \left\{ \frac{1}{b-a} \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \int_a^b |x-t| \left[\int_0^1 |f_i' \left[(1-\lambda) x + \lambda t \right]| \, d\lambda \right] dt \right\}$$

$$\leq \sum_{i=1}^{n} \left\{ \frac{1}{b-a} \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \int_a^b |x-t| \left[\int_0^1 (1-\lambda) |f_i'(x)| + \lambda |f_i'(t)| \, d\lambda \right] dt \right\}$$

$$= \sum_{i=1}^{n} \left\{ \frac{1}{b-a} \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \int_a^b |x-t| \frac{1}{2} \left(|f_i'(x)| + |f_i'(t)| \right) dt \right\}$$

$$\leq \frac{1}{2(b-a)} \int_a^b |x-t|\, dt \sum_{i=1}^{n} \left\{ \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \left(|f_i'(x)| + \|f_i'\|_\infty \right) \right\}$$

$$= \frac{(x-a)^2 + (b-x)^2}{4(b-a)} \sum_{i=1}^{n} \left\{ \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \left(|f_i'(x)| + \|f_i'\|_\infty \right) \right\}$$

$$= \frac{b-a}{2} \left[\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right] \sum_{i=1}^{n} \left\{ \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \left(|f_i'(x)| + \|f_i'\|_\infty \right) \right\}$$

which is the inequality (2.3).

(b) Since $|f_i'|$ $(i = 1, \cdots, n)$ are log-convex on $[a, b]$, from (2.7) we get that

$$|S_\alpha(f_1, \cdots, f_n)|$$

$$\leq \sum_{i=1}^{n} \left\{ \frac{1}{b-a} \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \int_a^b |x-t| \left[\int_0^1 |f_i'[(1-\lambda)x + \lambda t]|\, d\lambda \right] dt \right\}$$

$$\leq \sum_{i=1}^{n} \left\{ \frac{1}{b-a} \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \int_a^b |x-t| \left[\int_0^1 [|f_i'(x)|]^{1-\lambda} [|f_i'(t)|]^{\lambda}\, d\lambda \right] dt \right\}$$

$$= \sum_{i=1}^{n} \left\{ \frac{1}{b-a} \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \int_a^b |x-t| \left[|f_i'(x)| \int_0^1 \left[\frac{|f_i'(t)|}{|f_i'(x)|} \right]^{\lambda} d\lambda \right] dt \right\}$$

$$= \sum_{i=1}^{n} \left\{ \frac{1}{b-a} \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)|\, |f_i'(x)| \int_a^b |x-t| \left[\left(\frac{\frac{|f_i'(t)|}{|f_i'(x)|} - 1}{\ln \frac{|f_i'(t)|}{|f_i'(x)|}} \right) \right] dt \right\}$$

$$= \sum_{i=1}^{n} \left\{ \frac{1}{b-a} \prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)|\, |f_i'(x)| \int_a^b |x-t| \left[\left(\frac{D_i - 1}{\ln D_i} \right) \right] dt \right\}$$

which is the inequality (2.4), where D_i $(i = 1, \cdots, n)$ are defined as in (2.5)..

This completes the proof.

Let $\alpha = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$ in Theorem 1, then we have the following corollary:

Corollary 1. *Let* $f_i : [a, b] \to R$ $(i = 1, \cdots, n)$ *be absolutely continuous functions on* $[a, b]$.

(a) If $|f_i'|$ $(i=1,\cdots,n)$ *are convex on* $[a,b]$*, then*

$$|S_\alpha(f_1,\cdots,f_n)| \le \frac{b-a}{2n}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right] \times \sum_{i=1}^{n}\left[\prod_{j=1,j\neq i}^{n}|f_j(x)|\left(|f_i'(x)|+\|f_i'\|_\infty\right)\right],$$

for $x\in[a,b]$.

(b) If $|f_i'|$ $(i=1,\cdots,n)$ *are log-convex on* $[a,b]$*, then*

$$|S_\alpha(f_1,\cdots,f_n)| \le \sum_{i=1}^{n}\frac{1}{n}\left\{\frac{1}{b-a}\prod_{j=1,j\neq i}^{n}|f_j(x)|\,|f_i'(x)|\int_a^b|x-t|\left[\left(\frac{D_i-1}{\ln D_i}\right)\right]dt\right\}$$

for $x\in[a,b]$*, where* D_i $(i=1,\cdots,n)$ *are defined as in* (2.5).

Remark 1. *If we choose* $n=2$*, then Corollary 1 reduces to Theorem A.*

Theorem 2. *Let* $f_i:[a,b]\to R$ $(i=1,\cdots,n)$ *be absolutely continuous functions on* $[a,b]$.

(a) If $|f_i'|$ $(i=1,\cdots,n)$ *are convex on* $[a,x]$ *and* $[x,b]$*, then*

$$|S_\alpha(f_1,\cdots,f_n)| \le \sum_{i=1}^{n}\left(\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\,F_i(x)\right) \tag{2.8}$$

where

$$F_i(x)=\frac{b-a}{6}\left[|f_i'(a)|\left(\frac{x-a}{b-a}\right)^2+|f_i'(b)|\left(\frac{b-x}{b-a}\right)^2 + \left(1+4\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right)|f_i'(x)|\right] \tag{2.9}$$

for $x\in[a,b]$ *and* $i=1,\cdots,n$.

(b) If $|f_i'|$ $(i = 1, \cdots, n)$ *are log-convex on* $[a, x]$ *and* $[x, b]$*, then*

$$|S_\alpha(f_1, \cdots, f_n)| \leq \sum_{i=1}^{n} \left(\prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \, H_i(x) \right) \tag{2.10}$$

where

$$I_i = \frac{|f_i'(x)|}{|f_i'(a)|}, \quad J_i = \frac{|f_i'(x)|}{|f_i'(b)|} \ (i = 1, \cdots, n) \tag{2.11}$$

and

$$H_i(x) = (b-a)\left[|f_i'(a)| \left(\frac{x-a}{b-a}\right)^2 \frac{I_i \ln I_i + 1 - I_i}{(\ln I_i)^2} \right.$$

$$\left. + |f_i'(b)| \left(\frac{b-x}{b-a}\right)^2 \frac{J_i \ln J_i + 1 - J_i}{(\ln J_i)^2}\right] \tag{2.12}$$

for $x \in [a, b]$ *and* $i = 1, \cdots, n$.

Proof. Using (2.2), we have the identities

$$f_1(x) - \frac{1}{b-a}\int_a^b f_1(t)\,dt$$

$$= \frac{(x-a)^2}{b-a}\int_0^1 \lambda f_1'[(1-\lambda)a + \lambda x]\,d\lambda$$

$$- \frac{(b-x)^2}{b-a}\int_0^1 \lambda f_1'[\lambda x + (1-\lambda)b]\,d\lambda, \tag{2.13.1}$$

$$f_2(x) - \frac{1}{b-a}\int_a^b f_2(t)\,dt$$

$$= \frac{(x-a)^2}{b-a}\int_0^1 \lambda f_2'[(1-\lambda)a + \lambda x]\,d\lambda$$

$$- \frac{(b-x)^2}{b-a}\int_0^1 \lambda f_2'[\lambda x + (1-\lambda)b]\,d\lambda, \tag{2.13.2}$$

$$\vdots$$

$$f_n(x) - \frac{1}{b-a}\int_a^b f_n(t)\,dt$$
$$= \frac{(x-a)^2}{b-a}\int_0^1 \lambda f_n'[(1-\lambda)a + \lambda x]\,d\lambda$$
$$- \frac{(b-x)^2}{b-a}\int_0^1 \lambda f_n'[\lambda x + (1-\lambda)b]\,d\lambda, \tag{2.13.n}$$

for $x \in [a,b]$. Multiplying both sides of (2.13.i) by $\prod_{j=1, j\neq i}^{n} \alpha_i f_j(x)$ $(i = 1, \cdots, n)$ and adding the resulting identities, we get

$$S_\alpha(f_1, \cdots, f_n)$$
$$= \sum_{i=1}^{n}\left\{\prod_{j=1, j\neq i}^{n} \alpha_i f_j(x)\left[\frac{(x-a)^2}{b-a}\int_0^1 \lambda f_i'[(1-\lambda)a + \lambda x]\,d\lambda\right.\right.$$
$$\left.\left. - \frac{(b-x)^2}{b-a}\int_0^1 \lambda f_i'[\lambda x + (1-\lambda)b]\,d\lambda\right]\right\} \tag{2.14}$$

Using (2.14), we get that

$$|S_\alpha(f_1, \cdots, f_n)| \le \sum_{i=1}^{n}\left[\prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)|\, M_i(x)\right] \tag{2.15}$$

where

$$M_i(x) = \frac{(x-a)^2}{b-a}\int_0^1 \lambda\,|f_i'[(1-\lambda)a + \lambda x]|\,d\lambda$$
$$+ \frac{(b-x)^2}{b-a}\int_0^1 \lambda\,|f_i'[\lambda x + (1-\lambda)b]|\,d\lambda \tag{2.16}$$

for $x \in [a,b]$ and $i = 1, \cdots, n$.

(a) Since $|f_i'|$ $(i = 1, \cdots, n)$ are convex on $[a,x]$ and $[x,b]$, we have that

$$\int_0^1 \lambda\,|f_i'[(1-\lambda)a + \lambda x]|\,d\lambda$$
$$\le |f_i'(a)|\int_0^1 \lambda(1-\lambda)\,d\lambda + |f_i'(x)|\int_0^1 \lambda^2 d\lambda$$

$$= \frac{1}{6}\left|f_i'(a)\right| + \frac{1}{3}\left|f_i'(x)\right| \tag{2.17}$$

and

$$\int_0^1 \lambda \left|f_i'[\lambda x + (1-\lambda)b]\right| d\lambda$$
$$\leq \left|f_i'(x)\right| \int_0^1 \lambda^2 d\lambda + \left|f_i'(b)\right| \int_0^1 \lambda(1-\lambda)\, d\lambda$$
$$= \frac{1}{3}\left|f_i'(x)\right| + \frac{1}{6}\left|f_i'(b)\right| \tag{2.18}$$

where $x \in [a,b]$ and $i = 1, \cdots, n$.

From (2.16)-(2.18), we get that

$$M_i(x) \leq \frac{(x-a)^2}{b-a}\left[\frac{1}{6}\left|f_i'(a)\right| + \frac{1}{3}\left|f_i'(x)\right|\right]$$
$$+ \frac{(b-x)^2}{b-a}\left[\frac{1}{3}\left|f_i'(x)\right| + \frac{1}{6}\left|f_i'(b)\right|\right]$$
$$= \frac{b-a}{6}\left[\left|f_i'(a)\right|\left(\frac{x-a}{b-a}\right)^2 + \left|f_i'(b)\right|\left(\frac{b-x}{b-a}\right)^2\right.$$
$$\left. + 2\left|f_i'(x)\right|\left(\left(\frac{x-a}{b-a}\right)^2 + \left(\frac{b-x}{b-a}\right)^2\right)\right]$$
$$= F_i(x) \tag{2.19}$$

where $x \in [a,b]$ and $i = 1, \cdots, n$.

Using (2.15) and (2.19), we get the inequality (2.8).

(b) Since $|f_i'|$ $(i = 1, \cdots, n)$ are log-convex on $[a,x]$ and $[x,b]$, we have that

$$\int_0^1 \lambda \left|f_i'[(1-\lambda)a + \lambda x]\right| d\lambda$$
$$\leq \int_0^1 \lambda \left|f_i'(a)\right|^{1-\lambda} \left|f_i'(x)\right|^{\lambda} d\lambda$$
$$= \left|f_i'(a)\right| \int_0^1 \lambda I_i^{\lambda} d\lambda$$

$$= |f_i'(a)| \frac{I_i \ln I_i + 1 - I_i}{(\ln I_i)^2} \tag{2.20}$$

and

$$\begin{aligned} &\int_0^1 \lambda |f_i'[\lambda x + (1-\lambda) b]| \, d\lambda \\ &\le \int_0^1 \lambda |f_i'(x)|^{\lambda} |f_i'(b)|^{1-\lambda} \, d\lambda \\ &= |f_i'(b)| \int_0^1 \lambda J_i^{\lambda} d\lambda \\ &= |f_i'(b)| \frac{I_i \ln J_i + 1 - J_i}{(\ln J_i)^2} \end{aligned} \tag{2.21}$$

where $x \in [a,b]$ and $i = 1, \cdots, n$.

From (2.16), (2.20) and (2.21), we get

$$M_i(x) \le H_i(x) \tag{2.22}$$

where $x \in [a,b]$ and $i = 1, \cdots, n$.

Using (2.15) and (2.22), we get the inequality (2.10).

This completes the proof.

Let $\alpha = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$ in Theorem 2, then we have the following corollary:

Corollary 2.*Let* $f_i : [a,b] \to R$ $(i = 1, \cdots, n)$ *be absolutely continuous functions on* $[a,b]$.

(*a*) *If* $|f_i'|$ $(i = 1, \cdots, n)$ *are convex on* $[a,x]$ *and* $[x,b]$, *then*

$$|S_\alpha(f_1, \cdots, f_n)| \le \frac{1}{n} \sum_{i=1}^{n} \left(\prod_{j=1, j \ne i}^{n} |f_j(x)| \, F_i(x) \right)$$

where $F_i(x)$ *is defined as in* (2.9) *for* $x \in [a,b]$ *and* $i = 1, \cdots, n$.

(*b*) *If* $|f_i'|$ $(i = 1, \cdots, n)$ *are log-convex on* $[a,x]$ *and* $[x,b]$, *then*

$$|S_\alpha(f_1, \cdots, f_n)| \le \frac{1}{n} \sum_{i=1}^{n} \left(\prod_{j=1, j \ne i}^{n} |f_j(x)| \, H_i(x) \right)$$

where $H_i(x)$ *is defined as in* (2.12) *for* $x \in [a,b]$ *and* $i = 1, \cdots, n$.

Remark 2. *If we choose $n = 2$, then Corollary 2 reduces to Theorem B.*

Theorem 3. *Let $f_i : [a, b] \to R$ $(i = 1, \cdots, n)$ be absolutely continuous functions on $[a, b]$.*

(a) If $|f_i'|$ $(i = 1, \cdots, n)$ are convex on $[a, b]$, then

$$|T_\alpha(f_1, \cdots, f_n)| \leq \frac{1}{2(b-a)^2} \sum_{i=1}^{n} \left\{ \int_a^b \left[\prod_{j=1, j \neq i}^{n} |\alpha_i f_j(x)| \left(|f_i'(x)| + \|f_i'\|_\infty \right) E(x) \right] dx \right\} \quad (2.23)$$

where $E(x)$ is defined as in (1.7) for $x \in [a, b]$.

(b) If $|f_i'|$ $(i = 1, \cdots, n)$ are log-convex on $[a, b]$, then

$$|T_\alpha(f_1, \cdots, f_n)| \leq \frac{1}{(b-a)^2} \sum_{i=1}^{n} \left\{ \int_a^b \left[\prod_{j=1, j \neq i}^{n} |\alpha_i f_j(x)| \times \int_a^b \left(|x - t| \, |f_i'(x)| \left(\frac{D_i - 1}{\ln D_i} \right) \right) dt \right] dx \right\} \quad (2.24)$$

where D_i $(i = 1, \cdots, n)$ are defined as in (2.5).

Proof. From the hypotheses of f_i $(i = 1, \cdots, n)$, the identity (2.7) holds. Integrating both sides of (2.7) with respect to x from a to b and rewriting it, we have

$$T_\alpha(f_1, \cdots, f_n) = \frac{1}{(b-a)^2} \sum_{i=1}^{n} \left\{ \int_a^b \left[\prod_{j=1, j \neq i}^{n} \alpha_i f_j(x) \times \int_a^b (x - t) \left(\int_0^1 f_i' [(1-\lambda)x + \lambda t] \, d\lambda \right) dt \right] dx \right\}. \quad (2.25)$$

(a) Since $|f_i'|$ $(i = 1, \cdots, n)$ are convex on $[a, b]$, from (2.25) we get that

$$|T_\alpha(f_1, \cdots, f_n)|$$

$$\leq \frac{1}{(b-a)^2}\sum_{i=1}^{n}\left\{\int_a^b \left[\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)| \times \int_a^b \left(|x-t|\int_0^1 \left[(1-\lambda)|f_i'(x)|+\lambda|f_i'(t)|\right]d\lambda\right)dt\right]dx\right\} \tag{2.26}$$

$$= \frac{1}{(b-a)^2}\sum_{i=1}^{n}\left\{\int_a^b \left[\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)| \times \int_a^b \left(|x-t|\left[\frac{|f_i'(x)|+|f_i'(t)|}{2}\right]\right)dt\right]dx\right\}$$

$$\leq \frac{1}{2(b-a)^2}\sum_{i=1}^{n}\left\{\int_a^b \left[\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)| \times \left(|f_i'(x)|+\|f_i'\|_\infty\right)\int_a^b |x-t|\,dt\right]dx\right\}$$

$$= \frac{1}{2(b-a)^2}\sum_{i=1}^{n}\left\{\int_a^b \left[\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\left(|f_i'(x)|+\|f_i'\|_\infty\right)E(x)\right]dx\right\}$$

which is the inequality (2.23), where $E(x)$ is defined as in (1.7).

(b) Since $|f_i'|$ ($i=1,\cdots,n$) are log-convex on $[a,b]$, from (2.26) we get that

$$|T_\alpha(f_1,\cdots,f_n)|$$

$$\leq \frac{1}{(b-a)^2}\sum_{i=1}^{n}\left\{\int_a^b \left[\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)| \times \int_a^b \left(|x-t|\int_0^1 \left[\left[|f_i'(x)|\right]^{1-\lambda}\left[|f_i'(t)|\right]^{\lambda}\right]d\lambda\right)dt\right]dx\right\}$$

$$= \frac{1}{(b-a)^2}\sum_{i=1}^{n}\left\{\int_a^b \left[\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)| \times \int_a^b \left(|x-t|\,|f_i'(x)|\int_0^1 \left[\frac{|f_i'(t)|}{|f_i'(x)|}\right]^{\lambda}d\lambda\right)dt\right]dx\right\}$$

$$= \frac{1}{(b-a)^2} \sum_{i=1}^{n} \left\{ \int_a^b \left[\prod_{j=1, j\neq i}^{n} |\alpha_i f_j(x)| \right.\right.$$
$$\left.\left. \times \int_a^b \left(|x-t| \, |f_i'(x)| \left(\frac{D_i - 1}{\ln D_i} \right) \right) dt \right] dx \right\}$$

which is the inequality (2.24), where D_i $(i = 1, \cdots, n)$ are defined as in (2.5).

This completes the proof.

Let $\alpha = \left(\frac{1}{n}, \cdots, \frac{1}{n}\right)$ in Theorem 3, then we have the following corollary:

Corollary 3. *Let $f_i : [a, b] \to R$ $(i = 1, \cdots, n)$ be absolutely continuous functions on $[a, b]$.*

(a) If $|f_i'|$ $(i = 1, \cdots, n)$ are convex on $[a, b]$, then

$$|T_\alpha(f_1, \cdots, f_n)|$$
$$\leq \frac{1}{2n(b-a)^2} \sum_{i=1}^{n} \left\{ \int_a^b \left[\prod_{j=1, j\neq i}^{n} |f_j(x)| \left(|f_i'(x)| + \|f_i'\|_\infty\right) E(x) \right] dx \right\}$$

where $E(x)$ is defined as in (1.7) for $x \in [a, b]$.

(b) If $|f_i'|$ $(i = 1, \cdots, n)$ are log-convex on $[a, b]$
, then

$$|T_\alpha(f_1, \cdots, f_n)|$$
$$\leq \frac{1}{n(b-a)^2} \sum_{i=1}^{n} \left\{ \int_a^b \left[\prod_{j=1, j\neq i}^{n} |f_j(x)| \right.\right.$$
$$\left.\left. \times \int_a^b \left(|x-t| \, |f_i'(x)| \left(\frac{D_i - 1}{\ln D_i} \right) \right) dt \right] dx \right\}$$

where D_i $(i = 1, \cdots, n)$ are defined as in (2.5).

Remark 3. *If we choose $n = 2$, then Corollary 3 reduces to Theorem C.*

Theorem 4. *Let $f_i : [a, b] \to R$ $(i = 1, \cdots, n)$ be absolutely continuous functions on $[a, b]$.*

(a) If $|f_i'|$ $(i = 1, \cdots, n)$ are convex on $[a, b]$, then

$$|T_\alpha(f_1, \cdots, f_n)|$$

$$\leq \sum_{i=1}^{n}\left\{\int_a^b\left[\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\left\{\left(\frac{x-a}{b-a}\right)^2\left(\frac{1}{6}|f_i'(a)|+\frac{1}{3}|f_i'(x)|\right)\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{b-x}{b-a}\right)^2\left(\frac{1}{3}|f_i'(x)|+\frac{1}{6}|f_i'(b)|\right)\right\}\right]dx\right\}. \qquad (2.27)$$

(*b*) *If* $|f_i'|$ $(i=1,\cdots,n)$ *are log-convex on* $[a,x]$ *and* $[x,b]$*, then*

$$|T_\alpha(f_1,\cdots,f_n)|$$
$$\leq \sum_{i=1}^{n}\left\{\int_a^b\left[\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\left\{\left(\frac{x-a}{b-a}\right)^2|f_i'(a)|\frac{I_i\ln I_i+1-I_i}{(\ln I_i)^2}\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{b-x}{b-a}\right)^2|f_i'(b)|\frac{J_i\ln J_i+1-J_i}{(\ln J_i)^2}\right\}\right]dx\right\} \qquad (2.28)$$

where I_i*,* J_i $(i=1,\cdots,n)$ *are defined as in* (2.11).

Proof. From the hypotheses of f_i $(i=1,\cdots,n)$, the identity (2.14) holds. Integrating both sides of (2.14) with respect to x from a to b and rewriting it, we have

$$T_\alpha(f_1,\cdots,f_n)$$
$$=\sum_{i=1}^{n}\left\{\int_a^b\left(\prod_{j=1,j\neq i}^{n}\alpha_i f_j(x)\left[\frac{(x-a)^2}{b-a}\int_0^1\lambda f_i'[(1-\lambda)a+\lambda x]\,d\lambda\right.\right.\right.$$
$$\left.\left.\left.-\frac{(b-x)^2}{b-a}\int_0^1\lambda f_i'[\lambda x+(1-\lambda)b]\,d\lambda\right]dx\right)\right\}. \qquad (2.29)$$

(*a*) Since $|f_i'|$ $(i=1,\cdots,n)$ are convex on $[a,b]$, from (2.29) we get that

$$|T_\alpha(f_1,\cdots,f_n)|$$
$$\leq \sum_{i=1}^{n}\left\{\int_a^b\left(\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\left[\frac{(x-a)^2}{b-a}\int_0^1\lambda|f_i'[(1-\lambda)a+\lambda x]|\,d\lambda\right.\right.\right.$$
$$\left.\left.\left.+\frac{(b-x)^2}{b-a}\int_0^1\lambda|f_i'[\lambda x+(1-\lambda)b]|\,d\lambda\right]dx\right)\right\} \qquad (2.30)$$

$$\leq \sum_{i=1}^{n}\left\{\int_a^b\left(\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\right.\right.$$
$$\times\left[\frac{(x-a)^2}{b-a}\int_0^1\left(\lambda(1-\lambda)|f_i'(a)|+\lambda^2|f_i'(x)|\right)d\lambda\right.$$
$$\left.\left.\left.+\ \frac{(b-x)^2}{b-a}\int_0^1\left(\lambda^2|f_i'(x)|+\lambda(1-\lambda)|f_i'(b)|\right)d\lambda\right]dx\right)\right\}$$
$$=\sum_{i=1}^{n}\left\{\int_a^b\left[\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\left\{\left(\frac{x-a}{b-a}\right)^2\left(\frac{1}{6}|f_i'(a)|+\frac{1}{3}|f_i'(x)|\right)\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{b-x}{b-a}\right)^2\left(\frac{1}{3}|f_i'(x)|+\frac{1}{6}|f_i'(b)|\right)\right\}\right]dx\right\}$$

which is the inequality (2.27).

(b) Since $|f_i'|$ $(i=1,\cdots,n)$ are log-convex on $[a,x]$ and $[x,b]$, from (2.30) we get that

$$|T_\alpha(f_1,\cdots,f_n)|$$
$$\leq\sum_{i=1}^{n}\left\{\int_a^b\left(\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\left[\left(\frac{x-a}{b-a}\right)^2\int_0^1\left(\lambda\left[|f_i'(a)|\right]^{1-\lambda}\left[|f_i'(x)|\right]^{\lambda}\right)d\lambda\right.\right.\right.$$
$$\left.\left.\left.+\ \left(\frac{b-x}{b-a}\right)^2\int_0^1\left(\lambda\left[|f_i'(x)|\right]^{\lambda}\left[|f_i'(b)|\right]^{1-\lambda}\right)d\lambda\right]dx\right)\right\}$$
$$=\sum_{i=1}^{n}\left\{\int_a^b\left(\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\left[\left(\frac{x-a}{b-a}\right)^2\left(|f_i'(a)|\int_0^1\lambda I_i^{\lambda}d\lambda\right)\right.\right.\right.$$
$$\left.\left.\left.+\ \left(\frac{b-x}{b-a}\right)^2\left(|f_i'(b)|\int_0^1\lambda J_i^{\lambda}d\lambda\right)\right]dx\right)\right\}$$
$$=\sum_{i=1}^{n}\left\{\int_a^b\left(\prod_{j=1,j\neq i}^{n}|\alpha_i f_j(x)|\left[\left(\frac{x-a}{b-a}\right)^2|f_i'(a)|\frac{I_i\ln I_i+1-I_i}{(\ln I_i)^2}\right.\right.\right.$$
$$\left.\left.\left.+\left(\frac{b-x}{b-a}\right)^2|f_i'(b)|\frac{J_i\ln J_i+1-J_i}{(\ln J_i)^2}\right]dx\right)\right\}$$

which is the inequality (2.28), where $I_i,\ J_i\ \ (i=1,\cdots,n)$ are defined as in (2.11).

This completes the proof.

Let $\alpha=\left(\frac{1}{n},\cdots,\frac{1}{n}\right)$ in Theorem 4, then we have the following corollary:

Corollary 4. *Let $f_i:[a,b]\to R$ $(i=1,\cdots,n)$ be absolutely continuous functions on $[a,b]$.*

(a) If $|f_i'|$ $(i=1,\cdots,n)$ are convex on $[a,b]$, then

$$\begin{aligned}&|T_\alpha(f_1,\cdots,f_n)|\\&\quad\le\frac{1}{n}\sum_{i=1}^{n}\left\{\int_a^b\left[\prod_{j=1,j\neq i}^{n}|f_j(x)|\left\{\left(\frac{x-a}{b-a}\right)^2\left(\frac{1}{6}|f_i'(a)|+\frac{1}{3}|f_i'(x)|\right)\right.\right.\right.\\&\qquad\qquad\left.\left.\left.+\left(\frac{b-x}{b-a}\right)^2\left(\frac{1}{3}|f_i'(x)|+\frac{1}{6}|f_i'(b)|\right)\right\}\right]dx\right\}.\end{aligned}$$

(b) If $|f_i'|$ $(i=1,\cdots,n)$ are log-convex on $[a,x]$ and $[x,b]$, then

$$\begin{aligned}&|T_\alpha(f_1,\cdots,f_n)|\\&\quad\le\frac{1}{n}\sum_{i=1}^{n}\left\{\int_a^b\left[\prod_{j=1,j\neq i}^{n}|f_j(x)|\left\{\left(\frac{x-a}{b-a}\right)^2|f_i'(a)|\frac{I_i\ln I_i+1-I_i}{(\ln I_i)^2}\right.\right.\right.\\&\qquad\qquad\left.\left.\left.+\left(\frac{b-x}{b-a}\right)^2|f_i'(b)|\frac{J_i\ln J_i+1-J_i}{(\ln J_i)^2}\right\}\right]dx\right\}\end{aligned}$$

where $I_i,\ J_i\ \ (i=1,\cdots,n)$ are defined as in (2.11).

Remark 4. *If we choose $n=2$, then Corollary 4 reduces to Theorem D.*

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